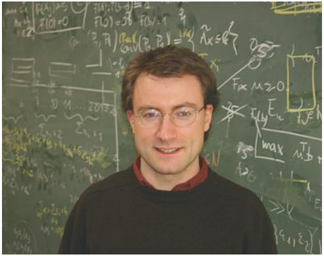
On the Foundations of Non-Linear and Multi-Objective Integer Optimization

Jesús A. De Loera, UC Davis

November 20, 2008

New results joint work with:



Raymond Hemmecke (Magdeburg) and Matthias Koeppe (UC Davis)

Le Menu

1. **APPETIZER:** What is Multi-Objective Integer Optimization and its difficulty.

2. **MAIN DISH:** Complexity of Multi-Objective Integer Linear Optimization in Fixed dimension

3. **DESSERT:** Graver and Gröbner bases methods

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Motivation

- In "traditional" optimization there is a single objective function. But, in reality, often there are competing optimization criteria, thus several objective functions are present.
- Usually, impossible to minimize all the objective functions simultaneously! Still, the many objective functions induce a partial order over the vectors in the feasible region!!
- The multiobjective optimization approach is to find the maximal (minimal) elements of a partially ordered set. Seminal work in the 1880's by Edgeworth and Pareto were aimed to defining economic equilibria in such situations.
- Here we study only multiobjective integer linear programs (MOILPs).

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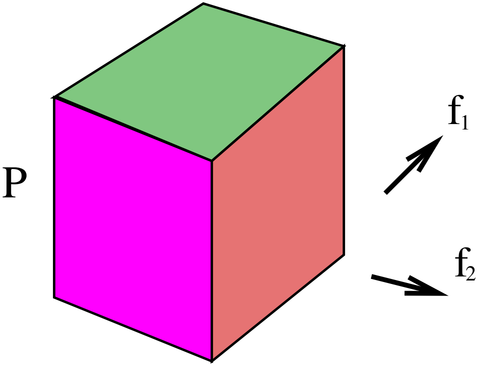
Let $A = (a_{ij})$ be an integral $m \times n$-matrix and $\mathbf{b} \in \mathbf{Z}^m$ defining a polytope $P = \{ \mathbf{u} \in \mathbf{R}^n : A\mathbf{u} \leq \mathbf{b} \}$.
Given k linear functionals $f_1, f_2, \ldots, f_k \in \mathbf{Z}^n$, consider the multiobjective integer linear programming problem

$$\begin{array}{rl} \text{vmin} & (f_1(\mathbf{u}), \ldots, f_k(\mathbf{u})) \\ \text{subject to} & A\mathbf{u} \leq \mathbf{b} \\ & \mathbf{u} \in \mathbf{Z}^n \end{array}$$

where vmin is defined as the problem of finding all Pareto optima and a corresponding Pareto strategy,

First, the strategies are the lattice points inside P.



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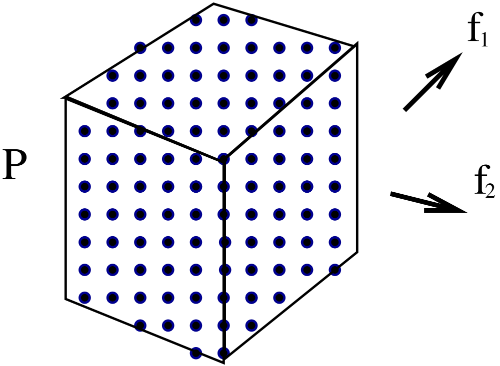
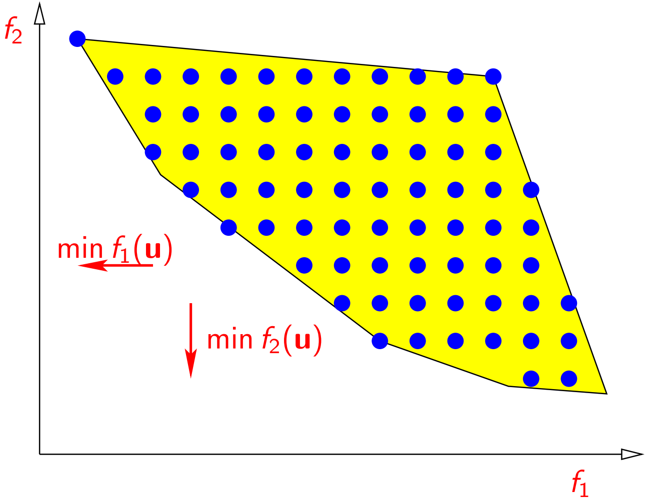


Outcome space

For a feasible point **u**,

$$\mathbf{f}(\mathbf{u}) = \left(f_1(\mathbf{u}), \ldots, f_k(\mathbf{u})\right) \in \mathbf{Z}^k$$

is called an outcome vector.



For a feasible point $\mathbf{u}$,

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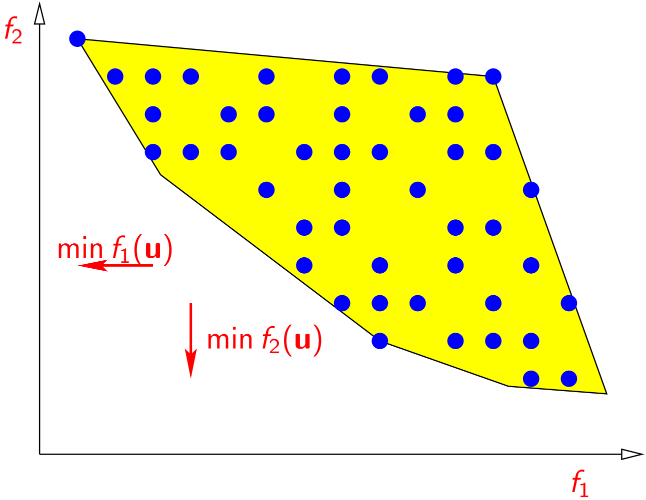
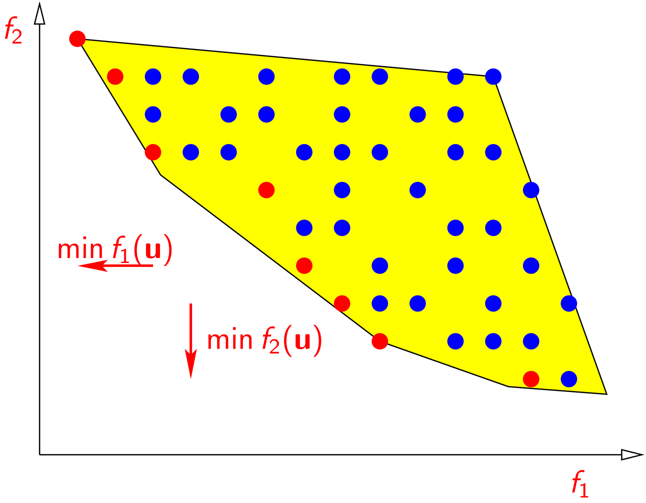
Important: note Integer projection of the feasible region (holes!)

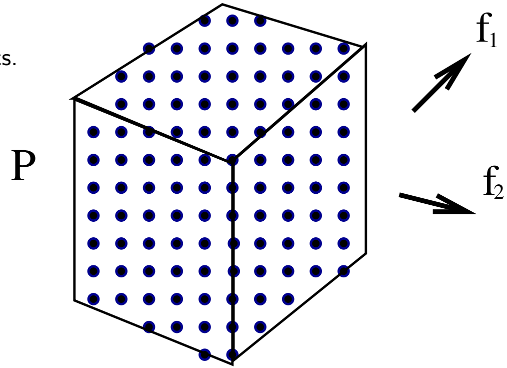




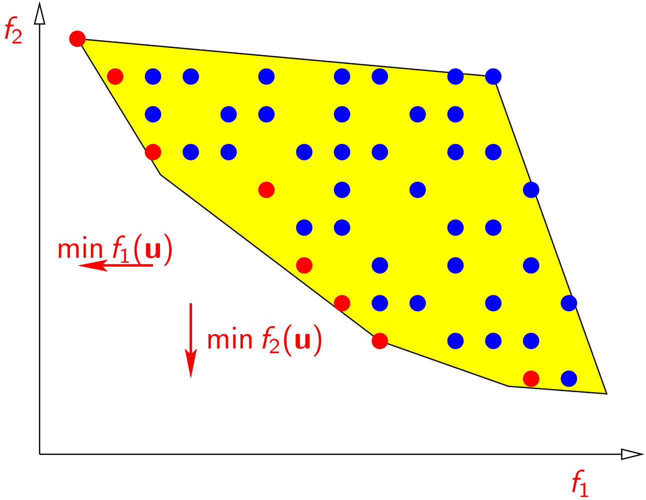
DEFINITION: An outcome vector is a Pareto optimum if and only if there is no other feasible point $\tilde{\mathbf{u}}$ such that $f_i(\tilde{\mathbf{u}}) \leq f_i(\mathbf{u})$ for all i and $f_j(\tilde{\mathbf{u}}) < f_j(\mathbf{u})$ for at least one index j.

Pareto Strategies

Each Pareto optimum is the projection of at least one of the strategy points. Those are the Pareto strategies



We wish to find the Pareto optima and Pareto Strategies

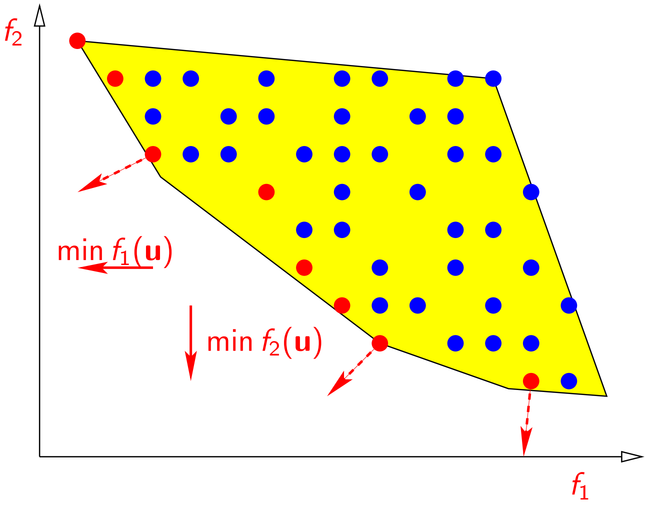


Popular technique

Variable weighting of the objective functions

Difficulty

Non-supported Pareto optima



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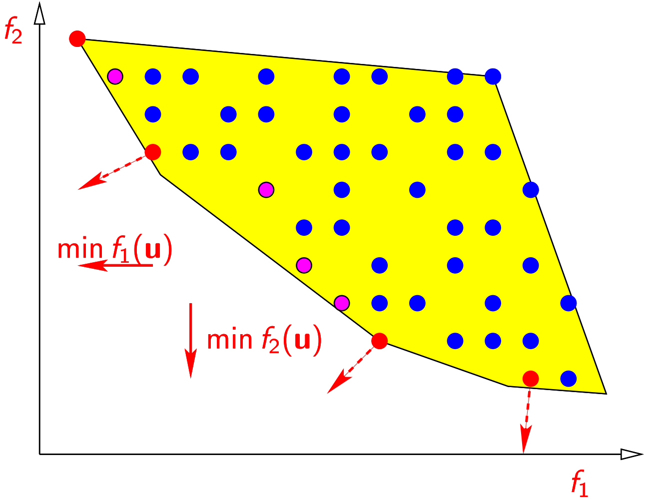
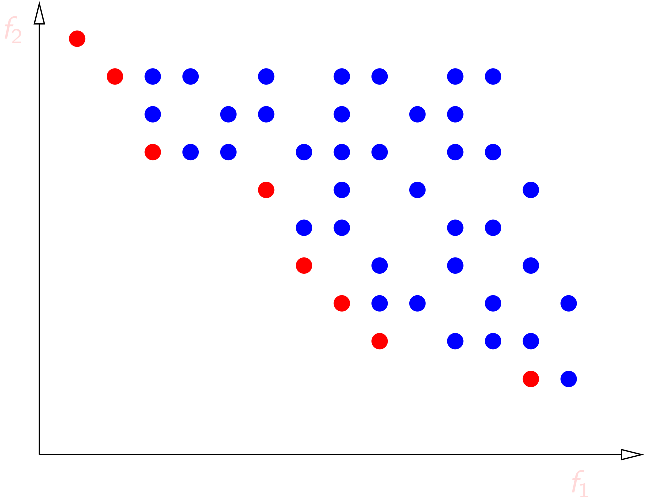


Global criteria

The quality of the compromise is often measured by the distance of a Pareto optimum $\mathbf{v}$ from a user-defined comparison point $\hat{\mathbf{v}}$ for some norm.

Many methods cannot guarantee to pick a Pareto optimum

Some people use a relaxation which optimizes over all feasible points on the outcome space, not just the Pareto optima. Worse! some people allow the holes to be optima.



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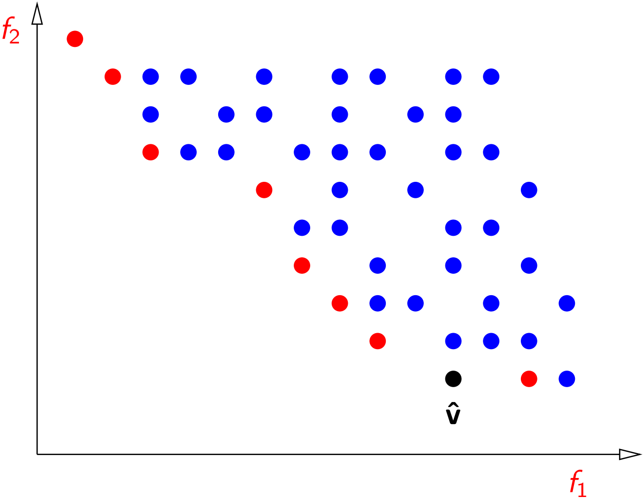

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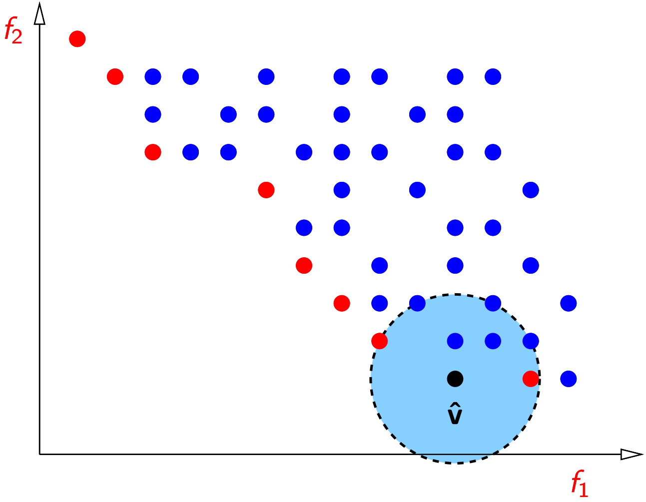

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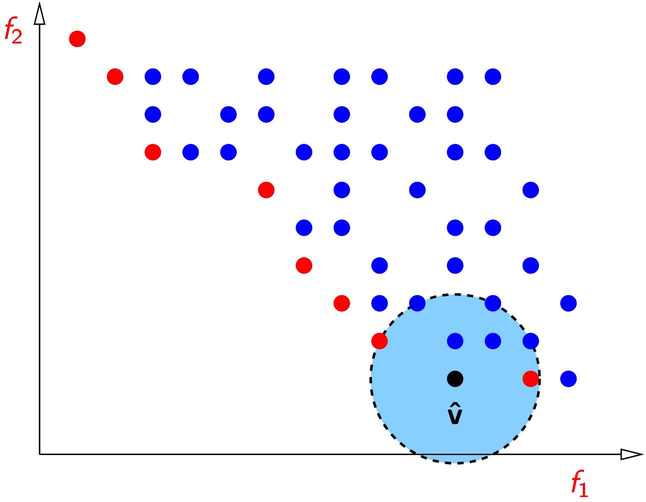

Reality sinks in

Complexity

- $\#\mathrm{P}$-hard to enumerate Pareto optima
- NP-hard to optimize a "global criterion"

More Sad news

Problem remains NP-hard even in the most structured cases!

- When number of objectives is TWO!
- matroid optimization.
- min-cost flow problems.
- transportation problems.

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Complexity Analysis in Fixed Dimension

Since the problem is NP-hard, turn to a refined complexity analysis that is valid for every class

$$\mathrm{MOILP}(n, k)$$

of multicriterion problems with

- a prescribed dimension n of the strategy space
- a prescribed dimension k of the outcome space

(In the spirit of Lenstra's 1983 polynomial-time algorithm for ILP in fixed dimension.)

Main results

Theorem (Counting and enumeration theorem)

Let the dimensions k and n be fixed.

Using the input data $A \in \mathbf{Z}^{m\times n}$, an m-vector $\mathbf{b}$, and linear functions $f_1, \ldots, f_k \in \mathbf{Z}^n$,

(i) *there exists a polynomial-time algorithm to exactly count the Pareto optima and the Pareto strategies;*

(ii) *there exists a polynomial-space polynomial-delay prescribed-order enumeration algorithm to generate the full sequence of Pareto optima ordered lexicographically.*

Theorem (Global-criterion optimization theorem)

Let the dimension n and the number k of objective functions be fixed.

(i) *There exists a polynomial-time algorithm to find a Pareto optimum $\mathbf{v}$ of (6) that minimizes the distance $\|\mathbf{v} - \hat{\mathbf{v}}\|$ from a prescribed point $\hat{\mathbf{v}} \in \mathbf{Z}^k$ for an arbitrary polyhedral norm.*

(ii) *There exists a fully polynomial-time approximation scheme for the problem of minimizing the Euclidean distance of a Pareto optimum from a prescribed comparison point $\hat{\mathbf{v}} \in \mathbf{Z}^k$.*

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Represent Sets of Lattice Points as Rational Function

Given $K \subset \mathbf{R}^d$ we define the sum

$$f(K) = \sum_{\alpha \in K \cap \mathbf{Z}^d} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}.$$

Think of the lattice points as monomials!!! EXAMPLE: $(7, 4, -3)$ is $z_1^7 z_2^4 z_3^{-3}$.

KEY IDEA: For a rational convex polyhedron, i.e. $K = \{x \in \mathbf{R}^n | Ax = b,\ Bx \leq b'\}$, where A, B are integral matrices and b, b' are integral vectors, The generating function $f(K)$, and thus **ALL** the lattice points of the polyhedron K, **can be written in a sum of rational functions!**

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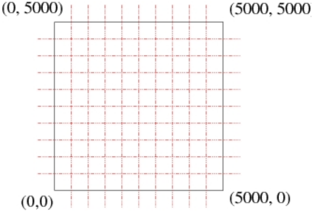
Example

Let P be the square with vertices $V_1 = (0,0)$, $V_2 = (5000,0)$, $V_3 = (5000,5000)$, and $V_4 = (0,5000)$.



The generating function $f(P)$ has over 25,000,000 monomials,
$f(P) = 1 + z_1 + z_2 + z_1^1 z_2^2 + z_1^2 z_2 + \cdots + z_1^{5000} z_2^{5000}$,

But it can be written using only four rational functions

$$\frac{1}{(1-z_1)(1-z_2)} + \frac{{z_1}^{5000}}{(1-{z_1}^{-1})(1-z_2)} + \frac{{z_2}^{5000}}{(1-{z_2}^{-1})(1-z_1)} + \frac{{z_1}^{5000}{z_2}^{5000}}{(1-{z_1}^{-1})(1-{z_2}^{-1})}$$

Also, $f(tP, z)$ is

$$\frac{1}{(1-z_1)(1-z_2)} + \frac{{z_1}^{5000\cdot t}}{(1-{z_1}^{-1})(1-z_2)} + \frac{{z_2}^{5000\cdot t}}{(1-{z_2}^{-1})(1-z_1)} + \frac{{z_1}^{5000\cdot t}{z_2}^{5000\cdot t}}{(1-{z_1}^{-1})(1-{z_2}^{-1})}$$

Theorem (Alexander Barvinok, 1994)

Let the dimension d be fixed. There is a polynomial-time algorithm for computing a representation of the generating function

$$g_P(z_1, \ldots, z_d) = \sum_{(\alpha_1,\ldots,\alpha_d)\in P\cap \mathbf{Z}^d} z_1^{\alpha_1} \cdots z_d^{\alpha_d} = \sum_{\boldsymbol{\alpha}\in P\cap \mathbf{Z}^d} \mathbf{z}^{\boldsymbol{\alpha}}$$

of the integer points $P \cap \mathbf{Z}^d$ of a polyhedron $P \subset \mathbf{R}^d$ (given by rational inequalities) in the form of a rational function.

Corollary

In particular,

$$|P \cap \mathbf{Z}^d| = g_P(\mathbf{1})$$

can be computed in polynomial time (in fixed dimension).

WARNING calculations have to be done using Complex Analysis!

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Theory by Barvinok–Woods, 2003

Theorem (Projection Theorem; Barvinok–Woods, 2003)

Assume the dimension n is a fixed constant. Consider a rational polytope $P \subset \mathbf{R}^n$ and a linear map $T : \mathbf{Z}^n \to \mathbf{Z}^k$. There is a polynomial time algorithm which computes a short representation of the generating function $f\big(T(P \cap \mathbf{Z}^n); \mathbf{x}\big)$.

Theorem (Boolean Operations Lemma; Barvinok–Woods, 2003)

Let m and ℓ be fixed integers. Let $S_1, S_2, \dots, S_m$ be finite subsets of $\mathbf{Z}^n$, given by their rational generating functions $g(S_i; \mathbf{x})$ for $i = 1, \dots, m$, with at most ℓ binomials in each denominator. Let a set $S \subseteq \mathbf{Z}^n$ be defined as a Boolean combination of $S_1, \dots, S_m$ (i.e., using any of the operations $\cup$, $\cap$, $\setminus$). Then there exists a polynomial time algorithm, which computes

$$g(S; \mathbf{x}) = \sum_{i \in I} \gamma_i \frac{\mathbf{x}^{\mathbf{c}_i}}{(1 - \mathbf{x}^{\mathbf{d}_{i1}}) \dots (1 - \mathbf{x}^{\mathbf{d}_{is}})}$$

where $s = s(\ell, m)$ is a constant, the γ_i are rational numbers, $\mathbf{c}_i, \mathbf{d}_{ij}$ are nonzero integer vectors, and I is a polynomial-size index set.

Theorem (FPTAS for maximizing non-negative polynomials over lattice point sets)

For all fixed integers k (dimension) and s (maximum number of binomials in the denominator), there exists an algorithm with running time polynomial in the encoding size of the problem and $\frac{1}{\epsilon}$ *for the following problem.*

Input: *Let* $V \subseteq \mathbf{Z}^k$ *be a finite set, given by a rational generating function in the form*

$$g(V;\mathbf{x}) = \sum_{i \in I} \gamma_i \frac{\mathbf{x}^{\mathbf{c}_i}}{(1-\mathbf{x}^{\mathbf{d}_{i1}})\dots(1-\mathbf{x}^{\mathbf{d}_{is_i}})}$$

where the the numbers s_i *of binomials in the denominators are at most* s*. Furthermore, let two vectors* $\mathbf{v}_{\mathrm{L}}, \mathbf{v}_{\mathrm{U}} \in \mathbf{Z}^k$ *be given such that* V *is contained in the box* $\{\, \mathbf{v} : \mathbf{v}_{\mathrm{L}} \le \mathbf{v} \le \mathbf{v}_{\mathrm{U}} \,\}$.

Let $f \in \mathbf{Q}[v_1, \dots, v_k]$ *be a polynomial with rational coefficients that is non-negative on* V*, given by a list of its monomials, whose coefficients are encoded in binary and whose exponents are encoded in unary.*

Finally, let $\epsilon \in \mathbf{Q}$.

Output: *Compute a point* $\mathbf{v}_\epsilon \in V$ *that satisfies*

$$f(\mathbf{v}_\epsilon) \ge (1-\epsilon) f^* \quad \textit{where} \quad f^* = \max_{\mathbf{v} \in V} f(\mathbf{v}).$$

A similar Theorem Holds for mixed integer case

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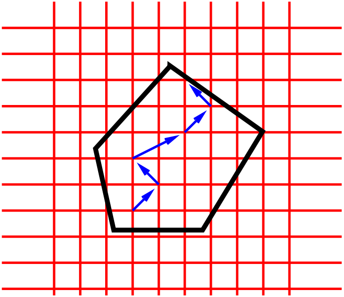

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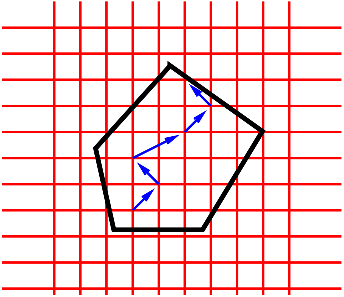

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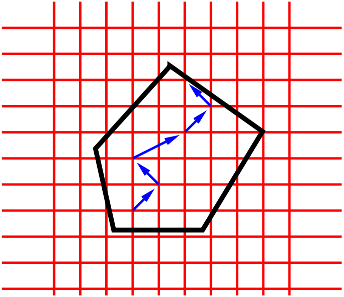

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- For a fixed cost vector c, we can visualize a Graver basis of of an integer program by creating a graph!!
- Here is how to construct it, consider

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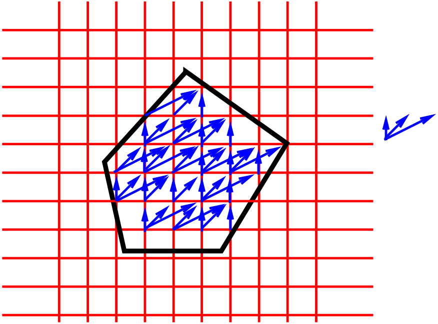

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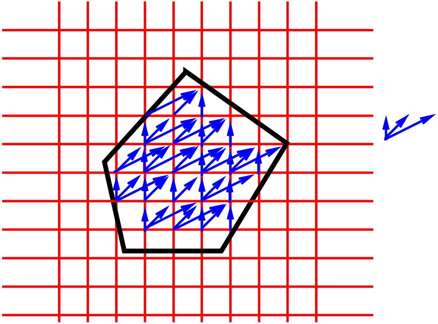

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A multiobjective version of Graver bases

- In the single objective theory, there is a **unique** sink for the directed graph. This is the optimal solution!
- **Theorem** (V. Blanco- J. Puerto 2008) The Graver basis builds a connected directed graph in every fiber of $L(b)$. The nodes of the graph are all the lattice points in the fiber and (γ, γ') is an edge of the directed graph if $\gamma' = \gamma - g_{ij}^k$ for some i, j and k.
- For each maximal chain in the b-fiber of $L(b)$, its directed graph has a unique final node at a Pareto-optimal solution.
- In the graph of a fiber $L(b)$ there exists a directed path from every feasible point α to each Pareto-optimal point, β, in the same fiber.
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- Jon is definitely right that these are the hardest problems: even for 10 variables these problems are undecidable!
- I disagree we are crazy, we are not crazy enough!!
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