

# Shock-Free Periodic Solutions in Gas Dynamics

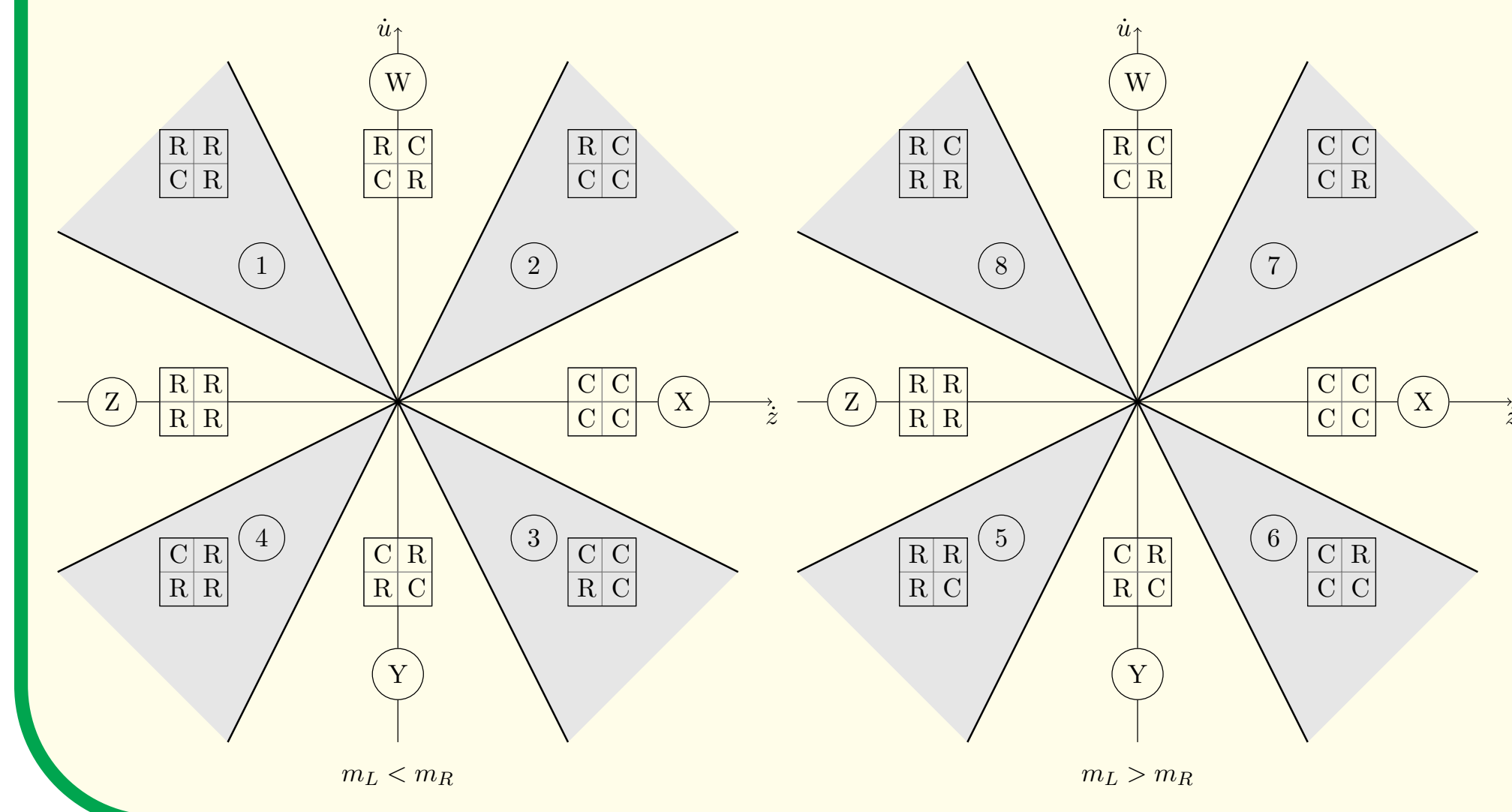
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## The Question

- Do there exist nontrivial time periodic solutions to the compressible Euler equations?
- Can shock formation be indefinitely delayed?
- Identify the structure of such solutions:
  - Linearizing yields sinusoidal solutions
  - Shocks incompatible with periodicity
  - Interactions generate reflections

**Our premise:** There should be wave patterns in which decay and wave generation are balanced.

## Labelled Block Regions



## Eigenvalue Problem

Nondimensionalize, maximize symmetry and evolve in **space** rather than time!

Let  $\mathcal{E}(\theta)$  be evolution over  $0 \leq y \leq \theta$

$$w_y + \sigma v_t = 0 \quad \text{and} \quad v_y + \sigma w_t = 0,$$

where  $\sigma = w^{-d}$ . Jump operator is a  $2 \times 2$  **constant** matrix  $\mathcal{J}$ , and  $\mathcal{S}$  is a  $\pi$ -shift.

**Fully nonlinear problem:** Find  $2\pi$ -periodic  $V = V(t)$  such that  $\mathcal{N}V(\cdot) = V(\cdot)$ , where

$$\mathcal{N} \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{E}(\theta) \cdot \mathcal{J} \cdot \mathcal{E}(\theta). \quad (1)$$

That is: **evolve** at one entropy level, **jump** to the other level, **evolve** again, and **jump** back to return half-period **shifted** data.

## Small Divisors

Need to solve the auxiliary equation, when the parameters are **nonresonant**.

**Theorem 1** For almost every  $\theta$ , the eigenvalues  $\lambda_n^\pm$  of  $2 \times 2$   $n$ -th mode matrix  $M_n$  satisfy

$$|\lambda_n^\pm - 1| \geq \frac{C}{n^r} \quad \text{for } n \geq 2,$$

In particular, (the projection of)  $\mathcal{M} - \mathcal{I}$  is invertible, but has **small divisors**.

So the inverse is unbounded, and the (classical) Implicit Function Theorem does not apply!

## Rarefaction vs. Compression

Use the  $3 \times 3$  Lagrangian equations:

$$\tau_t - u_x = 0, \quad u_t + p_x = 0, \quad S_t = 0;$$

with  $p = e^{S/c_\tau} \tau^{-\gamma}$ . Change variables:

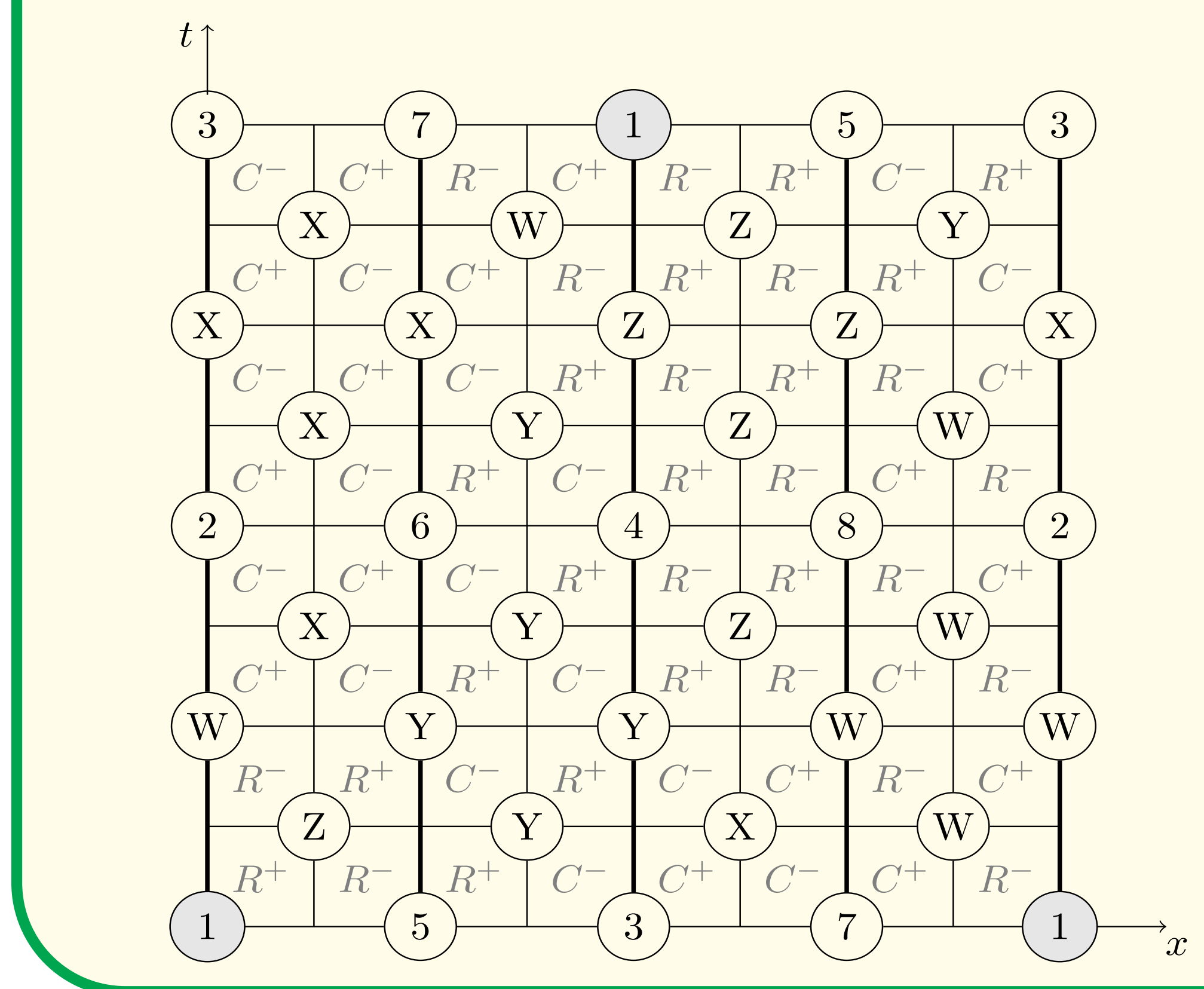
$$e^{S/2c_\tau} \rightsquigarrow m \quad \text{and} \quad \tau^{-(\gamma-1)/2} \rightsquigarrow z.$$

Riemann invariants are

$$s = u + m z, \quad \text{and} \quad r = u - m z;$$

The **local R/C character** of a solution is determined by signs of  $r_t$  and  $s_t$ .

## Simplest Periodic R/C Pattern



## Linearization

Linearize around constant state  $(1 \ 0)^t$ : linearized problem is

$$MV \equiv \mathcal{S} \cdot \mathcal{J}^{-1} \cdot \mathcal{L}(\theta) \cdot \mathcal{J} \cdot \mathcal{L}(\theta)V = V,$$

where  $\mathcal{L}$  is the **linear wave equation**.

**Fourier decomposition:** respects modes!

Linearized EVP for the  $n$ -th mode is

$$M_n q_n = q_n, \quad (2)$$

with  $2 \times 2$  **real** matrix

$$M_n = (-1)^n \mathcal{J}^{-1} R(n\theta) \mathcal{J} R(n\bar{\theta});$$

$R(\alpha)$  is  $2 \times 2$  rotation: nice geometric picture!

## Finite cutoff

**Theorem 2** Given any  $K \gg 1$ , there are periodic solutions  $V$  to the projected nonlinear problem,

$$\mathcal{P}^K (\mathcal{N} - \mathcal{I}) V = 0,$$

where  $\mathcal{N}$  is given in (1) and  $\mathcal{P}^K$  is projection onto the first  $K$  Fourier modes.

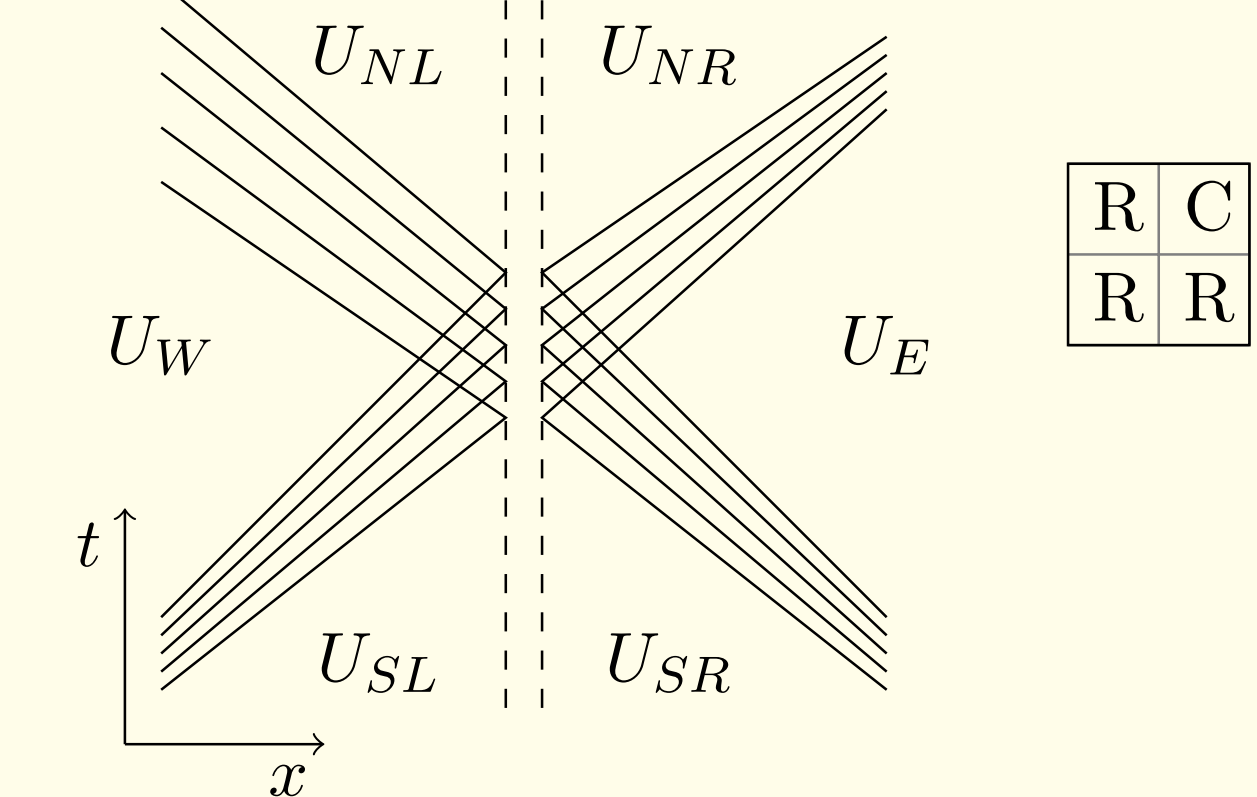
- Two-parameter family,  $\theta, \epsilon \ll 1$ ;
- Not uniform:  $\epsilon \rightarrow 0$  as  $K \rightarrow \infty$ .

## Entropy Jumps

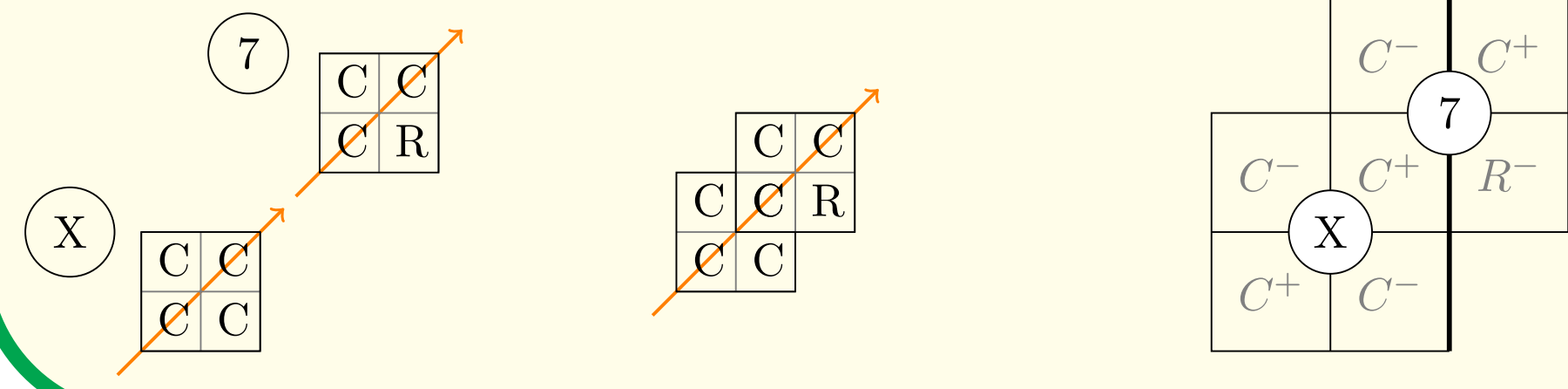
Rankine-Hugoniot yields:  $m_L \neq m_R$ ,

$$u_R = u_L, \quad m_R z_R = \left(\frac{m_L}{m_R}\right)^c m_L z_L.$$

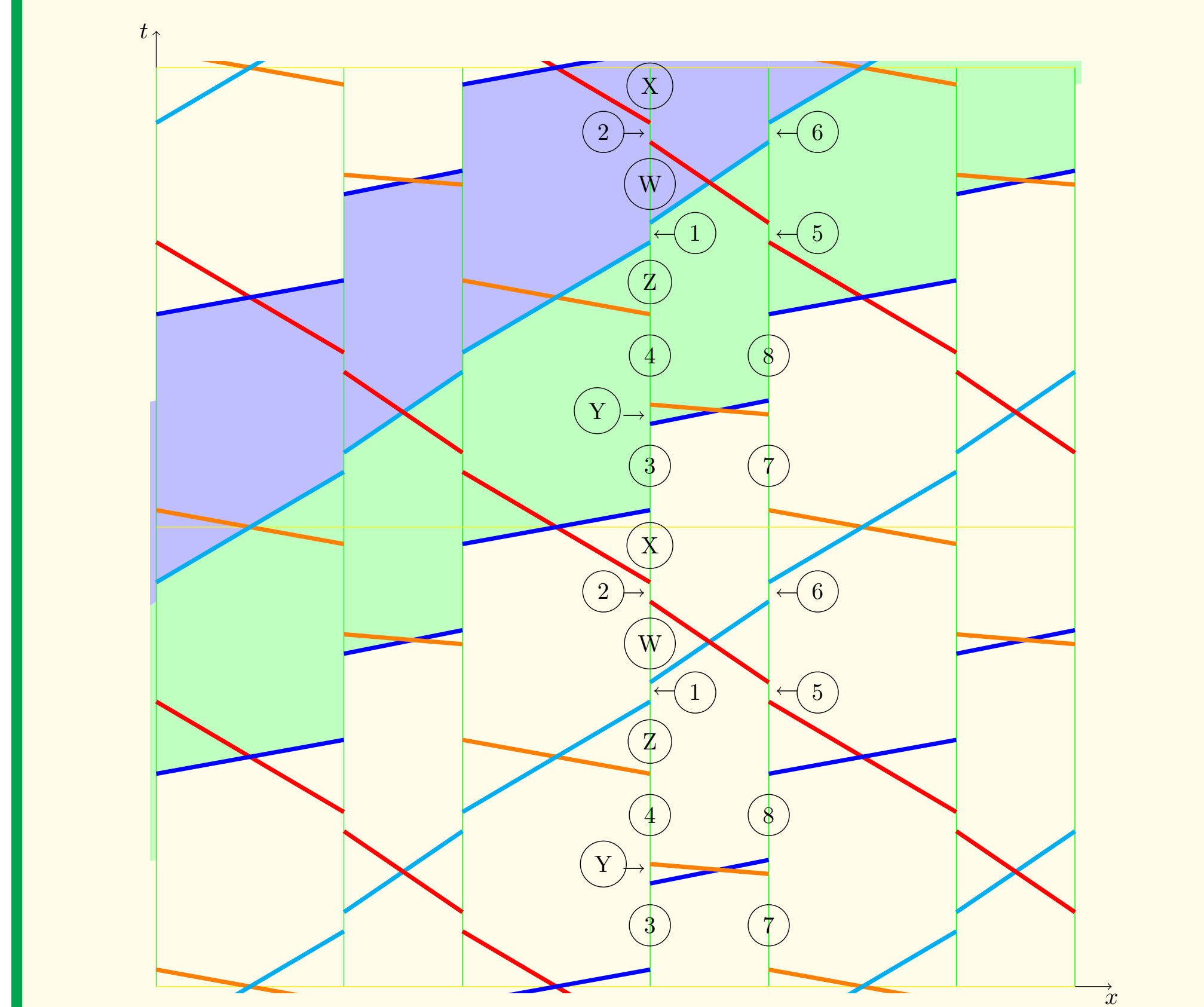
Thus, the R/C-character can change!



Get 16 blocks, to be **consistently** combined:



## Characteristic Picture



One forward **rarefaction** and **compression** shaded; Regions labelled: changes at entropy jumps.

## Bifurcation

Parameters: jump  $J$ , evolution  $\theta$ , amplitude  $\epsilon$ .

**Lemma 1** There is a one-mode solution  $q_1$  of (2), if we take  $J = \cot^2(\theta/2)$ .

**Our goal:** Choose the parameters so that the linearized one-mode solution  $q_1$  perturbs. Use amplitude  $\epsilon$  of  $q_1$ ,  $J$  and  $\theta$  as bifurcation parameters.

**Liapunov-Schmidt:** decompose into **bifurcation equation** and **auxiliary equation**.

When the parameters are nonresonant, the bifurcation equation can be solved.

## Nash-Moser

Small divisors and evolution regarded as **loss of derivatives**. **Weighted spaces** work for these separately, but not together.

We need a Newton-type method, with quadratic convergence, to cope with both losses.

**Main difficulty:** Get bounds for the nearby inverse linearized operators!

- Fourier methods break down!
- Want to use conservation law structure, such as genuine nonlinearity.
- May need to excise parameters.

## A Conjecture

Almost every nontrivial solution to the  $3 \times 3$  Euler equations will decay to a **nontrivial** time-periodic or almost periodic solution which consists of **balanced compressions and rarefactions** on some fixed varying entropy field.