

# Selected Topics in Approximate Solutions of Nonlinear Conservation Laws. High-Resolution Central Schemes.

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**Central schemes** offer a simple and versatile approach for computing approximate solutions of nonlinear systems of hyperbolic conservation laws and related PDEs. The solution of such problems often involves the spontaneous evolution of steep gradients. The multiscale aspect of these gradients poses a main computational challenge for their numerical solution. Central schemes utilize a minimal amount of information on the propagation speeds associated with the problems, in order to accurately detect these steep gradients. This information is then coupled with high-order, non-oscillatory reconstruction of the approximate solution in ‘the direction of smoothness’: that is, information of smoothness does not cross regions of steep gradients. The use of central stencils enable us to realize the reconstructed solutions through simple quadratures. In this manner, central schemes avoid the intricate and time-consuming details of the eigen-structure of the underlying PDEs, and in particular, the use of (approximate) Riemann solvers, dimensional splitting, etc. The resulting family of central schemes offers relatively simple, “black-box” solvers for a wide variety of problems governed by multi-dimensional systems of non-linear hyperbolic PDEs.

We highlight several features of this new class of central schemes.

**Scalar equations.** For both the second- and third-order schemes we prove variation bounds (– which in turn yield convergence with precise error estimates), as well as entropy and multidimensional  $L^\infty$ -stability estimates. **Systems of equations.** Extension to systems is carried out by *componentwise* application of the scalar framework. It is in this context that our central schemes offer a remarkable advantage over the corresponding upwind framework. **Multidimensional problems.** Since we bypass the need for (approximate) Riemann solvers, multidimensional problems are solved *without* dimensional splitting. In fact, the class of central schemes is utilized for a variety of nonlinear transport equations. A partial list of more than 120 references can be found in **CentPack**: <http://www.cscamm.umd.edu/centpack/>. CentPack is a collection of freely distributed C++ routines that implement a number of high-order, non-oscillatory central schemes for hyperbolic systems of conservation laws in one- and two-space dimensions,

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x + \mathbf{g}(\mathbf{u})_y = 0.$$

The efficiency and versatility of central schemes resides, mainly, in their simplicity: they eliminate the need for Riemann solvers and avoid dimensional splitting, yielding a generic formulation valid for any hyperbolic system that can be written in the above form. Only information specific to the model and problem to be solved needs to be provided; namely, a description of the flux functions  $\mathbf{f}(\mathbf{u})$  and  $\mathbf{g}(\mathbf{u})$ , and the appropriate initial and boundary conditions.

The numerical algorithm for the implementation of central schemes consists of two main steps: (i) a non-oscillatory polynomial reconstruction of point values from their cell averages, followed by (ii) the time evolution of the flux functions  $\mathbf{f}(\mathbf{u})$  and  $\mathbf{g}(\mathbf{u})$ .

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# 1 Introduction

In recent years, central schemes for approximating solutions of hyperbolic conservation laws, received a considerable amount of renewed attention. A family of high-resolution, non-oscillatory, *central* schemes, was developed to handle such problems. Compared with the 'classical' *upwind* schemes, these *central* schemes were shown to be both simple and stable for a large variety of problems ranging from one-dimensional scalar problems to multi-dimensional systems of conservation laws. They were successfully implemented for a variety of other related problems, such as, e.g., the incompressible Euler equations [25],[22],[20], [21], the magneto-hydrodynamics equations [42], viscoelastic flows—[20] hyperbolic systems with relaxation source terms [4],[34],[35] non-linear optics [33],[7], and slow moving shocks [17].

The family of high-order *central* schemes we deal with, can be viewed as a direct extension to the first-order, Lax-Friedrichs (LxF) scheme [9], which on one hand is robust and stable, but on the other hand suffers from excessive dissipation. To address this problematic property of the LxF scheme, a Godunov-like second-order central scheme was developed by Nessyahu and Tadmor (NT) in [29] (see also [38]). It was extended to higher-order of accuracy as well as for more space dimensions (consult [1], [16], [2], [3] and [21], for the two-dimensional case, and [37], [14], [28] and [24] for the third-order schemes).

The NT scheme is based on reconstructing, in each time step, a piecewise-polynomial interpolant from the cell-averages computed in the previous time step. This interpolant is then (exactly) evolved in time, and finally, it is projected on its staggered averages, resulting with the staggered cell-averages at the next time-step. The one- and two-dimensional second-order schemes, are based on a piecewise-linear MUSCL-type reconstruction, whereas the third-order schemes are based on the non-oscillatory piecewise-parabolic reconstruction [27],[28]. Higher orders are treated in [36].

Like *upwind* schemes, the reconstructed piecewise-polynomials used by the central schemes, also make use of non-linear limiters which guarantee the overall non-oscillatory nature of the approximate solution. But unlike the upwind schemes, central schemes avoid the intricate and time consuming Riemann solvers; this advantage is particularly important in the multi-dimensional setup, where no such Riemann solvers exist.

## 2 A Short Guide to Godunov-Type schemes

We want to solve the hyperbolic system of conservation laws

$$u_t + f(u)_x = 0 \tag{2.1}$$

by Godunov-type schemes. To this end we proceed in two steps. First, we introduce a small spatial scale,  $\Delta x$ , and we consider the corresponding (Steklov) sliding average of  $u(\cdot, t)$ ,

$$\bar{u}(x, t) := \frac{1}{|I_x|} \int_{I_x} u(\xi, t) d\xi, \quad I_x = \left\{ \xi \mid |\xi - x| \leq \frac{\Delta x}{2} \right\}.$$

The sliding average of (2.1) then yields

$$\bar{u}_t(x, t) + \frac{1}{\Delta x} \left[ f(u(x + \frac{\Delta x}{2}, t)) - f(u(x - \frac{\Delta x}{2}, t)) \right] = 0. \tag{2.2}$$

Next, we introduce a small time-step,  $\Delta t$ , and integrate over the slab  $t \leq \tau \leq t + \Delta t$ ,

$$\begin{aligned} \bar{u}(x, t + \Delta t) &= \bar{u}(x, t) \\ &\quad - \frac{1}{\Delta x} \left[ \int_{\tau=t}^{t+\Delta t} f(u(x + \frac{\Delta x}{2}, \tau)) d\tau - \int_{\tau=t}^{t+\Delta t} f(u(x - \frac{\Delta x}{2}, \tau)) d\tau \right]. \end{aligned} \tag{2.3}$$

We end up with an equivalent reformulation of the conservation law (2.1): it expresses the precise relation between the sliding averages,  $\bar{u}(\cdot, t)$ , and their underlying pointvalues,  $u(\cdot, t)$ . We shall use this reformulation, (2.3), as the starting point for the construction of Godunov-type schemes.

We construct an approximate solution,  $w(\cdot, t^n)$ , at the discrete time-levels,  $t^n = n\Delta t$ . Here,  $w(x, t^n)$  is a piecewise polynomial written in the form

$$w(x, t^n) = \sum p_j(x)\chi_j(x), \quad \chi_j(x) := 1_{I_j},$$

where  $p_j(x)$  are algebraic polynomials supported at the discrete cells,  $I_j = I_{x_j}$ , centered around the midpoints,  $x_j := j\Delta x$ . An *exact* evolution of  $w(\cdot, t^n)$  based on (2.3), reads

$$\begin{aligned} \bar{w}(x, t^{n+1}) &= \bar{w}(x, t^n) \\ &- \frac{1}{\Delta x} \left[ \int_{t^n}^{t^{n+1}} f(w(x + \frac{\Delta x}{2}, \tau)) d\tau - \int_{t^n}^{t^{n+1}} f(w(x - \frac{\Delta x}{2}, \tau)) d\tau \right]. \end{aligned} \quad (2.4)$$

To construct a Godunov-type scheme, we *realize* (2.4) — or at least an accurate approximation of it, at discrete gridpoints. Here, we distinguish between the main methods, according to their way of *sampling* (2.4): these two main sampling methods correspond to upwind schemes and central schemes.

## 2.1 Upwind schemes

Let  $\bar{w}_j^n$  abbreviates the cell averages,  $\bar{w}_j^n := \frac{1}{\Delta x} \int_{I_j} w(\xi, t^n) d\xi$ . By sampling (2.4) at the *mid-cells*,  $x = x_j$ , we obtain an evolution scheme for these averages, which reads

$$\bar{w}_j^{n+1} = \bar{w}_j^n - \frac{1}{\Delta x} \left[ \int_{\tau=t^n}^{t^{n+1}} f(w(x_{j+\frac{1}{2}}, \tau)) d\tau - \int_{\tau=t^n}^{t^{n+1}} f(w(x_{j-\frac{1}{2}}, \tau)) d\tau \right]. \quad (2.5)$$

Here, it remains to recover the *pointvalues*,  $\{w(x_{j+\frac{1}{2}}, \tau)\}_j$ ,  $t^n \leq \tau \leq t^{n+1}$ , in terms of their known cell averages,  $\{\bar{w}_j^n\}_j$ , and to this end we proceed in two steps:

- First, the *reconstruction* — we recover the pointwise values of  $w(\cdot, \tau)$  at  $\tau = t^n$ , by a reconstruction of a piecewise polynomial approximation

$$w(x, t^n) = \sum_j p_j(x)\chi_j(x), \quad \bar{p}_j(x_j) = \bar{w}_j^n. \quad (2.6)$$

- Second, the *evolution* —  $w(x_{j+\frac{1}{2}}, \tau \geq t^n)$  are determined as the solutions of the generalized Riemann problems

$$w_t + f(w)_x = 0, \quad t \geq t^n; \quad w(x, t^n) = \begin{cases} p_j(x) & x < x_{j+\frac{1}{2}}, \\ p_{j+1}(x) & x > x_{j+\frac{1}{2}}. \end{cases} \quad (2.7)$$

The solution of (2.7) is composed of a family of nonlinear waves — left-going and right-going waves. An exact Riemann solver, or at least an approximate one is used to distribute these nonlinear waves between the two neighboring cells,  $I_j$  and  $I_{j+1}$ . It is this distribution of waves according to their direction which is responsible for *upwind differencing*, consult Figure 2.1. We briefly recall few canonical examples for this category of upwind Godunov-type schemes.



- First, we use the given cell averages  $\{\bar{w}_j^n\}_j$ , to *reconstruct* the pointvalues of  $w(\cdot, \tau = t^n)$  as piecewise polynomial approximation

$$w(x, t^n) = \sum_j p_j(x) \chi_j(x), \quad \bar{p}_j(x_j) = \bar{w}_j^n. \quad (2.10)$$

In particular, the staggered averages on the right of (2.9) are given by

$$\bar{w}_{j+\frac{1}{2}}^n = \frac{1}{\Delta x} \left[ \int_{x_j}^{x_{j+\frac{1}{2}}} p_j(x) dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p_{j+1}(x) dx \right]. \quad (2.11)$$

The resulting central scheme (2.9) then reads

$$\begin{aligned} \bar{w}_{j+\frac{1}{2}}^{n+1} &= \frac{1}{\Delta x} \left[ \int_{x_j}^{x_{j+\frac{1}{2}}} p_j(x) dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} p_{j+1}(x) dx \right] + \\ &- \frac{1}{\Delta x} \left[ \int_{\tau=t^n}^{t^{n+1}} f(w(x_{j+1}, \tau)) d\tau - \int_{\tau=t^n}^{t^{n+1}} f(w(x_j, \tau)) d\tau \right]. \end{aligned} \quad (2.12)$$

- Second, we follow the *evolution* of the pointvalues along the mid-cells,  $x = x_j$ ,  $\{w(x_j, \tau \geq t^n)\}_j$ , which are governed by

$$w_t + f(w)_x = 0, \quad \tau \geq t^n; \quad w(x, t^n) = p_j(x) \quad x \in I_j. \quad (2.13)$$

Let  $\{a_k(u)\}_k$  denote the eigenvalues of the Jacobian  $A(u) := \frac{\partial f}{\partial u}$ . By hyperbolicity, information regarding the interfacing discontinuities at  $(x_{j \pm \frac{1}{2}}, t^n)$  propagates no faster than  $\max_k |a_k(u)|$ . Hence, the mid-cells values governed by (2.13),  $\{w(x_j, \tau \geq t^n)\}_j$ , remain free of discontinuities, at least for sufficiently small time step dictated by the CFL condition  $\Delta t \leq \frac{1}{2} \Delta x \cdot \max_k |a_k(u)|$ . Consequently,

since the numerical fluxes on the right of (2.12),  $\int_{\tau=t^n}^{t^{n+1}} f(w(x_j, \tau)) d\tau$ , involve only smooth integrands, they can be computed within any degree of desired accuracy by an appropriate quadrature rule.

It is the *staggered* averaging over the fan of left-going and right-going waves centered at the half-integrated interfaces,  $(x_{j+\frac{1}{2}}, t^n)$ , which characterizes the *central* differencing, consult Figure 2.2. A main feature of these central schemes – in contrast to upwind ones, is the computation of *smooth* numerical fluxes along the mid-cells,  $(x = x_j, \tau \geq t^n)$ , which avoids the costly (approximate) Riemann solvers. A couple of examples of central Godunov-type schemes is in order.

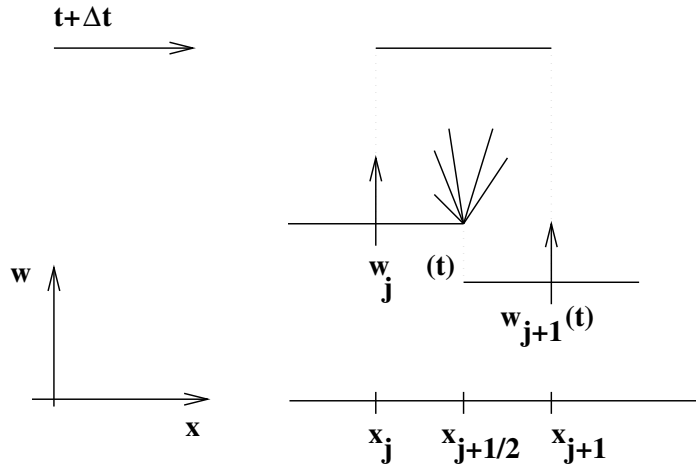


Figure 2.2: Central differencing by Godunov-type scheme.

The first-order Lax-Friedrichs (LxF) approximation is the forerunner for such central schemes — it is based on piecewise constant reconstruction,  $w(x, t^n) = \sum p_j(x) \chi_j(x)$  with  $p_j(x) = \bar{w}_j^n$ . The resulting central scheme, (2.12), then reads (with the usual fixed mesh ratio  $\lambda := \frac{\Delta t}{\Delta x}$ )

$$\bar{w}_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(\bar{w}_j + \bar{w}_{j+1}) - \lambda [f(\bar{w}_{j+1}) - f(\bar{w}_j)]. \quad (2.14)$$

Our main focus in the rest of this chapter is on non-oscillatory higher-order extensions of the LxF schemes.

### 3 Central schemes in one-space dimension

#### 3.1 The second-order Nessyahu-Tadmor scheme

In this section we overview the construction of high-resolution central schemes in one-space dimension. We begin with the reconstruction of the second-order, non-oscillatory Nessyahu and Tadmor (NT) scheme, [29]. To approximate solutions of (2.1), we introduce a piecewise-linear approximate solution at the discrete time levels,  $t^n = n\Delta t$ , based on linear functions  $p_j(x, t^n)$  which are supported at the cells  $I_j$  (see Figure 3.1),

$$w(x, t)|_{t=t^n} = \sum_j p_j(x, t^n) \chi_j(x) := \sum_j \left[ \bar{w}_j^n + w'_j \left( \frac{x - x_j}{\Delta x} \right) \right] \chi_j(x), \quad \chi_j(x) := 1_{I_j}. \quad (3.1)$$

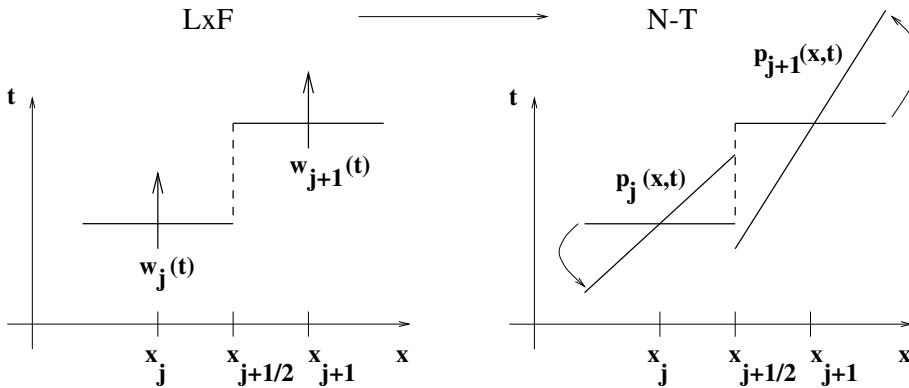


Figure 3.1: The second-order reconstruction

Second-order of accuracy is guaranteed if the discrete slopes approximate the corresponding derivatives,  $w'_j \sim \Delta x \cdot \partial_x w(x_j, t^n) + O(\Delta x)^2$ . Such a non-oscillatory approximation of the derivatives is possible, e.g., by using built-in non-linear limiters of the form

$$w'_j = MM \left\{ \theta(\bar{w}_{j+1}^n - \bar{w}_j^n), \frac{1}{2}(\bar{w}_{j+1}^n - \bar{w}_{j-1}^n), \theta(\bar{w}_j^n - \bar{w}_{j-1}^n) \right\}. \quad (3.2)$$

Here and below,  $\theta \in (0, 2)$  is a non-oscillatory limiter and  $MM$  denotes the Min-Mod function

$$MM\{x_1, x_2, \dots\} = \begin{cases} \min_i \{x_i\} & \text{if } x_i > 0, \forall i \\ \max_i \{x_i\} & \text{if } x_i < 0, \forall i \\ 0 & \text{otherwise.} \end{cases}$$

An *exact* evolution of  $w$ , based on integration of the conservation law over the staggered cell,  $I_{j+\frac{1}{2}}$ , then reads, (2.9)

$$\bar{w}_{j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} w(x, t^n) dx - \frac{1}{\Delta x} \int_{\tau=t^n}^{t^{n+1}} [f(w(x_{j+1}, \tau)) - f(w(x_j, \tau))] d\tau.$$

The first integral is the staggered cell-average at time  $t^n$ ,  $\bar{w}_{j+\frac{1}{2}}^n$ , which can be computed directly from the above reconstruction,

$$\bar{w}_{j+\frac{1}{2}}^n := \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} w(x, t^n) dx = \frac{1}{2}(\bar{w}_j^n + \bar{w}_{j+1}^n) + \frac{1}{8}(w'_j - w'_{j+1}). \quad (3.3)$$

The time integrals of the flux are computed by the second-order accurate mid-point quadrature rule

$$\int_{\tau=t^n}^{t^{n+1}} f(w(x_j, \tau)) d\tau \sim \Delta t \cdot f(w(x_j, t^{n+\frac{1}{2}})).$$

Here, the Taylor expansion is being used to predict the required mid-values of  $w$

$$\begin{aligned} w(x_j, t^{n+\frac{1}{2}}) &\sim w(x_j, t) + \frac{\Delta t}{2} w_t(x_j, t^n) \\ &= \bar{w}_j^n - \frac{\Delta t}{2} A(\bar{w}_j^n)(p_j(x_j, t^n))_x = \bar{w}_j^n - \frac{\lambda}{2} A_j^n w'_j. \end{aligned}$$

In summary, we end up with the central scheme, [29], which consists of a first-order *predictor step*,

$$w_j^{n+\frac{1}{2}} = \bar{w}_j^n - \frac{\lambda}{2} A_j^n w'_j, \quad A_j^n := A(\bar{w}_j^n), \quad (3.4)$$

followed by the second-order *corrector step*, (2.12),

$$\bar{w}_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(\bar{w}_j^n + \bar{w}_{j+1}^n) + \frac{1}{8}(w'_j - w'_{j+1}) - \lambda \left[ f(w_{j+1}^{n+\frac{1}{2}}) - f(w_j^{n+\frac{1}{2}}) \right]. \quad (3.5)$$

The *scalar* non-oscillatory properties of (3.4)-(3.5) were proved in [29], including the TVD property, cell entropy inequality,  $L^1_{loc}$ -error estimates, etc. Moreover, the numerical experiments, reported in [29], [2], [3], [42], [34], [35], [36], with one-dimensional *systems* of conservation laws, show that such second-order central schemes enjoy the same high-resolution as the corresponding second-order upwind schemes do. Thus, the excessive smearing typical to the first-order LxF central scheme is compensated here by the second-order accurate MUSCL reconstruction.

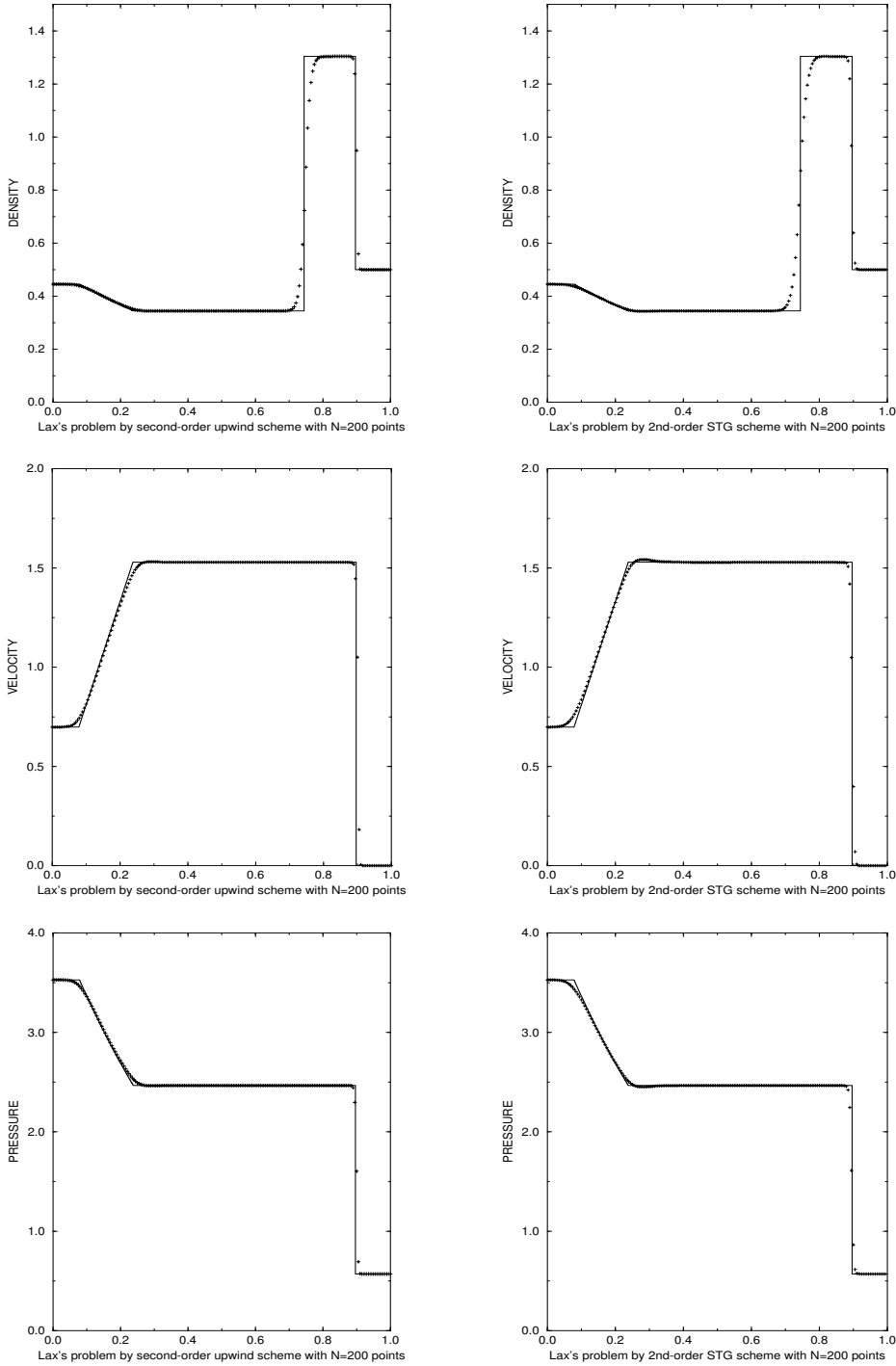


Figure 3.2: 2nd order: central (STG) vs. upwind (ULT) — Lax's Riemann problem

In figure 3.2 we compare, side by side, the upwind ULT scheme of Harten, [12], with our central scheme (3.4)-(3.5). The comparable high-resolution of this so called Lax's Riemann problem is evident.

At the same time, the central scheme (3.4)-(3.5) has the advantage over the corresponding upwind schemes, in that no (approximate) Riemann solvers, as in (2.7), are required. Hence, these Riemann-free central schemes provide an efficient high-resolution alternative in the one-dimensional case, and a particularly advantageous framework for multidimensional computations, e.g., [3], [2], [16]. This advantage in

the multidimensional case will be explored in the next section. Also, *staggered* central differencing, along the lines of the Riemann-free Nessyahu-Tadmor scheme (3.4)-(3.5), admits simple efficient extensions in the presence of general source terms, [8], and in particular, stiff source terms. Indeed, it is a key ingredient behind the relaxation schemes studied in [18].

It should be noted, however, that the component-wise version of these central schemes might result in deterioration of resolution at the computed extrema. The second-order computation presented in figure 3.2 below demonstrates this point. (this will be corrected by higher order central methods). Of course, this – so called extrema clipping, is typical to high-resolution upwind schemes as well; but it is more pronounced with our central schemes due to the built-in extrema-switching to the dissipative LxF scheme. Indeed, once an extrema cell,  $I_j$ , is detected (by the limiter), it sets a zero slope,  $w'_j = 0$ , in which case the second-order scheme (3.4)-(3.5) is reduced back to the first-order LxF, (2.14).

### 3.2 The third-order central scheme

Following the framework outlined in §3.1, the upgrade to third-order central scheme consists of two main ingredients:

- (i) A third-order accurate, piecewise-quadratic polynomial reconstruction which enjoys desirable non-oscillatory properties;
- (ii) An appropriate quadrature rule to approximate the numerical fluxes along cells' interfaces.

Following [28], we proceed as follows. The piecewise-parabolic reconstruction takes the form

$$p_j(x) = w_j^n + w'_j \left( \frac{x - x_j}{\Delta x} \right) + \frac{1}{2} w''_j \left( \frac{x - x_j}{\Delta x} \right)^2. \quad (3.6)$$

Here,  $w''_j$  are the (pointvalues of) the *reconstructed second derivatives*

$$w''_j := \theta_j \Delta_+ \Delta_- \bar{w}_j^n; \quad (3.7)$$

$w'_j$  are the (pointvalues of) the *reconstructed slopes*,

$$w'_j := \theta_j \Delta_0 \bar{w}_j^n; \quad (3.8)$$

and  $w_j^n$  are the *reconstructed pointvalues*

$$w_j^n := \bar{w}_j^n - \frac{w''_j}{24}. \quad (3.9)$$

Observe that, starting with third- (and higher-) order accurate methods, pointwise values *cannot* be interchanged with cell averages,  $w_j^n \neq \bar{w}_j^n$ .

Here,  $\theta_j$  are appropriate nonlinear limiters which guarantee the non-oscillatory behavior of the third-order reconstruction; its precise form can be found in [27], [28]. They guarantee that the reconstruction (3.6) is non-oscillatory in the sense that  $N(w(\cdot, t^n))$  — the number of extrema of  $w(x, t^n)$ , does not exceed that of its piecewise-constant projection,  $N(\Sigma \bar{w}_j^n \chi_j(\cdot))$ ,

$$N(w(\cdot, t^n)) \leq N(\Sigma \bar{w}_j^n \chi_j(\cdot)). \quad (3.10)$$

Next we turn to the evolution of the piecewise-parabolic reconstructed solution. To this end we need to evaluate the staggered averages,  $\{\bar{w}_{j+\frac{1}{2}}^n\}$ , and to approximate the interface fluxes,  $\left\{ \int_{\tau=t^n}^{t^{n+1}} f(w(x_j, \tau)) d\tau \right\}$ .

With  $p_j(x) = w_j^n + w'_j \left( \frac{x - x_j}{\Delta x} \right) + \frac{1}{2} w''_j \left( \frac{x - x_j}{\Delta x} \right)^2$  specified in (3.6)-(3.9), one evaluates the staggered averages of the third order reconstruction  $w(x, t^n) = \Sigma p_j(x) \chi_j(x)$

$$\bar{w}_{j+\frac{1}{2}}^n = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} w(x, t^n) dx = \frac{1}{2} (\bar{w}_j + \bar{w}_{j+1}) + \frac{1}{8} (w'_j - w'_{j+1}). \quad (3.11)$$

Remarkably, we obtain here the same formula for the staggered averages as in the second-order cases, consult (3.3); the only difference is the use of the new limited slopes in (3.8),  $w'_j = \theta_j \Delta_0 \bar{w}_j^n$ .

Next, we approximate the (exact) numerical fluxes by Simpson's quadrature rule, which is (more than) sufficient for retaining the overall third-order accuracy,

$$\frac{1}{\Delta x} \int_{\tau=t^n}^{t^{n+1}} f(w(x_j, \tau)) d\tau \sim \frac{\lambda}{6} \left[ f(w_j^n) + 4f(w_j^{n+\frac{1}{2}}) + f(w_j^{n+1}) \right]. \quad (3.12)$$

This in turn, requires the three approximate *pointvalues* on the right,  $w_j^{n+\beta} \sim w(x_j, t^{n+\beta})$  for  $\beta = 0, \frac{1}{2}, 1$ . Following our approach in the second-order case, [29], we use Taylor expansion to *predict*

$$w_j^n = \bar{w}_j^n - \frac{w_j''}{24}; \quad (3.13)$$

$$\begin{aligned} \dot{w}_j^n &\equiv (\Delta x \cdot \partial_t) w(x_j, t^n) = -\Delta x \cdot \partial_x f(w(x_j, t^n)) = \\ &= -a(w_j^n) \cdot w'_j; \end{aligned} \quad (3.14)$$

$$\begin{aligned} \ddot{w}_j^n &\equiv (\Delta x \cdot \partial_t)^2 w(x_j, t^n) = \\ &= \Delta x \cdot \partial_x [a(w_j^n) \Delta x \cdot \partial_x f(w(x_j, t^n))] = \\ &= a^2(w_j^n) w_j'' + 2a(w_j^n) a'(w_j^n) (w'_j)^2. \end{aligned} \quad (3.15)$$

In summary of the scalar setup, we end up with a two step scheme where, starting with the reconstructed pointvalues

$$w_j^n = \bar{w}_j^n - \frac{w_j''}{24}, \quad (3.16)$$

we *predict* the pointvalues  $w_j^{n+\beta}$  by, e.g. Taylor expansions,

$$w_j^{n+\beta} = w_j^n + \lambda \beta \dot{w}_j^n + \frac{(\lambda \beta)^2}{2} \ddot{w}_j^n, \quad \beta = \frac{1}{2}, 1; \quad (3.17)$$

this is followed by the *corrector* step

$$\begin{aligned} \bar{w}_{j+\frac{1}{2}}^n &= \frac{1}{2} (\bar{w}_j^n + \bar{w}_{j+1}^n) + \frac{1}{8} (w'_j - w'_{j+1}) + \\ &- \frac{\lambda}{6} \left\{ \left[ f(w_{j+1}^n) + 4f(w_{j+1}^{n+\frac{1}{2}}) + f(w_{j+1}^{n+1}) \right] \right. \\ &\quad \left. - \left[ f(w_j^n) + 4f(w_j^{n+\frac{1}{2}}) + f(w_j^{n+1}) \right] \right\}. \end{aligned} \quad (3.18)$$

In figure 3.2 we revisit the so called Woodward-Colella problem, [43], where we compare the second vs. the third-order results. The improvement in resolving the density field is evident.

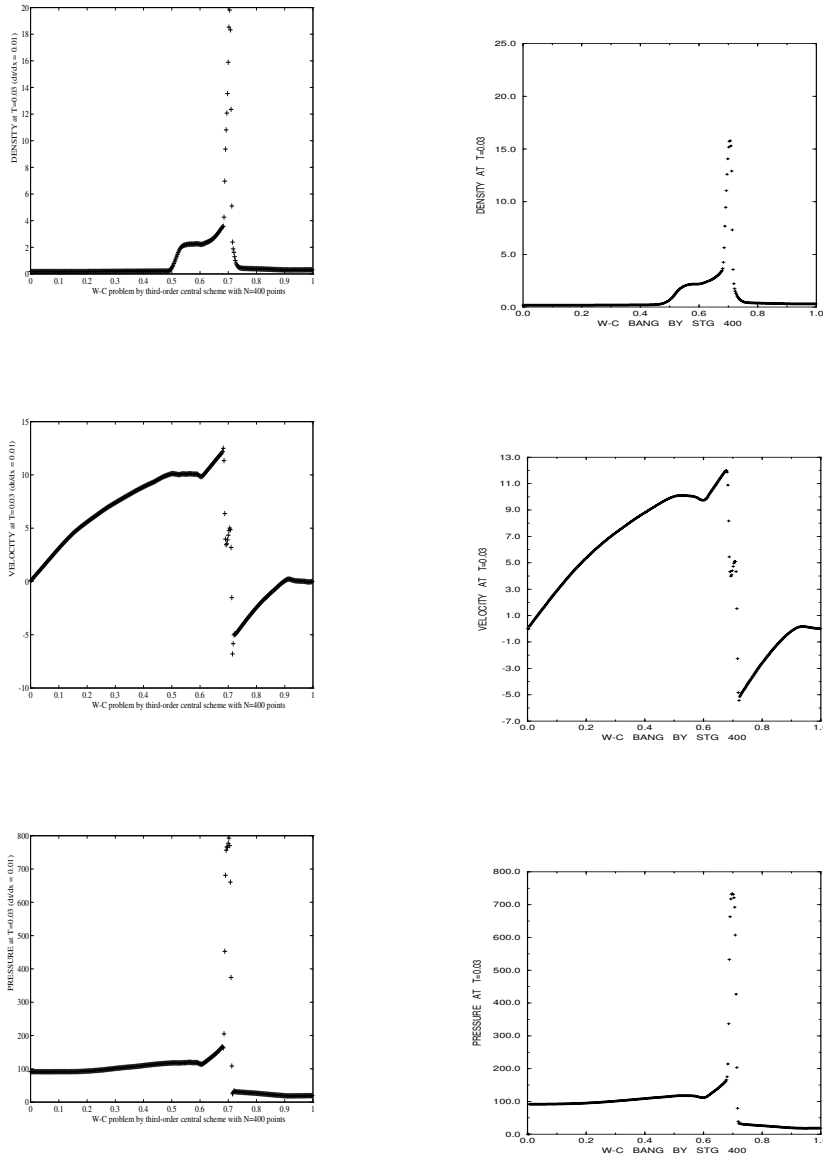


Figure 3.3:  $3^{rd}$  vs.  $2^{nd}$  order central schemes — Woodward-Colella problem at  $t = 0.03$

We conclude this section with several remarks.

*Remarks.*

1. Stability.

We briefly mention the stability results for the scalar central schemes. In the second order case, the NT scheme was shown to be both TVD and entropy stable in the sense of satisfying a cell entropy inequality – consult [29]. The third-order scalar central scheme is stable in the sense of satisfying the NED property, (3.10), namely

**Theorem 3.1** ([28]) *Consider the central scheme (3.16),(3.17),(3.18), based on the third-order accurate quadratic reconstruction, (3.6)-(3.9). Then it satisfies the so-called Number of Extrema*

*Diminishing (NED) property, in the sense that*

$$N \left( \sum_{\nu} \bar{w}_{\nu+\frac{1}{2}}^{n+1} \chi_{\nu+\frac{1}{2}}(x) \right) \leq N \left( \sum_{\nu} \bar{w}_{\nu}^n \chi_{\nu}(x) \right). \quad (3.19)$$

2. Source terms, radial coordinates, ...

Extensions of the central framework which deal with both, stiff and non-stiff source terms can be found in [34],[35], [8], [4]. In particular, Kupferman in [20],[21] developed the central framework within the radial coordinates which require to handle both – variable coefficients + source terms.

3. Higher order central schemes.

We refer to [36], where a high-order ENO reconstruction is realized by a staggered cell averaging. Here, intricate Riemann solvers are replaced by high order quadrature rules. and for this purpose, one can effectively use the RK method (rather than the Taylor expansion outlined above):

4. Taylor vs. Runge-Kutta.

The evaluations of Taylor expansions could be substituted by the more economical Runge-Kutta integrations; the simplicity becomes more pronounced with *systems*. A particular useful approach in this context was proposed in [36], using the natural continuous extensions of RK schemes.

5. Systems.

One of the main advantages of our central-staggered framework over that of the upwind schemes, is that expensive and time-consuming characteristic decompositions can be avoided. Specifically, all the non-oscillatory computations can be carried out with diagonal limiters, based on a *component-wise* extension of the scalar limiters outlined above.

## 4 Central schemes in two space dimensions

Following the one dimensional setup, one can derive a non-oscillatory, two-dimensional central scheme. Here we sketch the construction of the second-order two-dimensional scheme following [16] (see also [2],[1]). For the two-dimensional third-order accurate scheme, we refer to [24].

We consider the two-dimensional hyperbolic system of conservation laws

$$u_t + f(u)_x + g(u)_y = 0. \quad (4.1)$$

To approximate a solution to (4.1), we start with a two-dimensional linear reconstruction

$$w(x, y, t^n) = \sum_{j,k} p_{j,k}(x, y) \chi_{j,k}(x, y), \quad p_{j,k}(x, y) = \bar{w}_{j,k}^n + w'_{j,k} \left( \frac{x - x_j}{\Delta x} \right) + w^{\backslash}_{j,k} \left( \frac{y - y_k}{\Delta y} \right). \quad (4.2)$$

Here, the discrete slopes in the  $x$  and in the  $y$  direction approximate the corresponding derivatives,  $w'_{j,k} \sim \Delta x \cdot w_x(x_j, y_k, t^n) + O(\Delta x)^2$ ,  $w^{\backslash}_{j,k} \sim \Delta y \cdot w_y(x_j, y_k, t^n) + O(\Delta y)^2$ , and  $\chi_{j,k}(x, y)$  is the characteristic function of the cell  $C_{j,k} := \left\{ (\xi, \eta) \mid |\xi - x_j| \leq \frac{\Delta x}{2}, |\eta - y_k| \leq \frac{\Delta y}{2} \right\} = I_j \otimes J_k$ . Of course, it is essential to reconstruct the discrete slopes,  $u'$  and  $u^{\backslash}$ , with built in *limiters*, which guarantee the non-oscillatory character of the reconstruction; the family of min-mod limiters is a prototype example

$$u'_{j,k} = MM \left\{ \theta(\bar{w}_{j+1,k}^n - \bar{w}_{j,k}^n), \frac{1}{2}(\bar{w}_{j+1,k}^n - \bar{w}_{j-1,k}^n), \theta(\bar{w}_{j,k}^n - \bar{w}_{j-1,k}^n) \right\} \quad (4.3')$$

$$u^{\backslash}_{j,k} = MM \left\{ \theta(\bar{w}_{j,k+1}^n - \bar{w}_{j,k}^n), \frac{1}{2}(\bar{w}_{j,k+1}^n - \bar{w}_{j,k-1}^n), \theta(\bar{w}_{j,k}^n - \bar{w}_{j,k-1}^n) \right\}. \quad (4.3'')$$

An exact evolution of this reconstruction, which is based on integration of the conservation law over the staggered volume yields

$$\begin{aligned}
\bar{w}_{j+\frac{1}{2},k+\frac{1}{2}}^{n+1} &= \int_{C_{j+\frac{1}{2},k+\frac{1}{2}}} w(x, y, t^n) dx dy + \\
&- \lambda \left\{ \int_{\tau=t^n}^{t^{n+1}} \int_{y \in J_{k+\frac{1}{2}}} [f(w(x_{j+1}, y, \tau)) - f(w(x_j, y, \tau))] dy d\tau \right\} + \\
&- \mu \left\{ \int_{\tau=t^n}^{t^{n+1}} \int_{x \in I_{j+\frac{1}{2}}} [g(w(x, y_{k+1}, \tau)) - g(w(x, y_k, \tau))] dx d\tau \right\}. \tag{4.4}
\end{aligned}$$

The exact averages at  $t^n$  – consult the floor plan in Figure 4.1 yields

$$\begin{aligned}
\bar{w}_{j+\frac{1}{2},k+\frac{1}{2}}^n &:= \int_{C_{j+\frac{1}{2},k+\frac{1}{2}}} w(x, y, t^n) dx dy = \tag{4.5} \\
&= \frac{1}{4} (\bar{w}_{jk}^n + \bar{w}_{j+1,k}^n + \bar{w}_{j,k+1}^n + \bar{w}_{j+1,k+1}^n) + \\
&+ \frac{1}{16} \left\{ (u'_{jk} - u'_{j+1,k}) + (u'_{j,k+1} - u'_{j+1,k+1}) + \right. \\
&\quad \left. + (u''_{jk} - u''_{j,k+1}) + (u''_{j+1,k} - u''_{j+1,k+1}) \right\}.
\end{aligned}$$

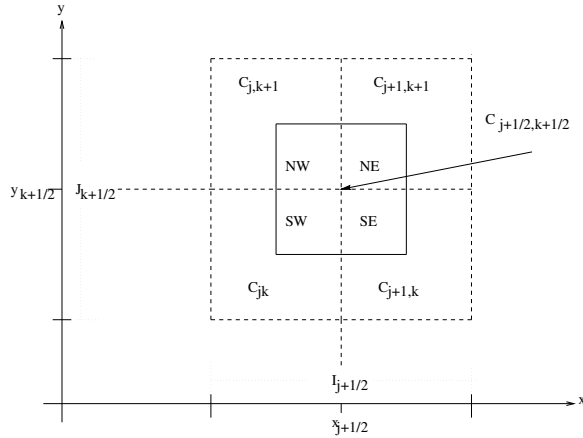


Figure 4.1: Floor plan of the staggered grid.

So far everything is *exact*. We now turn to *approximate* the four fluxes on the right of (4.4), starting with the one along the East face, consult figure 4.2,  $\int_{t^n}^{t^{n+1}} \int_{J_{k+\frac{1}{2}}} f(w(x_{j+1}, y, \tau)) dy d\tau$ . We use the midpoint quadrature rule for second-order approximation of the temporal integral,  $\int_{y \in J_{k+\frac{1}{2}}} f(w(x_{j+1}, y, t^{n+\frac{1}{2}})) dy$ ; and, for reasons to be clarified below, we use the second-order rectangular quadrature rule for the spatial integration across the  $y$ -axis, yielding

$$\int_{t^n}^{t^{n+1}} \int_{y \in J_{k+\frac{1}{2}}} f(w(x_{j+1}, y, \tau)) dy d\tau \sim \frac{1}{2} \left[ f(w_{j+1,k}^{n+\frac{1}{2}}) + f(w_{j+1,k+1}^{n+\frac{1}{2}}) \right]. \tag{4.6}$$

In a similar manner we approximate the remaining fluxes.

These approximate fluxes make use of the midpoint values,  $w_{jk}^{n+\frac{1}{2}} \equiv w(x_j, y_k, t^{n+\frac{1}{2}})$ , and it is here that we take advantage of utilizing these midvalues for the spatial integration by the rectangular rule. Namely,

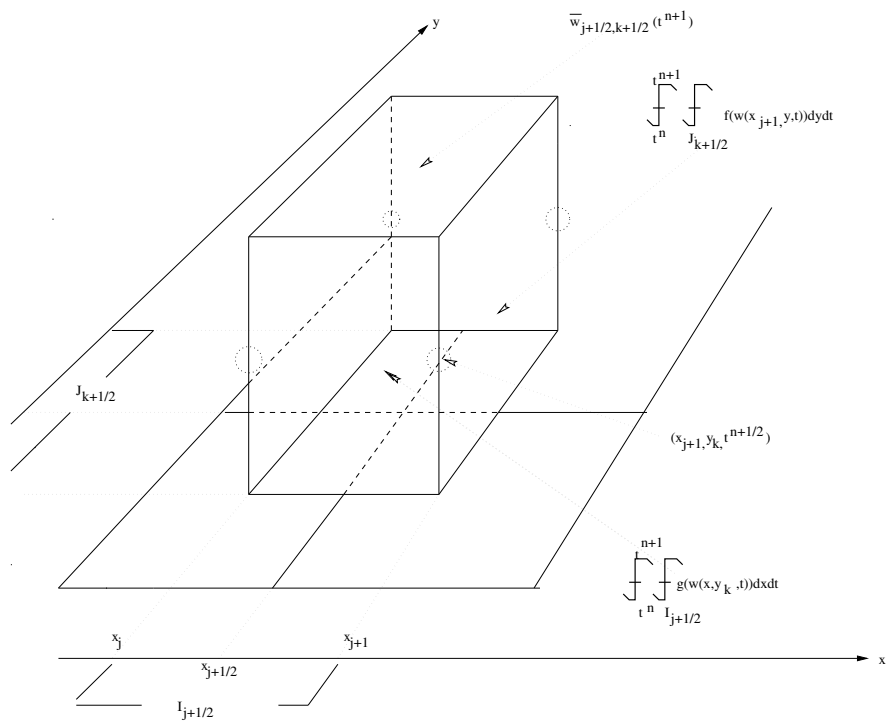


Figure 4.2: The central, staggered stencil.

since these midvalues are secured at the smooth center of their cells,  $C_{jk}$ , bounded away from the jump discontinuities along the edges, we may use Taylor expansion,  $w(x_j, y_k, t^{n+\frac{1}{2}}) = \bar{w}_{jk}^n + \frac{\Delta t}{2} w_t(x_j, y_k, t^n) + \mathcal{O}(\Delta t)^2$ . Finally, we use the conservation law (4.1) to express the time derivative,  $w_t$ , in terms of the spatial derivatives,  $f(w)'$  and  $g(w)'$ ,

$$w_{jk}^{n+\frac{1}{2}} = \bar{w}_{jk}^n - \frac{\lambda}{2} f(w)'_{jk} - \frac{\mu}{2} g(w)'_{jk}. \quad (4.7)$$

Here,  $f(w)'_{jk} \sim \Delta x \cdot f(w(x_j, y_k, t^n))_x$  and  $g(w)'_{jk} \sim \Delta y \cdot g(w(x_j, y_k, t^n))_y$ , are one-dimensional discrete slopes in the  $x$ - and  $y$ -directions, of the type reconstructed in (4.3')-(4.3''); for example, multiplication by the corresponding Jacobians  $A$  and  $B$  yields

$$f(w)'_{jk} = A(\bar{w}_{jk}^n) u'_{jk}, \quad g(w)'_{jk} = B(\bar{w}_{jk}^n) u'_{jk}.$$

Equipped with the midvalues (4.7), we can now evaluate the approximate fluxes, e.g., (4.6). Inserting these values, together with the staggered average computed in (4.6), into (4.4), we conclude with new staggered averages at  $t = t^{n+1}$ , given by

$$\begin{aligned} \bar{w}_{j+\frac{1}{2}, k+\frac{1}{2}}^{n+1} &= \frac{1}{4}(\bar{w}_{jk}^n + \bar{w}_{j+1, k}^n + \bar{w}_{j, k+1}^n + \bar{w}_{j+1, k+1}^n) + \\ &+ \frac{1}{16}(u'_{jk} - u'_{j+1, k}) - \frac{\lambda}{2} \left[ f(w_{j+\frac{1}{2}, k}^{n+\frac{1}{2}}) - f(w_{j, k+\frac{1}{2}}^{n+\frac{1}{2}}) \right] \\ &+ \frac{1}{16}(u'_{j, k+1} - u'_{j+1, k+1}) - \frac{\lambda}{2} \left[ f(w_{j+\frac{1}{2}, k+1}^{n+\frac{1}{2}}) - f(w_{j, k+1}^{n+\frac{1}{2}}) \right] \\ &+ \frac{1}{16}(u'_{jk} - u'_{j, k+1}) - \frac{\mu}{2} \left[ g(w_{j, k+\frac{1}{2}}^{n+\frac{1}{2}}) - g(w_{j, k}^{n+\frac{1}{2}}) \right] \\ &+ \frac{1}{16}(u'_{j+1, k} - u'_{j+1, k+1}) - \frac{\mu}{2} \left[ g(w_{j+1, k+\frac{1}{2}}^{n+\frac{1}{2}}) - g(w_{j+1, k}^{n+\frac{1}{2}}) \right]. \end{aligned} \quad (4.8)$$

In summary, we end up with a simple two-step predictor-corrector scheme which could be conveniently expressed in terms on the one-dimensional staggered averaging notations

$$\langle w_{j,\cdot} \rangle_{k+\frac{1}{2}} := \frac{1}{2}(w_{j,k} + w_{j,k+1}), \quad \langle w_{\cdot,k} \rangle_{j+\frac{1}{2}} := \frac{1}{2}(w_{j,k} + w_{j+1,k}).$$

Our scheme consists of a *predictor step*

$$w_{j,k}^{n+\frac{1}{2}} = w_{j,k}^n - \frac{\lambda}{2} f'_{j,k} - \frac{\mu}{2} g'_{j,k}, \quad (4.9)$$

followed by the *corrector step*

$$\begin{aligned} \bar{w}_{j+\frac{1}{2}, k+\frac{1}{2}}^{n+1} &= \langle \frac{1}{4}(\bar{w}_{j,\cdot}^n + \bar{w}_{j+1,\cdot}^n) + \frac{1}{8}(w'_{j,\cdot} - w'_{j+1,\cdot}) - \lambda(f_{j+\frac{1}{2},\cdot}^{n+\frac{1}{2}} - f_{j,\cdot}^{n+\frac{1}{2}}) \rangle_{k+\frac{1}{2}} + \\ &+ \langle \frac{1}{4}(\bar{w}_{\cdot,k}^n + \bar{w}_{\cdot,k+1}^n) + \frac{1}{8}(w'_{\cdot,k} - w'_{\cdot,k+1}) - \mu(g_{\cdot,k+\frac{1}{2}}^{n+\frac{1}{2}} - g_{\cdot,k}^{n+\frac{1}{2}}) \rangle_{j+\frac{1}{2}}. \end{aligned}$$

In figures 4.3 taken from [16], we present the two-dimensional computation of a double-Mach reflection problem; in figure 4.4 we quote from [42] the two-dimensional computation of MHD solution of Kelvin-Helmholtz instability due to shear flow. The computations are based on our second-order central scheme. It is remarkable that such a simple 'two-lines' algorithm, with no characteristic decompositions and no dimensional splitting, approximates the rather complicated double Mach reflection problem with such high resolution. Couple of remarks are in order.

- The two-dimensional computation is more sensitive to the type of limiter than in the one-dimensional framework [29]. In the context of the double Mach reflection problem, the  $MM_2$  (consult (3.2) with  $\theta = 2$ ) seems to yield the sharper results.
- No effort was made to optimize the boundary treatment. The staggered stencils require a different treatment for even-odd cells intersecting with the boundaries. The lack of boundary resolution could be observed at the bottom of the two Mach stems.

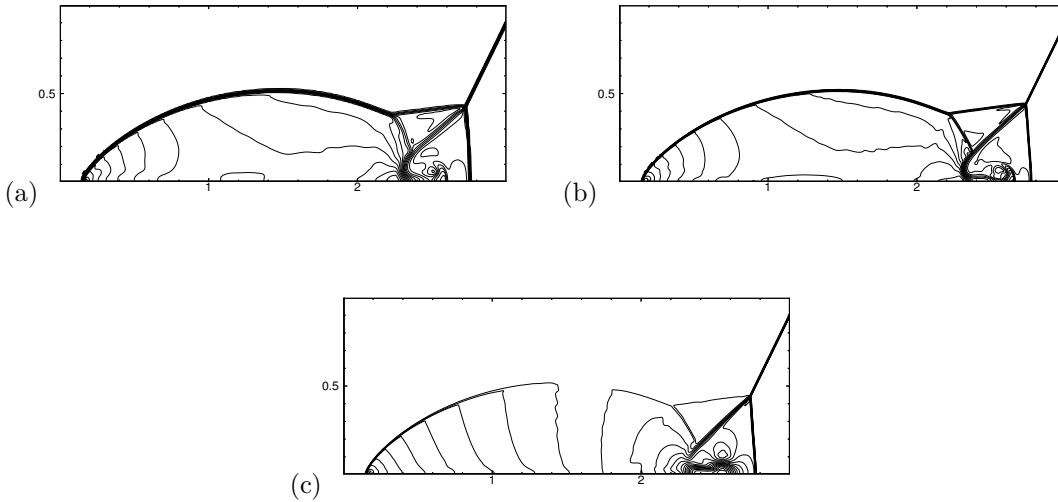


Figure 4.3: Double Mach reflection problem computed with the central scheme using  $MM_2$  limiter with CFL=0.475 at  $t = 0.2$  (a) density computed with  $480 \times 120$  cells (b) density computed with  $960 \times 240$  cells (c) x-velocity computed with  $960 \times 240$  cells

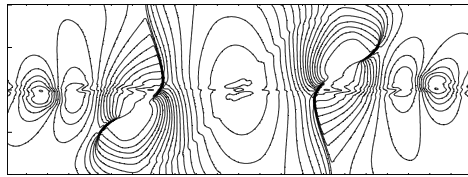


Figure 4.4: Kelvin-Helmholtz instability due to shear flow. Transverse configuration ( $B$  perpendicular to  $v$ ). Pressure contours at  $t = 140$

We conclude this section with brief remarks on further results related to central schemes.

*Remarks.*

1. Simplicity.

Again, we would like to highlight the simplicity of the central schemes, which is particularly evident in the multidimensional setup: no characteristic information is required – in fact, even the exact Jacobians of the fluxes are not required; also, since no (approximate) Riemann solvers are involved, the central schemes require no dimensional splitting; as an example we refer to the approximation of the incompressible equations by central schemes, [22, 25]; the results in [7] provide another example of a *weakly* hyperbolic multidimensional system which could be efficiently solved in term of central schemes, by avoiding dimensional splitting.

2. Non-staggering. We refer to [15] for a non-staggered version of the central schemes.

3. Stability.

The following maximum principle holds for the nonoscillatory scalar central schemes:

**Theorem 4.1** ([16]) *Consider the two-dimensional scalar scheme (4.7-4.8), with minmod slopes,  $u'$  and  $u''$ , in (4.3'-4.3''). Then for any  $\theta < 2$  there exists a sufficiently small CFL number,  $C_\theta$  (– e.g.  $C_1 = (\sqrt{7} - 2)/6 \sim 0.1$ ), such that if the CFL condition is fulfilled,*

$$\max(\lambda \cdot \max_u |f_u(u)|, \mu \cdot \max_u |g_u(u)|) \leq C_\theta,$$

then the following local maximum principle holds

$$\min_{\substack{|p-(j+\frac{1}{2})|=\frac{1}{2} \\ |q-(k+\frac{1}{2})|=\frac{1}{2}}} \{\bar{w}_{p,q}^n\} \leq \bar{w}_{j+\frac{1}{2},k+\frac{1}{2}}^{n+1} \leq \max_{\substack{|p-(j+\frac{1}{2})|=\frac{1}{2} \\ |q-(k+\frac{1}{2})|=\frac{1}{2}}} \{\bar{w}_{p,q}^n\}. \quad (4.10)$$

4. Third-order accuracy. Extensions to third-order accuracy in two space dimensions can be found in [24].

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