# Water network design by MINLP

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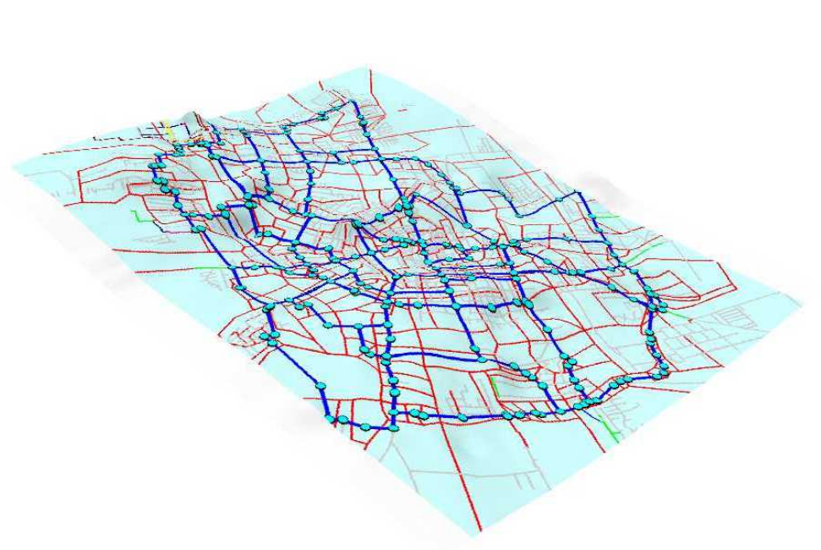
## Introduction

Water Distribution Network (WDN) optimal design: choice of a diameter for each pipe, with other fixed design properties (e.g., the topology and pipe lengths).



MINLP problem:

- discrete variables: set of commercially-available diameters;
- hydraulic constraints on water flows and pressures;
- minimize the cost (function of the selected diameters).

## Notation

Sets:

**E** = set of pipes; **N** = set of junctions; **S** = set of source junctions ($S \subset N$).

Parameters for each pipe $e \in E$:

**len(e)** = length of pipe $e$;
**k(e)** = roughness coeff. of pipe $e$;
$\mathbf{d_{min}(e)}, \mathbf{d_{max}(e)}$ = min and max diam. of pipe $e$;
$\mathbf{v_{max}(e)}$ = max speed of water in pipe $e$;
$\mathfrak{D}(\mathbf{e},\mathbf{r}), \mathfrak{C}(\mathbf{e},\mathbf{r})$ = value and cost of the $r$th discrete diameter for pipe $e$ ($r = 1, \ldots, r_e$).

Parameters for each junction $i \in N \setminus S$:

**dem(i)** = demand at junction $i$;
**elev(i)** = physical elevation of junction $i$;
$\mathbf{ph_{min}(i)}, \mathbf{ph_{max}(i)}$ = min and max pressure head at junction $i$.

Parameters for each source junction $i \in S$:

$\mathbf{h_s(i)}$ = fixed hydraulic head of source junction $i$;

Variables:

**Q(e)** = flow in pipe $e$ ($e \in E$);
**H(i)** = hydraulic head of junction $i$ ($i \in N$);
**D(e)** = diameter of pipe $e$ ($e \in E$).

## A preliminary continuous model

$$
\begin{aligned}
\min \sum_{e \in E} len(e) \cdot C_e(D(e)) & \\
\sum_{e \in \delta_-(i)} Q(e) - \sum_{e \in \delta_+(i)} Q(e) = dem(i) & \qquad (\forall\, i \in N \setminus S)\\
-\frac{\pi}{4} v_{max}(e) D^2(e) \le Q(e) \le \frac{\pi}{4} v_{max}(e) D^2(e) & \qquad (\forall\, e \in E)\\
H(i) - H(j) = \frac{\operatorname{sgn}(Q(e))|Q(e)|^{1.852} \cdot 10.7 \cdot len(e)}{k(e)^{1.852} \cdot D(e)^{4.87}} & \qquad (\forall\, e = (i,j) \in E)\\
d_{min}(e) \le D(e) \le d_{max}(e) & \qquad (\forall e \in E)\\
ph_{min}(i) + elev(i) \le H(i) \le ph_{max}(i) + elev(i) & \qquad (\forall\, i \in N \setminus S)\\
H(i) = h_s(i) & \qquad (\forall\, i \in S).
\end{aligned}
$$

with $\delta_+(i)$ (resp. $\delta_-(i)$) = set of pipes with tail (resp. head) at junction $i$ ($i \in N$).

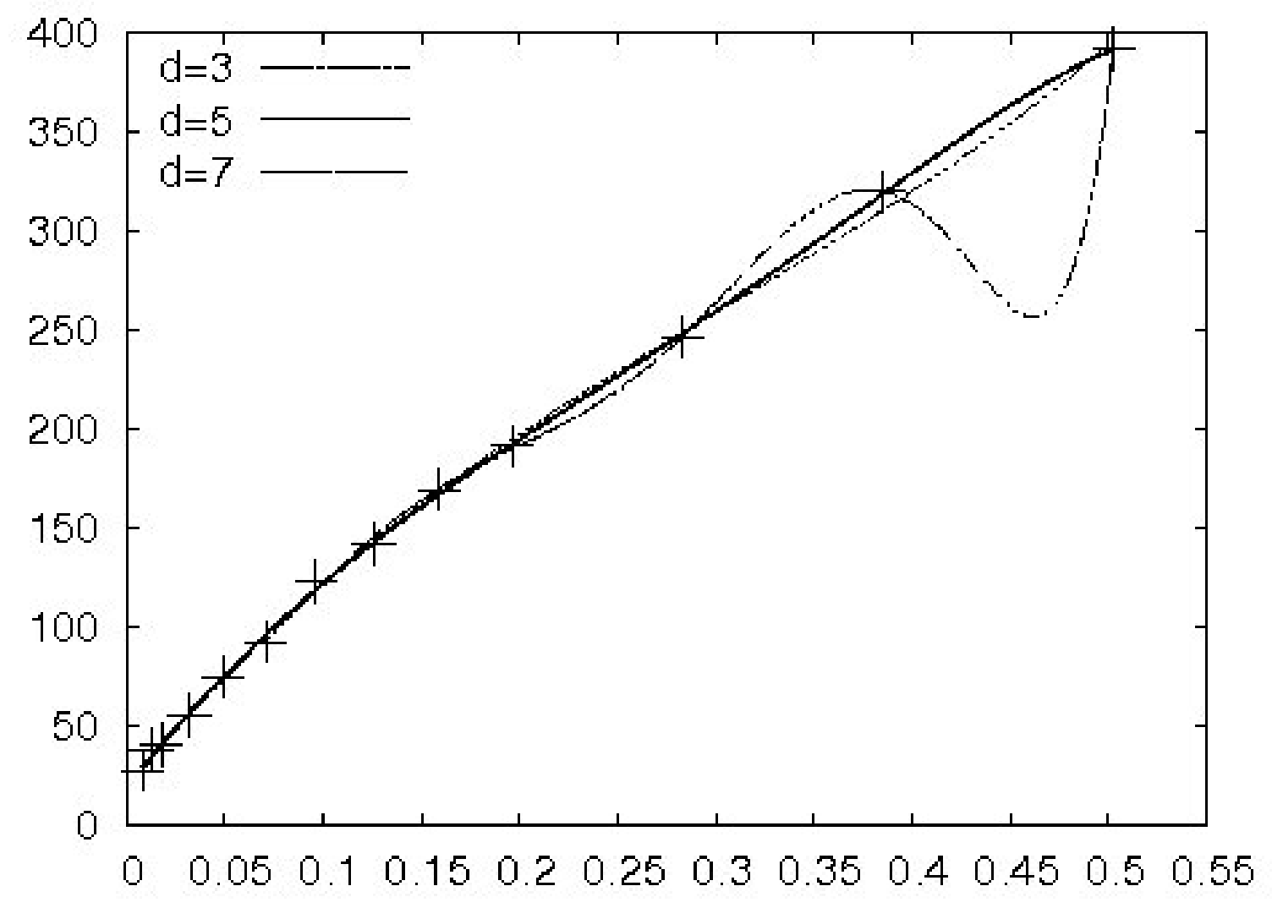
Issues:

- Continuous objective function $C_e(D(e))$?
- Nondifferentiability! See $\operatorname{sgn}(Q(e))|Q(e)|^{1.852}$
- Reformulating with $A(e)$ instead of $D(e)$!

## Continuous objective function

Fitting a polynomial to the input discrete cost data to make a smooth working continuous cost function $C_e()$. We want to minimize the relative error, for example, our least-squares fit for arc $e$ minimizes:

$$
\sum_{r=1}^{r_e} \frac{\left[\mathfrak{C}(e,r) - \left(\sum_{j=0}^{d} \beta(j,e) \left(\frac{\pi}{4}\mathfrak{D}(e,r)^2\right)^j\right)\right]^2}{\mathfrak{C}(e,r)^2} = \sum_{r=1}^{r_e} \left[1 - \left(\frac{\sum_{j=0}^{d} \beta(j,e)\left(\frac{\pi}{4}\mathfrak{D}(e,r)^2\right)^j}{\mathfrak{C}(e,r)}\right)\right]^2
$$

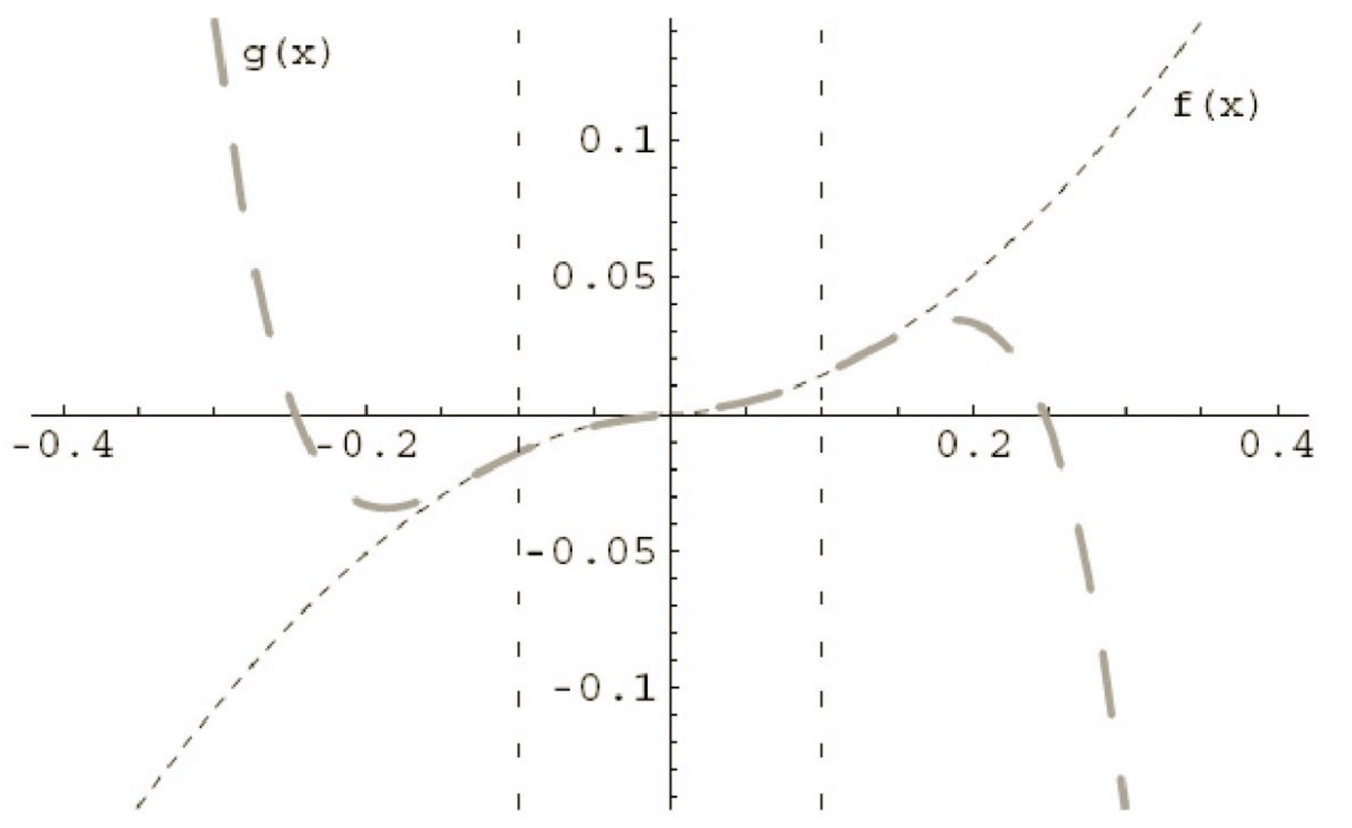


with $d$ = the degree of the polynomial.

## Smoothing the non-differentiability

Approximate $\operatorname{sgn}(Q(e))|Q(e)|^{1.852}$ near 0 with a smooth function $g(x) = ax + bx^3 + cx^5$ having $f(x) = g(x)$, $f'(x) = g'(x)$ and $f''(x) = g''(x)$ at $x = |\delta|$

$$
\begin{aligned}
g(x) = & \left(\frac{3\delta^{p-5}}{8} + \frac{1}{8}(p-1)p\delta^{p-5} - \frac{3}{8}p\delta^{p-5}\right) x^5\\
& + \left(-\frac{5\delta^{p-3}}{4} - \frac{1}{4}(p-1)p\delta^{p-3} + \frac{5}{4}p\delta^{p-3}\right) x^3\\
& + \left(\frac{15\delta^{p-1}}{8} + \frac{1}{8}(p-1)p\delta^{p-1} - \frac{7}{8}p\delta^{p-1}\right) x\,.
\end{aligned}
$$



## Discretizing the area

New set of binary variables $X(e,r)$ ($e \in E$, $r = 1, \ldots, r_e$)

$A(e) = \sum_{r=1}^{r_e} \frac{\pi}{4}\mathfrak{D}^2(e,r)X(e,r)$, $\sum_{r=1}^{r_e} X(e,r) = 1$ and $\{X(e,r) | r = 1, \ldots, r_e\}$ = SOS of type 1 ($e \in E$).

## The MINLP model

$$
\begin{aligned}
\min \sum_{e \in E} len(e) \cdot C_e(A(e)) & \\
\min \sum_{e \in E} len(e) \sum_{r=1}^{r_e} \mathfrak{C}(e,r) \cdot X(e,r) & \\
\sum_{e \in \delta_-(i)} Q(e) - \sum_{e \in \delta_+(i)} Q(e) = dem(i) & \qquad (\forall\, i \in N \setminus S)\\
-v_{max}(e)A(e) \le Q(e) \le v_{max}(e)A(e) & \qquad (\forall\, e \in E)\\
\frac{(H(i) - H(j)) \cdot k(e)^{1.852} \cdot A(e)^{2.435}}{10.7 \cdot len(e) \cdot \left(\frac{\pi}{4}\right)^{2.435}} = \begin{cases} Q(e)^{1.852} & \text{if } Q(e) \ge \delta \\ g(Q(e)) & \text{if } -\delta < Q(e) < \delta \\ (-Q(e))^{1.852} & \text{if } Q(e) \le -\delta \end{cases} & \qquad (\forall\, e = (i,j) \in E)\\
A(e) = \sum_{r=1}^{r_e} \frac{\pi}{4}\mathfrak{D}^2(e,r)X(e,r) & \qquad (\forall e \in E)\\
\sum_{r=1}^{r_e} X(e,r) = 1 & \qquad (\forall e \in E)\\
\frac{\pi}{4}d^2_{min}(e) \le A(e) \le \frac{\pi}{4}d^2_{max}(e) & \qquad (\forall e \in E)\\
ph_{min}(i) + elev(i) \le H(i) \le ph_{max}(i) + elev(i) & \qquad (\forall\, i \in N \setminus S)\\
H(i) = h_s(i) & \qquad (\forall\, i \in S).
\end{aligned}
$$

## Bonmin branch-and-bound

**Bonmin** (Basic Open-source Nonlinear Mixed INteger programming): open-source code for MINLP problems, exact method for convex MINLPs, heuristic method for nonconvex MINLPs.
Branch-and-bound algorithm: solve NLP relaxation at each node of the search tree and branch on variables.
Options for nonconvex MINLPs:

- different starting points for root/each node;
- continue branching even if the solution value to the current node is worse than the best-known solution.

**Bonmin modifications**:

I.1 Properly evaluating the objective value of integer feasible solutions through the definition of 2 objective functions: LB and UB objective function.
I.2 Ad-hoc definition of the cutoff_decr option value

## Computational results

Characteristics of the 50 continuous solutions at the root node.

| | mean | min | max | std dev | coeff var |
|---|---|---|---|---|---|
| shamir | 413,507.00 | -2.922 | 62.827 | 37,735.10 | 0.0912563 |
| hanoi | 6,112,600.00 | -2.271 | 2.114 | 88,473.50 | 0.0144740 |
| blacksburg | 114,534.00 | -0.975 | 7.084 | 1,659.97 | 0.0144932 |
| New York | 83,480,900.00 | -53.278 | 34.331 | 12,024,900.00 | 0.1440440 |
| foss_poly_0 | 78,080,900.00 | -12.094 | 51.136 | 11,096,800.00 | 0.1421190 |
| foss_iron | 181,977.00 | -0.757 | 4.207 | 9,081.52 | 0.0169336 |
| foss_poly_1 | 33,076.40 | -19.459 | 55.683 | 5,470.00 | 0.1653750 |
| pescara | 1,846,930.00 | -1.338 | 16.622 | 66,672.00 | 0.0360989 |
| modena | 2,567,680.00 | -0.103 | 0.421 | 1,920.57 | 0.0007480 |

MINLP model results (part 1). Time limit 7200".

| | MINLP | $\text{MINLP}^{\circ}$ | $\text{fit[MINLP}^{\circ}]$ | |
|---|---|---|---|---|
| shamir | 423,696.31 | 419,000.00 | 0.000 | |
| hanoi | 6,109,620.78 | 6,109,620.90 | 0.000 | |
| blacksburg | 118,251.06 | 118,251.09 | 0.000 | |
| New York | 39,570,174.42 | 39,307,799.72 | 0.541 | ✓ |
| foss_poly_0 | 70,842,869.58 | 70,680,507.90 | 0.000 | |
| foss_iron | 181,865.00 | 178,494.14 | 0.024 | ✓ |
| foss_poly_1 | 29,062.82 | 29,202.99 | 0.000 | |
| pescara | 1,883,480.00 | 1,837,440.40 | 0.187 | ✓ |
| modena | 2,620,189.45 | 2,580,379.53 | 0.000 | |

MINLP model results (part 2). Time limit 7200".

| | $\text{MINLP}^{\circ}$ | time | % dev. $\text{MINLP}^{h}$ | # fit | # true |
|---|---|---|---|---|---|
| shamir | 419,000.00 | 1 | 0.000 | 2 | 2 |
| hanoi | 6,109,620.90 | 357 | 0.000 | 8 | 8 |
| blacksburg | 118,251.09 | 1,540 | 0.178 | 6 | 6 |
| New York | 39,307,799.72 | 3 | 0.000 | 5 | 6 |
| foss_poly_0 | 70,680,507.90 | 1,500 | 0.058 | 6 | 8 |
| foss_iron | 178,494.14 | 3,070 | 0.350 | 4 | 5 |
| foss_poly_1 | 29,202.99 | 6,772 | 0.274 | 6 | 5 |
| pescara | 1,837,440.40 | 6,701 | 0.447 | 7 | 21 |
| modena | 2,580,379.53 | 964 | 0.000 | 2 | 2 |

MINLP model results: 1 vs. 2 objective functions. Time limit 7200".

| | $\text{MINLP}^{\circ}$ | time | % dev. $\text{MINLP}^{*}$ | time |
|---|---|---|---|---|
| shamir | 419,000.00 | 1 | 0.00 | 3 |
| hanoi | 6,109,620.90 | 357 | 0.00 | 1,059 |
| blacksburg | 118,251.09 | 1,540 | 0.00 | 1,384 |
| New York | 39,307,799.72 | 3 | 0.00 | 217 |
| foss_poly_0 | 70,680,507.90 | 1,500 | 0.00 | 2,502 |
| foss_iron | 178,494.14 | 3,070 | 0.00 | 5,584 |
| foss_poly_1 | 29,202.99 | 6,772 | -0.35 | 681 |
| pescara | 1,837,440.40 | 6,701 | 1.17 | 576 |
| modena | 2,580,379.53 | 964 | 1.69 | 106 |

## Literature comparison

MINLP results compared with literature results.

| | Savic and Walters (1997) | | Savic and Walters (1997) | | Dandy et al. (1996) | | Cunha et Sousa (1999) | |
|---|---|---|---|---|---|---|---|---|
| | SW99 rel. | $\text{MINLP}^{a}$ | SW99 res. | $\text{MINLP}^{a}$ | DSM96 | $\text{MINLP}^{a}$ | CS99 | $\text{MINLP}^{a}$ |
| hanoi | 6.073 e+06 | 6.052 e+06 | 6.195 e+06 | 6.183 e+06 | — | — | 6.056 e+06 | 6.056 e+06 |
| New York | 37.13 e+06 | 36.68 e+06 | 40.42 e+06 | 40.47 e+06 | 38.8 e+06 | 38.8 e+06 | — | — |

## Conclusions

MINLP formulation proposed. Effective solutions for real-world instances found in reasonable computing times. Successfull approach because:

1. Availability of software for finding good solutions to MINLP problems;
2. Easy interface via the modeling language AMPL;
3. Special-purpose modeling tricks;
4. Bonmin modifications: dealing with nonconvex MINLPs and multiple objective functions.

Mixed-Integer Nonlinear Optimization: Algorithmic Advances and Applications (IMA, University of Minnesota) Nov 18th, 2008