IV Multidimensional behavior: flow in a duct.

IVA. Cellular/spinning instabilities.
Recall: Detonation can exhibit spinning, cellular instabilities, depending on cross-sectional geometry.

[Kasimov-Stewart] [Engel-Ak relocate] (ZMD)

Shocks: Mach stem formation (cellular waves) at high energies. [Artola-Ngda] phase transition/criticality. (Euler)

(OTHERW</body>
IVb. Shocks in the whole space.

(i) Inviscid case (Euler) [Erpenbeck 65, Majda 86]:

\[ U^{-}\left(x,t\right) \rightarrow U^{+}\left(x,t\right) \]

Free-boundary problem, linear well-posedness ("Normal mode")

\[ \begin{array}{c}
U_{t} + \varepsilon \sum_{j} F_{j} U_{x_{j}} = 0 \\
\Rightarrow U_{t} + \sum_{j} A_{j} U_{x_{j}} = 0, \quad \text{Lopatinski condition}
\end{array} \]

\[ \Delta(L, \vec{\gamma}) := \det \left( R_{-}^{-1} R_{+}^{-1}, \lambda I_{n} + \sum_{j} \gamma_{j} \Gamma_{j} \right) \]

\[ R_{j}^{-1} \text{e-basis of } \left( \lambda + \sum_{i} \gamma_{i} A_{j} \right) A_{i}^{-1} \]

Zeros in spectra, \( \vec{\gamma} = (\vec{\gamma}_{2}, \ldots, \vec{\gamma}_{d}) \) = Fourier frequencies. Homogeneous, deg. one — NOT LINEAR!
Uniform stability condition (Kreiss, Morse): \( \Delta \) has no zeroes for \( |\xi, \lambda| = 1, \ Re \lambda > 0. \)

Thus [Morse 86] Assumption uniform stab., + appropriate compatibility conditions, short-time well-posedness (see also [Metivier, Metivier-Mokrane])

Weak stab. (Morse): \( \Delta \) has no zeroes for \( (\xi, \lambda) \neq 1, \ Re \lambda > 0, \) but has zeroes \( \lambda = i \tau. \)

Interpretation - surface waves on fluid (see [Hunter])
Remarkable Fact [Magda, Erpenbeck]:
Fix $U^-$, left state, vary $U^+$
(Hugoniot curve):

Then, region of weak stability occupies open set (!) (Not a boundary...)

Strong instability: $\Delta$ has zero $\forall \text{Re}_R > 0$.

($\Rightarrow$ exponential instability $\Rightarrow e^{\text{Re}_R t}$.)

Boundary is transition to one-dim. instability [Serre, Faure, ...].

(Wave splitting)
[Artola-Maya, Maya-Rosales]: Cellular instabilities/mass star formation occurring in weak star regime, weakly nonlinear.

[Contonnel, Contonnel-Scacci]: Nonlinear well-pseudodense, also in weakly stable regime (Nash-Moser iteration).

\[ \Rightarrow \] Conclusions unclear re/cellular instability, question how to rigorously formulate the problem (transition - bifurcation unclear).
(ii) Viscous stability.

Std. case of spectral gap:

\[ \lambda_x(\tau) \]

\[ \text{Portrait for } \tau = 0 \] (one-dim. case).

Then, std. matrix/operator pert. theory:

\[ \Rightarrow \text{Top } \lambda \text{-value } \lambda_x(\tau) \text{ perturb} \]

analytically from \( \lambda_x(0) = 0 \).

\[ \begin{cases} \lambda_x(\tau) - i \epsilon \geq a_i j_j j_j j_j \geq b_i \epsilon \end{cases} \]

Projecting at, \( \Rightarrow \)

scalar convection diff. eqn. for

perturbation of the front.
For system of conservation laws, intuition

Completely wrong!

**RATHER,**

Thm. [Z-Serre].

\[ D(\xi, \eta) - \delta \Delta(\xi, \eta) \]
+ \( o(\| \xi \| ) \)

\( \gamma = \) transversality coeff., \( D = \) viscous stab. determinant, similar as 1-dim. case.

\[ \text{Inviscid spectrum} \]
\[ \text{Viscous spectrum} \]

*Cor.* Viscosity can DESTABILIZE but

not stabilize on inviscid shock wave \( (*) \)

[Hyperbolic instability \( \Rightarrow \) Viscous instability]
The refined stability condition (viscous correction)

Weak stability (inviscid):

- inviscid spectrum
- viscous spectrum

\[ \text{?} \quad \text{OR?} \quad \text{?} \]

\[ \text{Re} \lambda \]

(stable)

(unstable)

Convexity condition defining stability at viscous level:

\[ \gamma(x(r, t)) = 1 \text{ To } r = B(x, t) r^2. \]

"Effective viscosity" along front, real positive.
Sharp transition boundary.

In [201]: Refined stab. suff. for nonlinear stability in dim d=3, gas-dynamic case.

(open for general case - though see important work of [Nguyen 08].)

See also [Onis-Meibner-Williams-2], small vix land.

Q - How could violation relate to cellular instab ???

Noninductive (and incorrect - )
IVc. Shocks in a duct.

Model:

\[ \frac{\partial}{\partial t} u + \sum_j \xi_j \frac{\partial}{\partial x_j} u = \Delta_x u, \]
\[ x_1 \in \mathbb{R}, \quad x_2 \in [-L, L], \text{ periodic BC.} \]

**Obs.** Planar shock wave still a solution (remove char. boundary-layer.)

\[ \text{F.T.} \Rightarrow \]
\[ \hat{\xi}_k \frac{\partial}{\partial t} \hat{h} + \sum_j \hat{\xi}_j \frac{\partial}{\partial x_j} \hat{h} = \hat{\Delta}_x \hat{h}, \]
\[ \hat{\xi}_k \hat{A}_j \hat{G} + (\hat{A}_j \hat{G}) = -\hat{\Delta}_x \hat{G} + \hat{G}_{xx}, \]
\[ \sum = \hat{n}_2, \hat{G} \]

Same as whole space but (discrete)
For $\delta = 0$: 1-dim. case,

\[ \hat{u}_t + (A \cdot \hat{u}) x_1 = \hat{u} x_1 x_1. \]

For $\delta \neq 0$: Complicated, but essential spectrum shifted to left half-plane:

\[ \hat{u}_t = - (A \cdot \hat{u}) x_1 + \hat{u} x_1 x_1 + \int_{\mathbb{T}} \hat{u} A \hat{u}^* - |\hat{u}|^2 \hat{u} \]

\[ \text{shift to left} \]

\[ \text{Top eigenvalue only unknown} \]

\[ \Rightarrow \text{Projecting out unstable eigenvalues,} \]

\[ |e^{2 \delta t}| e^{2 \delta z} \leq C e^{-\delta t}, \quad \delta > 0. \]

\[ \forall \delta \leq 0 \text{ (less unstable part)} \]
Conclusion: Linear behavior is one-dim behavior (averaged behavior) + exponentially decaying transverse part. ( = bounded transverse direction, Poincaré inequality.)

Corollary: Conditional Stability, Bifurcation theorems, follow by (minor modification of) one-dimensional arguments. (See course notes for details.)

(RIGorous Framework STRAIGHTforward)

Remark. Interesting limits as $L \to \infty$ (whole space), $L \to 0$ (one-dim. limit).
IVd. Refined stab. condition and cellular bifurcations.

Loss of stability/crossing of e-values:

(\tilde{\gamma} = 0) One-dimensional instability, already treated ("pulsating wave").

(\tilde{\gamma} \neq 0) Transverse instability, roughly eigenmode of Laplacian on duct geometry ("spinning", "cellular", depending).

(i) involves variation in \((x_2, x_d)\).

(ii) Necessarily occurs in pair

\((\tilde{\gamma}, i\tau), (-\tilde{\gamma}, -i\tau)\) at transition point \(- \Rightarrow \text{Hopf}, \text{provided } T \to 0 -- \)
Rmk. Reflective/rotational symmetry (e.g., gas dynamics) implies more complicated bifurcation. (Well studied elsewhere, see [Golubitsky], [Sandstede-Scheel], [Wulff]... Not essentially different mathematically...)

Assume for simplicity this does not occur.

Refined stability condition + bifurcation. More accurate picture (neutral/weak inviscid stab):

Refined stability condition + bifurcation. More accurate picture (neutral/weak inviscid stab):

\[ \text{Re} \]

\[ \beta > 0 \text{ (stable)} \]

\[ \text{Re} \]

\[ \beta < 0 \text{ (unstable)} \]

(Here, inviscid mode \((30,10)\), \(T_0 = 0\) (Hopf))
Recall that the spectral curve is discretized \( \textit{(sampled at } \tilde{s}_n = \left( \frac{2\pi}{L} \right)_n \text{)} \).

\[ \text{First unstable mode} \]

\[ \tilde{s} = 0, \text{ translational } e\text{-value (harmless)} \]

\[ \exists \text{ Hopf bifurcation} (\text{cellular/spiral type}). \]

CONCLUSION: As \( \beta \downarrow \) through zero, part, \( \beta \uparrow \) in transverse mode

\[ \text{Bifurcation point } \beta^* \approx 0. \]
More: Cascade of Hopf bifurcations as $\beta$ decreases further, successively higher transverse frequencies...

(Bif. at $\xi_2$ mode)

Remarks: $\beta$ compatible [Benzoni, Caurel, Serre, 70]

- Still speculative however...

- ALTERNATIVE is bif. at high frequency again prior to strong instability/wave splitting.
Doesn't answer:

- Mechanism.
- Structure (finer structure).

Connect w/ Artola-Majda picture? (inviscid theory)

NOTE: In related, detonation context, ZND (inviscid) numerics give excellent resolution of structure.

Eg. Spinning detonation has "rotor;" small detonation front at combustion head (hot spot):
Closing Remarks.

Extended analysis of single fundamental example involving absence of spectral gap.

Already leads to rich class of problems.

OPEN: Systematic catalog of phenomena (now doable, but organizationally challenging).

- Numerical proof (again, doable in principle - but challenging).

- Understanding of mechanisms for stab./metal. (e.g., hints from pressureless limit, asymptotic models), analytic proofs for special structure (speculative).
MORE GENERAL: Understand

Stability/bifurcation of general (genuinely multi-d) stationary flow in exterior domain. (important for theory and numerics)

Eg. vortex shedding (periodic), flow about a cylinder. (mentioned already as motivation by Hopf)

( + ETC... )
THANKS,

to organizers, audience.

I.M.A., July 2009