Well-posedness results for the transport equation, and applications to the chromatography system

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The $k \times k$ chromatography system

$$
\begin{cases}
\partial_t u_1 + \partial_x \left(\dfrac{u_1}{1 + u_1 + u_2 + \cdots + u_k} \right) = 0 \\[2ex]
\partial_t u_2 + \partial_x \left(\dfrac{u_2}{1 + u_1 + u_2 + \cdots + u_k} \right) = 0 \\[2ex]
\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\[2ex]
\partial_t u_k + \partial_x \left(\dfrac{u_k}{1 + u_1 + u_2 + \cdots + u_k} \right) = 0
\end{cases}
$$

- $u_1 \geq 0,\ u_2 \geq 0,\ \ldots\ u_k \geq 0$

The chromatography system

$$\begin{cases} \partial_t u_1 + \partial_x \left(\dfrac{u_1}{1 + u_1 + u_2} \right) = 0 \\ \\ \partial_t u_2 + \partial_x \left(\dfrac{u_2}{1 + u_1 + u_2} \right) = 0 \end{cases}$$

- $t \in [0, +\infty[, \; x \in [0, +\infty[$
- $u_1 \geq 0, \; u_2 \geq 0$

Well-posedness of continuity equation

$$\begin{cases} \partial_t w + \operatorname{div}_x \big[b(t,x) w \big] = 0 \\ w(0,x) = \bar{w}(x) \end{cases}$$

Hypotheses:

- b bounded, $b \in BV([0,+\infty[\times\mathbb{R}^d, \mathbb{R}^d)$
- $\bar{w}$ is bounded
- $\exists\ \rho \in BV([0,+\infty[\times\mathbb{R}^d)$ such that

$$\frac{1}{C} \leq \rho \leq C \quad \partial_t \rho + \operatorname{div}_x(b\rho) = 0 \quad \text{and technical hypotheses}$$

Then $\exists!$ solution $w \in \mathcal{C}^0([0,+\infty[; L^\infty - w^*)$

Chromatography: main result I

Hypotheses:

- $\bar{u}_1, \bar{u}_2 \in L^\infty(\mathbb{R})$;
- $\bar{u}_1 \geq 0, \ \bar{u}_2 \geq 0$
- $\bar{u}_1 + \bar{u}_2 \in BV(\mathbb{R})$

Then:

- Existence + Uniqueness: $\exists!$ distributional solution (u_1, u_2) satisfying
  1. (u_1, u_2) is entropy admissible
  2. $(u_1, u_2) \in \mathcal{C}^0([0, +\infty[; L^\infty - w^*)$
  3. $u_1(t, \cdot) \geq 0$, $u_2(t, \cdot) \geq 0$ for every t
- $(u_1, u_2) \in \mathcal{C}^0([0, +\infty[; L^1_{\mathrm{loc}} - s)$
- $u_1(t, \cdot) + u_2(t, \cdot) \in BV(\mathbb{R})$ for every t

Chromatography: main result II

Hypotheses:

- $\bar{u}_1, \bar{u}_2 \in L^\infty(\mathbb{R})$;
- $\bar{u}_1 \geq 0, \;\bar{u}_2 \geq 0$
- for every $R > 0$, there exists $\delta_R > 0$ such that

$$\bar{u}_1 + \bar{u}_2 \geq \delta_R > 0 \quad \text{in }]-R, R[$$

Then:

- Existence + Uniqueness: $\exists!$ distributional solution (u_1, u_2) satisfying
  1. (u_1, u_2) is entropy admissible
  2. (u_1, u_2) is bounded
  3. $(u_1, u_2) \in \mathcal{C}^0([0, +\infty[; L^1_{\text{loc}} - s)$
- $u_1(t, \cdot), \;u_2(t, \cdot) \in L^\infty(\mathbb{R})$ for every t
- $u_1(t, \cdot) \geq 0, \;u_2(t, \cdot) \geq 0$ for every t
- for every $t, \;R > 0$, there exists $\delta_{t,R} > 0$ such that

$$\bar{u}_1 + \bar{u}_2 \geq \delta_{t,R} > 0 \quad \text{in }]-R, R[$$