

# The Difference Between $5 \times 5$ Doubly Nonnegative and Completely Positive Matrices

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Absolute cleanliness is Godliness! Who else but God gave man Love that can spark mere dust to life! Poetry, uniting All-One! All brave! All life! Who else but God! "Listen Children Eternal Father Eternally One!" Einstein, 1939, after Nazis & Commies united, proposed spacebombs that destroy all, unless we finally teach the Moral ABC's the real Rabbi Hillel taught Jesus to unite all in All-One-God-Faith. As teach astronomers Abraham-Israel-Moses-Buddha-Hillel-Jesus-Spinoza-Paine-Sagan & Mohammed, inspired every 76 years, 6000 years by the Messenger of God's Law, the Messiah, Halley's Comet: "WE'RE ALL ONE OR NONE!" "THERE IS NO GOD BUT GOD!" "TEACH LOVE THY ENEMY!" "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!" "Israel! Moses-Buddha-Jesus-Mohammed: ONE! ALL-ONE!"

1st: If I'm not for me, who am I? Nobody! 2nd: Yet, if I'm only for me, what am I? Nothing! 3rd: If not now, when? Once More: Unless constructive-selfish I work hard, like Mark Spitz, perfecting first, make absolute nothing to help me! 4th: Only hard work-God's Law can save us, but if we teach only our clan? We're all hated then! So, we must teach friend & enemy, the whole Human race, the full-truth, hard-work, free speech, press-&profitsharing Moral ABC's All-One-God-Faith, lighting-like, 6-billion strong, for we're All-One or none! All-One-God-Faith, as teach the African shepherd-astronomers Abraham & Israel, for 6000 years, since the year 1: "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!" - WE'RE ONE! ALL-ONE! EXCEPTIONS ETERNALLY? NONE! ABSOLUTE NONE!

5th: Whatever unites mankind is better than whatever divides us! Yet, if absolute-unselfish I am not for me, I'm not but classless, raceless, starving masses, never free nor brave! Only if constructive-selfish I work hard perfecting first me, like Mark Spitz-arctic owls-penguin-pilot-cat-swallow-beaver, bee, can I teach the Moral ABC that unites all mankind free! Lightning-like unites the Human race! For we're ALL-ONE OR NONE! ALL-ONE! "listen children eternal father eternally one!" EXCEPTIONS ETERNALLY? ABSOLUTE NONE!

6th: Absolute cleanliness is Godliness! Balanced food for body-mind-soul-spirit is our medicinal Full-truth our God, half-truth our enemy, hard work our salvation, unity our goal, free speech our weapon, All-One our soul, self-discipline the key to love, uniting All-One above! Above! So, when your fellow man you measure, take him at his best, with that lever, lift him higher, overlook the rest! For we're All-One or none! As teach for 6000 years astronomers Abraham & Israel: "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!"

7th: Each swallow works hard to be perfect pilot-provider-builder-trainer-teacher-lover-mate, no half-true hate! So, each day like a bird, perfect thyself first! Have courage and smile my friend. Think and act 10 years ahead! And the man without fault? He's dead! Do one thing at a time, work hard! Get done! Then teach friend & enemy the Moral ABC that unites all mankind free! Uniting One! All-One! Face the world with a smile, life is always worthwhile! To the fearless are given crowns, keep out the past, disappointments won't last! Help unite mankind, or we're wandering clowns! Diligent preparation, precede . . . spectacular restoration!! So, help teach the whole Human race, the Moral ABC's All-One-God-Faith, lightning-like, for we're All-One or none! All-One!!

8th: More good is caused by evil than by good, do what's right! Enlarge the positive! Replace the negative with the Moral ABC's ALL-ONE-GOD-FAITH, that lightning-like unites the Human race! FOR WE'RE ALL ONE OR NONE! "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!" WE'RE ALL ONE OR NONE! EXCEPTIONS ETERNALLY? NONE!

9th: Free speech is man's only weapon against half-truth, that denies free speech to smear-stay-slender-tax-enslave. Full-truth, our only God, unites all mankind brave, if 10 men guard free speech, brave! "If ever one religion unites all mankind, it will be by omitting all irrelevancies & redundancies, added unto the Faith in One-Almighty, all-embracing, ever-loving, ever-evolving, ever-recreating Eternal God, and by ABSOLUTE NOTHING ELSE" America's founding father, Thomas Paine, 8 books suppressed since 1799!

10th: Thank God we don't descend down from perfect Adam & Eve to sinful sinner, brother's keeper, divided slave! Thank God! United, hard-working trained brave, from dust we ascend up! Thank God for that! Our brother's teacher of the Moral ABC, mason-tent & sandal-maker Hillel, taught Carpenter Jesus to unite all mankind free! With it, every Human being created on God's Spaceship Earth, can evolve united, inspired-raised-trained-skilled-disciplined, guided lightning-like by a new birth! Without it . . . we destroy God's Spaceship Earth!

11th: Essene, Chinese and other birth control methods must reduce birth or Easter Isle type overpopulation destroys God's Spaceship Earth!

12th: A great teacher, must first, a self-supporting hardworker be, like Alesen-Baech-Carnegie-Cousteau-Hammer-Liebman-Paine-Pike-Sanger-Spinoza-Strauss-Szasz-Wilke-Yadin-Zamenhof, or he'll turn our greatest teaching into spades, to bury our people! "All people!", added Carpenter Jesus entering manhood! Manhood! but, for 2000 years, we Rabbits never teach the Moral ABC the real Rabbi Hillel taught Jesus, to unite the Human race in our Eternal Father's great ALL-ONE-GOD-FAITH! For we're ALL-ONE OR NONE! ALL-ONE! For example: Einstein's Rabbi Levy evicted us from his University Princeton "Hillel Foundation", when we asked "You agree, of course, that Hillel's hard work-speech-press-&-profitsharing Moral ABC, unites the Human race, in our Eternal Father's great ALL-ONE-GOD-FAITH! The exact opposite to Marxist-Socialism, that does demoralize-divide-decay the whole human race today!

13th: "Knowing the full truth that unites the human race & not teaching all is deathly guilt!", learned carpenter Jesus from Mason Rabbi Hillel! But Marx, innocent grandson of 2 Rabbits, learned only half truth! Ashamed, Marx wrote 1844: "One World Without Jews", to kill all Jews, 1948: "The Communist Manifesto", to kill the rich, existing Nazi-Psycho-Communism, 66 million murders! As Mao wrote in Redbook 51: "Marxist-Communism, once in power, is utterly unworkable, has less value than cowding, its power is the gun!" What an apology we Rabbits ever Israel, Marx, Mao, all mankind, for not teaching Astronomy's great All-One-God-Faith, that with just 6 words eternally unites the human race! As teaches African-shepherd Astronomer Israel for 6000 years, "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!" For one God's Spaceship Earth, with Bomb & Gun, we're All-One or none! All-One! All-One! All-One! Exceptions eternally? Absolute none!

14th: When half-truth is gone & we are dust, the full-truth we print, protect & teach alone lives on! Full-truth is God, it must! Help teach the whole Human race, the Moral ABC of All-One-God-Faith, lightning-like 6 billion strong & in our Eternal Father's Kingdom, we're All-One! "Listen Children Eternal Father Eternally One!"

15th: Americanism is the knowledge of, the loyalty inspiration or attachment to the 'Army of Principles of All-One-God-Faith'. That lightning-like unites the Human race! For we're all-one or none! Exceptions eternally? Absolute None!" Noah Webster.

16: "We can no longer live half-slave, half-free! We unite the Human race in All-One-God-Faith or perish by half-truth hate! For we're All-One or None!" Abraham Lincoln

17: "Government, like fire, is our most useful servant, if fully controlled by us, its Citizens; and government, exactly like fire, becomes our most destructive master if not fully controlled by the open majority of its productive working Citizens! Only united-armed-vigilant & trained, Swiss 600 year Peace & Freedom reigned! Exception? None!" From 'The Army of Principles' by George Washington.

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21: The trouble is that the wrong people are always the most energetic, united & intense; driving the hard-workers to lose in self-defense! That fact alone brings Hitlers & Stalins to power & that will only change when we rally-raise-train-evolve-unite the whole Human race with the Moral ABC in All-One-God-Faith!

22: The intensity of man's emotions is a greater driving force and more decisive than the sum total of his education, his money, plus the size of his brain! Proof: Einstein!

23: The whole World is our country, our Fatherland, because all mankind are born its Citizens! We're all Brothers and Sisters because One, ever-loving Eternal Father is our only God, our only Religion, that Reunites God's legions! For we're ALL-ONE OR NONE. "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!"

24: There are brave souls who dare to breathe love to never kill in bitter, hate. Who dare to hear the mighty truth, reverberating through the long years, that faith-love-courage conquer fear & teamwork heal a nation's tears. Though food & fire sweep the old earth's sod, & raging wars and evils wreck its calm, still through the awful tumult there is God our glorious world within His voice! Among the burning stars, the moon, the sun that have not faded because of that great night, with other planet's wars are one held in His hand, kept in His steadfast sight. Amidst the cannons you can hear God's voice: "Replicate half truth, our real enemy, that age old hate" with full truth, hard work, God's Law uniting mankind in All-One-God-Faith! For centuries man struggles half asleep, half living, small & jealous, bickering with mountains of red tape to awaken, the right God chose giving his great reward for hard work, Poetry, uniting Love, evolving man above the ape! Machine age man is full of sense & nonsense, fear, greed & jealousy destroy this every land! Today, the whole wide world craves love-faith-courage, united by the Moral ABC we stand!

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"The 2nd Coming of God's Law", Mohammed's Arabs, 1948, found Israel Essene Scrolls & Einstein's "Hillel" prove that as certain as no 6-year-old, can grow up free without the abc, so certain can no 12-year old survive free without the Moral ABC mason, tent & sandal-maker, Rabbi Hillel taught carpenter Jesus to unite all mankind free in our Eternal Father's great All-One-God-Faith! For we're All-One or none! "Listen Children Eternal Father Eternally One!" One!

NO DETERGENTS

TO SIMPLIFY & ENJOY LIFE MORE, DILUTE 1/2 OZ. OR 2 SQUIRTS OF THIS PURE CASTILE SOAP WITH 2 GALLONS, OR SINKFUL HOT WATER. THEN TOWEL MASSAGE A FACIAL PACK, THEN WRING TOWEL OUT & FINGER TIP MASSAGE YOUR HAIR & SCALP. ENJOY THE CREAMY EMOLLIENT LATHER ON BABY, BATH, BEACH, BODY, DENTURES, MINT DEODORANT, SHAVING.

100% VEGAN. HEALTH IS OUR GREATEST WEALTH.

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"ALL-ONE!"

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## Dr. Bronner's Magic Soaps

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Contains: Water, Saponified Coconut-Hemp-Olive Oils (with retained Glycerin), Olive Fatty Acids, Peppermint Oil, Rosemary Extract

**100% Cruelty Free in Bars and Liquid**

Peppermint, Lavender, Almond, Eucalyptus, Tea Tree, Aloe Vera Baby-Hind

For everyday bodywashing: Get wet and pour soap full-strength onto washcloth-loofah. Lather up, scrub down, rinse off, and tingle fresh & clean!

Use Dr. Bronner's for Shave-Shampoo-Massage-Dental-Soap Bath! Pure Castile Soap for Dispensers-Uniforms-Diapers-Babies-Beach! Dash in Hot Water. Towel Massage body always toward your heart! Soap clouds when cold -50 F. Put in warm room/water: clears at ~70 F.

**Warning! Keep Out of Eyes! Wash Out with Water! Don't Drink Soap! Dilute! Dilute! or Wet Skin Well! OK!**

NO FOAMING AGENTS

NO SYNTHETIC PERFUMING OILS & 100% VEGETARIAN SUPER MILD CASTILE HAS OUTSTANDING WATER SOFTENING & CLEANSING POWERS. PREFERABLE TO HARSH SOAP & DEFOAMING SYNTHETICS. IT DOES NOT CUT DIRT BUT DISSOLVES IT! IS THE MILDDEST, MOST PLEASANT SOAP YOU EVER USED OR MONEY BACK! ENJOY BODY RUB TO STIMULATE BODY MIND-SOUL-SPIRIT AND TEACH THE ESSENCE MORAL ABC UNITING ALL FREE IN THE SHEPHERD-ASTRONOMER ISRAEL'S GREATEST ALL-ONE-GOD-FAITH!

WE'RE ALL-ONE OR NONE! GUARANTEED SINCE 1948.

DILUTE: ENJOY 18 DIFFERENT USES! GUARANTEED NO SYNTHETIC PERFUMING OILS & 100% VEGETARIAN SUPER MILD CASTILE HAS OUTSTANDING WATER SOFTENING & CLEANSING POWERS. PREFERABLE TO HARSH SOAP & DEFOAMING SYNTHETICS. IT DOES NOT CUT DIRT BUT DISSOLVES IT! IS THE MILDDEST, MOST PLEASANT SOAP YOU EVER USED OR MONEY BACK! ENJOY BODY RUB TO STIMULATE BODY MIND-SOUL-SPIRIT AND TEACH THE ESSENCE MORAL ABC UNITING ALL FREE IN THE SHEPHERD-ASTRONOMER ISRAEL'S GREATEST ALL-ONE-GOD-FAITH!

100% BIODEGRADABLE!

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NO ANIMAL TESTED



Enjoy only 2 cosmetics, enough sleep & Dr. Bronner's 'Magic Soap' to clean body-mind-soul-spirit instantly uniting One! All-One! Absolute cleanliness is Godliness! For facial packs, scalp & soothing body rub, add dash on bath towel in sink of hot water. Wring out. Lay over face & scalp. Massage with fingertips. Repeat 3 or 4 times 'til arms, legs & all are rubbed, always towards the heart. Rinse towel in plain hot water and massage again. Breathe deep! Health is Wealth. Within 9 minutes you feel fresh, mint-clean, saving 90% of your hot water & soap, ready to help teach the whole Human race the Moral ABC of All-One-God-Faith! For we're All-One or none! ALL-ONE! ALL-ONE! ALL-ONE!

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### FROM THE ARMY OF PRINCIPLES BY THOS. PAINE, 1799.

1st: A human being works hard to teach love his enemy, to help unite all mankind free, or that being is not yet Human; so, go the second mile, hold the other cheek brave, not meek! For we're All-One or none! All-One! Exceptions eternally none! ABSOLUTE NONE!

2nd: Small minds decay! Average minds delay! Great minds teach All-One today! Win victory and all stand by you; give us! All deny you! Remember, the only difference between the brave and the coward, the Brave has an ideal to fight for, such as teaching the Moral ABC, that at once unites the human race in All-One-God-Faith! As teach Abraham & Israel, inspired by the Messenger of God's Law, the sign of the Messiah, Halley's Comet, the Blazing Star of Buddha-Bethlehem-Mohammed! "LISTEN CHILDREN ETERNAL FATHER ETERNALLY ONE!"

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### THE MORAL ABC, INTRODUCED BY KIPLING'S "IF" & SOAPMAKER BRONNER

If you can talk to crowds & keep your virtue, or walk with kings & not lose that common touch! If neither loving friend nor enemy can hurt you; if all men count with you, but none too much! If you can work hard to teach each unforgiving minute All-ONE-GOD-FAITH mason Hillel taught carpenter Jesus to unite the Human race, come hell, hate, ban, you'll enjoy God's spaceship Earth & do great work within it; & which is more my son, you'll be a man! A man! "Sure, East is East and West is West & never the twain shall meet!" But there is neither East nor West, nor border, breed nor birth, once the Moral ABC unites all mankind free on God's spaceship Earth! Then & only then, no matter how rough the trip, how charged with punishment the scroll, you are the captain of thy ship, the master of thy soul!

Each day, like a bird, perfect thyself first, to have courage & smile my friend! Think & act 10 years ahead! And the man without fault? He's dead! Do one thing at a time! Work hard! Get done! Then teach friend & enemy 'How to Work & How to Love' for God is Love, uniting All-One above in God's Eternal All-One-God-Faith!

To dream that impossible dream! To reach that unreachable star! 'Til All-One, All-One we are! To fight that unbeatable foe! To go where the brave dare not go! To right the unrightable wrong! To love pure, chaste, from afar! To try 'til your arms are too weary! To reach that unreachable star! 'Til All-One, All-One we are! For this is my goal! To reach that unreachable star! No matter how hopeless, no matter how far! To fight for the right without question or pause, to be willing to march into hell for a heavenly cause! For I know that if I follow this glorious quest, my heart will lie peaceful & calm when I'm laid to my rest! And I know that the world will be better for this, that one man, tortured, blindeed, covered with scars, still strove with his last ounce of courage, to reach that unreachable star 'til united All-One we are! These are the days my friend, we know they'll never end! We'll work-sing-dance-love, marching on! We live God's Law today! We win Free Speech OK! With 10 men & full-truth, our only God, we rally-raise-unite All-One! All-One!

144: When half-truth is gone & we are dust, the full-truth we print, protect & teach alone lives on! Full-truth is God, it must! Help teach the whole Human race, the Moral ABC of All-One-God-Faith, lightning-like 6 billion strong & in our Eternal Father's Kingdom, we're All-One! "Listen Children Eternal Father Eternally One!"

Consider the cones of  $n \times n$  Completely Positive (CPP) and Doubly Nonnegative (DNN) matrices:

$$\mathcal{C}_n := \{X \in \mathcal{S}_n : X = NN^T \text{ for some } N \geq 0\},$$
$$\mathcal{D}_n := \{X \in \mathcal{S}_n : X \succeq 0, X \geq 0\}.$$

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- Known that  $\mathcal{C}_n = \mathcal{D}_n$  for  $n \leq 4$ , but **not** for  $n \geq 5$ .

## Extreme Rays of $\mathcal{D}_5$

For a symmetric matrix  $X$  let  $G(X)$  be the undirected graph on vertices  $\{1, \dots, n\}$  induced by the nonzero off-diagonal elements of  $X$ . A complete characterization of extreme rays of  $\mathcal{D}_5$  based on  $\text{rank}(X)$  and  $G(X)$  is known.

**Theorem 1.** [Hamilton-Jester and Li, 1996] *Suppose  $X \in \mathcal{D}_5$ . Then  $X \in \text{Ext}(\mathcal{D}_5)$  if and only if one of the following holds:*

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- (i)  $\text{rank}(X) = 1$ , in which case  $X \in \mathcal{C}_5$ ;
- (ii)  $\text{rank}(X) = 3$  and  $G(X)$  is a 5-cycle, in which case  $X \notin \mathcal{C}_5$ .

$$P^T X P = \begin{pmatrix} + & + & & + \\ + & + & + & \\ & + & + & + \\ & & + & + & + \\ + & & & + & + \end{pmatrix}.$$

We call  $X \in \mathcal{D}_n \setminus \mathcal{C}_n$  a *bad* matrix, and  $X \in \text{Ext}(D_n)$ ,  $X \notin \mathcal{C}_n$  an *extremely bad* matrix. Let  $\mathcal{E}_5$  be the set of extremely bad  $5 \times 5$  matrices, and  $\mathcal{B}_5 = \text{Cone}(\mathcal{E}_5)$ . It follows that

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We prove that in fact

$$\mathcal{D}_5 = \mathcal{C}_5 + \mathcal{E}_5,$$

that is, any  $X \in \mathcal{D}_5 \setminus \mathcal{C}_5$  differs from a CPP matrix by a *single* extremely bad matrix.

## CPP Reduction

**Definition 1.** A matrix  $X \in \mathcal{D}_n$  is **CPP-reducible** if there are  $0 \neq Y \in \mathcal{C}_n$  and  $Z \in \mathcal{D}_n$  so that  $X = Y + Z$ . If no such  $Y, Z$  exist then  $X$  is **CPP-irreducible**.

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**Theorem 4.** Let  $X \in \mathcal{D}_n$ . Then  $X$  is CPP-reducible iff there exists a partition  $(I, J)$  of  $\{1, \dots, n\}$  such that  $I \neq \emptyset$ ,  $X_{II} > 0$ , and there is an  $f \in \text{Range}(X)$  with  $f \geq 0$ ,  $f_J = 0$ ,  $f_I \neq 0$ .

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For any  $I$ , conditions in Theorem 4 can be checked using the LP

$$\begin{aligned} & \max e^T f \\ & \text{s. t. } Xw = f, \quad e^T f \leq 1, \\ & \quad f \geq 0, \quad f_J = 0. \end{aligned} \tag{1}$$

Let  $f^*$  be an optimal solution. If  $e^T f^* = 1$  then  $X$  is CPP-reducible, and can take  $Y := \varepsilon^* f^* (f^*)^T$  and  $Z := X - \varepsilon^* f^* (f^*)^T$ , where  $\varepsilon^* = \text{argmax}\{\varepsilon > 0 : X - \varepsilon f^* (f^*)^T \in \mathcal{D}_n\}$ .

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**Algorithm 1** CPP Reduction of a DNN Matrix

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**Input:**  $X \in \mathcal{D}_n$

1: Set  $Y_0 := 0$  and  $Z_0 := X$ .

2: **for**  $k = 0, 1, 2, \dots$  **do**

3: Find a partition  $(I, J)$  of  $\{1, \dots, n\}$  such that:

(i)  $I \neq \emptyset$ ,

(ii)  $[Z_k]_{II} > 0$ ,

(iii) the optimal value of (1) – with  $X$  replaced by  $Z_k$  – is one. Let  $f_k$  be an optimal solution of (1).

If no such partition, set  $Y := Y_k$ ,  $Z := Z_k$  and **STOP**.

4: Let  $\varepsilon_k = \operatorname{argmax}\{\varepsilon > 0 : Z_k - \varepsilon f_k f_k^T \in \mathcal{D}_n\}$ .

5: Set  $Y_{k+1} := Y_k + \varepsilon_k f_k f_k^T$  and  $Z_{k+1} := Z_k - \varepsilon_k f_k f_k^T$ .

6: **end for**

7: Set  $Y := Y_{k+1}$  and  $Z := Z_{k+1}$ .

**Output:**  $Y \in \mathcal{C}_n$  and  $Z \in \mathcal{D}_n$  with  $X = Y + Z$ .

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**Theorem 6.** *Let  $X \in \mathcal{D}_5$ . Then  $X$  is **CPP-irreducible** if and only if  $X$  is **extremely bad**. Moreover, if  $X \in \mathcal{D}_5 \setminus \mathcal{C}_5$  then there are  $Y \in \mathcal{C}_5$  and  $Z \in \mathcal{E}_5$  so that  $X = Y + Z$ .*

Proof of Theorem 6 uses new “explicit” representation for matrices in  $\mathcal{E}_5$ :

**Theorem 7.**  *$X \in \mathcal{E}_5$  if and only if there exists a permutation matrix  $P$ , a positive-diagonal matrix  $\Lambda$  and a  $5 \times 3$  matrix*

$$R = \begin{pmatrix} 1 & 0 & 0 \\ r_{21} & r_{22} & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -r_{22} \\ 1 & 0 & -r_{21} \end{pmatrix} \quad (r_{21} > 0, r_{22} > 0)$$

*such that  $P^T X P = \Lambda R R^T \Lambda$ .*

## Separating an $X \in \mathcal{E}_5$ from $\mathcal{C}_5$

We also use Theorem 7 to construct a **cut** that separates a given  $X \in \mathcal{E}_5$  from  $\mathcal{C}_5$ . Let  $H$  be the **Horn matrix**

$$H := \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

$H$  is a **copositive** matrix which cannot be represented as the sum of a positive semidefinite and a nonnegative matrix, i.e.,  $H \in \mathcal{C}_5^* \setminus \mathcal{D}_5^*$ .

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**Theorem 8.** *Let  $X \in \mathcal{E}_5$  with  $P^T X P = \Lambda R R^T \Lambda$  its representation provided by Theorem 7. Define  $w := -(P^T X P \circ H)^{-1} e > 0$  and  $K := P \text{Diag}(w) H \text{Diag}(w) P^T$ . Then  $K \bullet Y \geq 0$  for all  $Y \in \mathcal{C}_5$ , but  $K \bullet X < 0$ .*

**Application:** Computing the **maximum stable set** in a graph.

Let  $A$  be the adjacency matrix of a graph  $G$  on  $n$  vertices, and let  $\alpha$  be the maximum size of a stable set. It is known that

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- Unique solution of problem in (3) is an  $X \in \mathcal{E}_5$ . Adding cut from Theorem 8 and re-solving increases solution value from  $1/\sqrt{5}$  to  $1/2$ ; i.e. **gap between (3) and (2) is closed**.

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**Open Problem 2:** Give complete “inner” description of  $\mathcal{C}_5^* \Leftrightarrow$  give complete “outer” description of  $\mathcal{C}_5$ .

To dream that impossible dream! To reach that unreachable star! 'Til All-One, All-One we are! To fight that unbeatable foe! To go where the brave dare not go! To right the unrightable wrong! To love pure, chaste, from afar! To try 'til your arms are too weary! To reach that unreachable star! 'Til All-One, All-One we are! For this is my goal! To reach that unreachable star! No matter how hopeless, no matter how far! To fight for the right without question or pause, to be willing to march into hell for a heavenly cause! For I know that if I follow this glorious quest, my heart will lie peaceful & calm when I'm laid to my rest! And I know that the world will be better for this, that one man, tortured, blinded, covered with scars, still strove with his last ounce of courage, to reach that unreachable star 'til united All-One we are! These are the days my friend, we know they'll never end! We'll work-sing-dance-love, marching on! We live God's Law today! We win Free Speech OK! With 10 men & full-truth, our only God, we rally-raise-unite All-One! All-One!