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Branch-and-Cut for mixed 0-1 problems

- Relaxation hierarchies

- Cut generation

- Algorithm

Outer Approximation based Branch-and-Bound

- Subdifferential

- Second Order Cone Subproblems

- Linear Outer Approximations

- Algorithm

Computational Results



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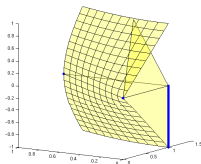
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Mixed 0-1 second order cone problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \succeq 0 \\ & (x)_j \in \{0, 1\} \quad \forall j \in J \subseteq \{1, \dots, n\}, \end{aligned}$$

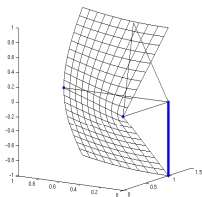


$x \succeq 0$:

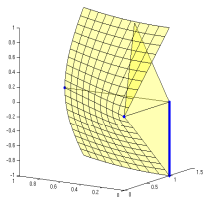
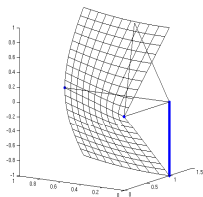
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{noc} \end{pmatrix}, x_i \in \mathbb{R}^{k_i}, \sum_{i=1}^{noc} k_i = n,$$

$$(x_i)_1 \geq \|(x_i)_{2:k_i}\|_2, \quad (i = 1, \dots, noc)$$

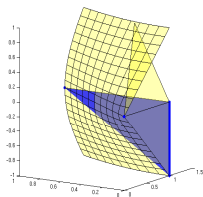
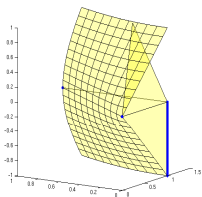
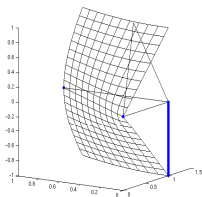
- ▶ binary feasible set $C^0 = \{x \in \mathbb{R}^n : Ax = b, x \succeq 0, x_j \in \{0, 1\} \quad \forall j \in J\}$,



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- ▶ convex hull of the binary feasible set $\text{conv}(C^0)$





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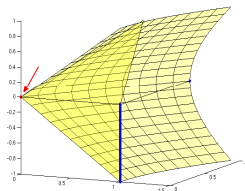
Computational Results

Goal: A tight convex relaxation of the binary feasible set C^0 !

Approach: Lift-and-project-techniques: generalizations of hierarchies of relaxations by Lovasz and Schrijver

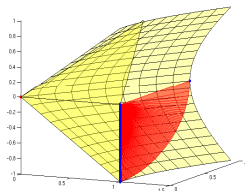
Describe $\text{conv}(C \cap \{x_j \in \{0, 1\}\})$ for a $j \in J$ by **lifting**:

$$M_j(C) = \left\{ \begin{array}{l} (x, u_0, u_1, \lambda_0, \lambda_1) : \\ x = \lambda_0 u_0 + \lambda_1 u_1, \\ \lambda_0 + \lambda_1 = 1, \lambda_0, \lambda_1 \geq 0, \\ u_0 \in C, \quad (u_0)_j = 0 \\ u_1 \in C, \quad (u_1)_j = 1 \end{array} \right\},$$



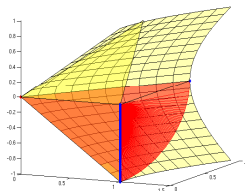
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Convex representation of $M_j(C)$

Use the substitution $v^0 := \lambda^0 u^0$ and $v^1 := \lambda^1 u^1$

$$M_j(C) = \left\{ (x, u^0, u^1, \lambda^0, \lambda^1) : \begin{array}{l} \lambda^0 u^0 + \lambda^1 u^1 = x \\ \lambda^0 + \lambda^1 = 1, \lambda^0, \lambda^1 \geq 0 \\ Au^0 - b = 0, Au^1 - b = 0 \\ u^0 \succeq 0, u^1 \succeq 0 \\ (u^0)_k \in [0, 1] \quad (k \in J, k \neq j) \\ (u^1)_k \in [0, 1] \quad (k \in J, k \neq j) \\ (u^0)_j = 0, (u^1)_j = 1 \end{array} \right\},$$

Convex representation of $M_j(C)$

Use the substitution $v^0 := \lambda^0 u^0$ and $v^1 := \lambda^1 u^1$

$$\tilde{M}_j(C) = \left\{ (x, u^0, u^1, \lambda^0, \lambda^1) : \begin{array}{l} \lambda^0 u^0 + \lambda^1 u^1 = x \\ \lambda^0 + \lambda^1 = 1, \lambda^0, \lambda^1 \geq 0 \\ A\lambda^0 u^0 - \lambda^0 b = 0, A\lambda^1 u^1 - \lambda^1 b = 0 \\ \lambda^0 u^0 \succeq 0, \lambda^1 u^1 \succeq 0 \\ (\lambda^0 u^0)_k \in [0, \lambda^0] \quad (k \in J, k \neq j) \\ (\lambda^1 u^1)_k \in [0, \lambda^1] \quad (k \in J, k \neq j) \\ \lambda^0 (u^0)_j = 0, \lambda^1 (u^1)_j = \lambda^1 \end{array} \right\},$$

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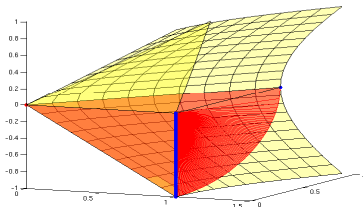
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Projection on x :

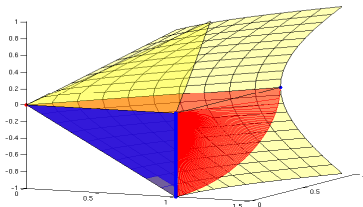
$$P_j(C) := \{x : (x, v^0, v^1, \lambda^0, \lambda^1) \in \tilde{M}_j(C)\}.$$

Higher order lifting

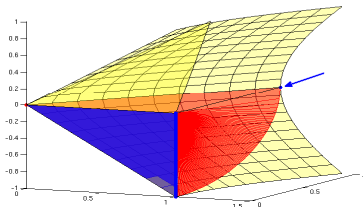
Make a **higher order lifting** for indices $B \subseteq J \rightsquigarrow \tilde{M}_B(C)$



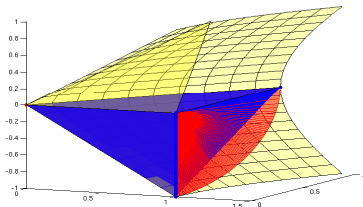
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Projection on x :

$$P_B(C) := \{x : (x, (v^{0j}, v^{1j}, \lambda^{0j}, \lambda^{1j})_{(\forall j \in B)}) \in \tilde{M}_B(C)\}.$$

Add an additional valid semidefinite constraint

Add convex relaxation of $x_j = x_j^2$, $j \in B$:

$$\tilde{M}_B^+(C) = \left\{ (x, v^{0j}; v^{1j}, \lambda^{0j}, \lambda^{1j}) \in \tilde{M}_B(C), V_B^1 - x_B x_B^T \succeq_{sd} 0 \right\},$$

with $V_B^1 := (v_B^{11}, \dots, v_B^{1|B|})$.

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Projection on x :

$$P_B^+(C) := \{x : (x, (v^{k0}, v^{k1}, \lambda^{k0}, \lambda^{k1}) (\forall k \in B)) \in \tilde{M}_B^+(C)\},$$



Theorem [Stubbs, Mehrotra '99]

Let $B \subseteq J = \{i_1, \dots, i_p\}$,

$P_B^1(C) := P_B(C)$ and $P_B^t(C) := P_B(P_B^{t-1}(C))$ for $t \geq 2$.

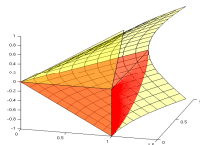
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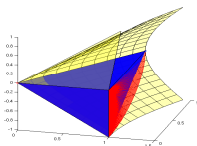
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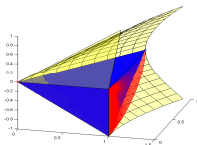
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- ▶ $P_{i_1}(P_{i_2}(\dots P_{i_p}(C))) = \text{conv}(C^0)$
- ▶ $C^0 \subseteq P_B^+(C) \subseteq P_B(C) \subseteq \bigcap_{j \in B} \text{conv}(C \cap \{x_j \in \{0, 1\}\})$,
- ▶ for $B = J$: $(P_J^+)^p(C) = P_J^p(C) = \text{conv}(C^0)$,



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Goal: Find convex inequalities

$$x^T Q x + \alpha^T x \geq \beta,$$

(Q negative semidefinite, $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$), that are valid for all $x \in C^0$.

Theorem (Stubbs, Mehrotra '99)

Let $\bar{x} \notin P_B(C)$ and \hat{x} be the optimal solution of the minimum distance problem

$$\min_{x \in P_B(C)} f(x) = \|x - \bar{x}\|,$$

then there exists a subgradient ξ of f at \hat{x} such that

$$\xi^T(x - \hat{x}) \geq 0$$

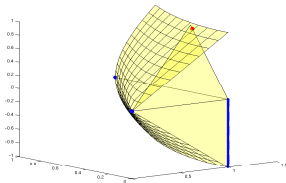
is a valid linear inequality for all $x \in C^0$, that cuts off \bar{x} .



Proposition (Subgradient cuts for SOCP)

Let $\bar{x} \notin P_B(C)$. The minimum distance problem $\min_{x \in P_B(C)} \|x - \bar{x}\|_2$, can be written as a second order cone program. If \hat{x} is its optimal solution, then

$$(\hat{x} - \bar{x})^T x \geq (\hat{x} - \bar{x})^T \hat{x}$$



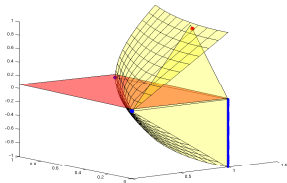
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Theorem (Cezik, Iyengar '05, D. '08)

Let $\text{int}(\text{conv}(C^0)) \neq \emptyset$, $B \subseteq J$, $V_B^1 := (v_B^{11}, \dots, v_B^{1|B|})$, then

$$Q \bullet V_B^1 + \alpha^T x \geq \beta \text{ with } Q = Q^T = (q^1, \dots, q^{|B|})$$

is valid for all $(x, (v^{1k}, v^{0k}, \lambda^{1k}, \lambda^{0k}) \forall k \in B) \in \tilde{M}_B^+(C)$

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is valid for all $(x, (v^{1k}, v^{0k}, \lambda^{1k}, \lambda^{0k}) \forall k \in B) \in \tilde{M}_B^+(C)$ if and only if there exist

$$y^k = (y^{1,k}, y^{2,k}, y^{3,k}, y^{4,k}, y^{5,k}, y^{6,k}) \in \mathbb{R}^{n+2+3m}$$

$$s^k = (s_{v^{k0}}, s_{v^{k1}}, s_{\lambda^{k0}}, s_{\lambda^{k1}}) \in \mathbb{R}^{2n+2},$$

$$\forall k \in B$$

$$s_x \in \mathbb{R}^n, \quad S^6 = (s^{6,1}, \dots, s^{6,|B|+1}) \in \mathbb{R}^{|B|+1, |B|+1} :$$



$$(1) \quad - \sum_{k=1}^{|B|} y^{1,k} + (e_{i_1}, \dots, e_{i_{|B|}}) \begin{pmatrix} s_{|B|+1}^{6,1} \\ \dots \\ s_{|B|+1}^{6,|B|} \end{pmatrix} + (e_{i_1}, \dots, e_{i_{|B|}}) s_{1:|B|}^{6,|B|+1} + s_x = \alpha, s_x \succeq 0,$$

$$(2) \quad I^n y^{1,k} + y_k^3 e_{i_k}^n + A^T y^{5,k} + s_{v,k0} = 0, s_{v,k0} \succeq 0,$$

$$(3) \quad I^n y^{1k} + y_k^4 e_{i_k}^n + A^T y^{6,k} + \tilde{E} y^{7,k} + (e_{i_1}, \dots, e_{i_{|B|}}, 0) s^{6,k} + s_{v,k1} = q^k, s_{v,k1} \succeq 0,$$

$$(4) \quad y_k^2 - b^T y^{5,k} + s_{\lambda,k0} = 0, s_{\lambda,k0} \geq 0,$$

$$(5) \quad y_k^2 - y_k^4 - b^T y^{6,k} + s_{\lambda,k1} = 0, s_{\lambda,k1} \geq 0,$$

$$\forall k \in B,$$

$$(6) \quad s^6 \succeq_{sd} 0,$$

$$(7) \quad \sum_{k=1}^{|B|} y_k^2 - s_{|B|+1}^{6,|B|+1} = \beta.$$

$$Q \bullet V_B^1 + \alpha^T x \geq \beta \quad \Leftrightarrow! \quad (1)-(7)$$



Proof: Consider

$$(P) \quad \begin{array}{ll} \min & Q \bullet V_B^1 + \alpha^T x \\ \text{s.t.} & (x, (v^{1k}, v^{0k}, \lambda^{1k}, \lambda^{0k}) \forall k \in B) \in \tilde{M}_B^+(C) \end{array}$$

with dual problem (D). $\text{int}(\text{conv}(C^0)) \neq \emptyset \Rightarrow \text{int}(\tilde{M}_B^+(C)) \neq \emptyset$.

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' \Leftarrow ': A feasible point of (D) with dual objective value β satisfies conditions (1)-(7)
 \Rightarrow primal objective $Q \bullet V_B^1 + \alpha^T x$ is bounded below by β .

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' \Rightarrow ': (i) $\text{int}(\tilde{M}_B^+(C)) \neq \emptyset$ and primal objective bounded below \Rightarrow (D) solvable
(ii) dual objective unbounded below
(iii) continuity of dual objective and convexity of (D)
 \Rightarrow for every $Q \bullet V_B^1 + \alpha^T x \geq \beta$ exists a dual feasible point with objective value β (a point satisfying (1)-(7))



How to get convex inequalities $x_B^T Q x_B + \alpha^T x \geq \beta$ in x ?



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- (1) Let (Q, α, β) satisfy conditions (1-7): $\Rightarrow Q \bullet V_B^1 + \alpha^T x \geq \beta$,
- (2) Guarantee $x_B^T Q x_B \geq V_B^1 \bullet Q \Rightarrow$ (using (1)) $x_B^T Q x_B + \alpha^T x \geq \beta$

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- (2) Guarantee $x_B^T Q x_B \geq V_B^1 \bullet Q \Rightarrow$ (using (1)) $x_B^T Q x_B + \alpha^T x \geq \beta$
- (3) Guarantee $-Q \succeq_{sd} 0$: $\Rightarrow x_B^T Q x_B + \alpha^T x \geq \beta$ is convex.

Goal: Solve a conic program to generate a valid cut!

Lemma (Cezik, Iyengar '05)

The convex quadratic inequality $x_B^T Q x_B + \alpha^T x \geq \beta$, with $-Q \succeq_{sd} 0$ is valid for $P_B(C)^+$, if (Q, α, β) satisfy (1-7).

Proof:

$V_B^1 \succeq_{sd} x_B x_B^T$ and $-Q \succeq_{sd} 0$ imply $x_B^T Q x_B \geq V_B^1 \bullet Q \square$

Lemma

- a) *The inequality $\alpha^T x \geq \beta$ is valid for $P_B(C)$, if there exists $(Q = 0, \alpha, \beta)$, satisfying (1) - (5) with $S^6 = 0$ and (7).*

Lemma

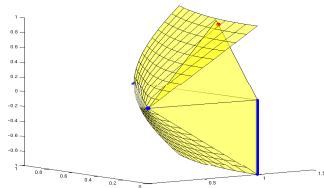
- a) The inequality $\alpha^T x \geq \beta$ is valid for $P_B(C)$, if there exists $(Q = 0, \alpha, \beta)$, satisfying (1) - (5) with $S^6 = 0$ and (7).
- b) The convex quadratic inequality $x_B^T Q x_B + \alpha^T x \geq \beta$, $Q = \text{diag}(q_1, \dots, q_{|B|})$, with $q_i \leq 0$ is valid for $P_B(C)$, if (Q, α, β) satisfy (1-5) with $S^6 = 0$ and (7).

Proof: We can show, that $x_B^T Q x_B \geq V_B^1 \bullet Q$ holds for $(x, (v^{1k}, v^{0k}, \lambda^{1k}, \lambda^{0k}) \forall k \in B) \in \tilde{M}_B(C) \square$

Linear cut $\alpha^T x - \beta \geq 0$ generating SOCP

Let $\bar{x} \in C$ be a fractional solution.

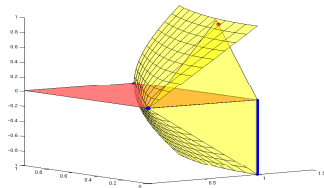
$$\begin{array}{ll} \min & \alpha^T \bar{x} - \beta \\ \text{s.t.} & (0, \alpha, \beta) \text{ satisfy (1)-(5), (7)} \\ & S^6 = 0 \end{array}$$



Linear cut $\alpha^T x - \beta \geq 0$ generating SOCP

Let $\bar{x} \in C$ be a fractional solution.

$$\begin{aligned} \min \quad & \alpha^T \bar{x} - \beta \\ \text{s.t.} \quad & (0, \alpha, \beta) \text{ satisfy (1)-(5), (7)} \\ & S^6 = 0 \end{aligned}$$

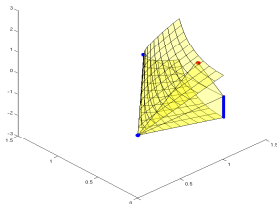


Diagonal quadratic cut $x_B^T Q x_B + \alpha^T x - \beta \geq 0$ generating SOCP



Let $\bar{x} \in C$ be a fractional solution.

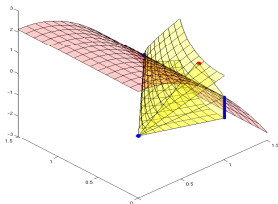
$$\begin{aligned} \min \quad & \bar{x}_B^T Q \bar{x}_B + \alpha^T \bar{x} - \beta \\ \text{s.t.} \quad & (Q, \alpha, \beta) \text{ satisfy (1)-(5), (7)} \\ & S^6 = 0 \\ & Q = \text{diag}(q_1, \dots, q_l) \leq 0 \end{aligned}$$



Diagonal quadratic cut $x_B^T Q x_B + \alpha^T x - \beta \geq 0$ generating SOCP

Let $\bar{x} \in C$ be a fractional solution.

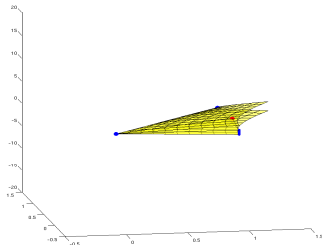
$$\begin{aligned} \min \quad & \bar{x}_B^T Q \bar{x}_B + \alpha^T \bar{x} - \beta \\ \text{s.t.} \quad & (Q, \alpha, \beta) \text{ satisfy (1)-(5), (7)} \\ & S^6 = 0 \\ & Q = \text{diag}(q_1, \dots, q_l) \leq 0 \end{aligned}$$



Quadratic cut $x_B^T Q x_B + \alpha^T x - \beta \geq 0$ generating semidefinite Problem

Let $\bar{x} \in C$ be a fractional solution.

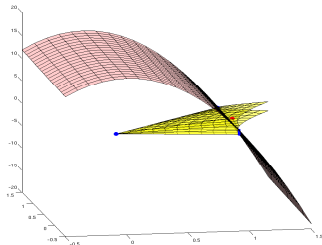
$$\begin{aligned} \min \quad & \bar{x}_B^T Q \bar{x}_B + \alpha^T \bar{x} - \beta \\ \text{s.t.} \quad & (Q, \alpha, \beta) \text{ satisfy (1)-(7)} \\ & -Q \succeq_{sd} 0 \end{aligned}$$



Quadratic cut $x_B^T Q x_B + \alpha^T x - \beta \geq 0$ generating semidefinite Problem

Let $\bar{x} \in C$ be a fractional solution.

$$\begin{aligned} \min \quad & \bar{x}_B^T Q \bar{x}_B + \alpha^T \bar{x} - \beta \\ \text{s.t.} \quad & (Q, \alpha, \beta) \text{ satisfy (1)-(7)} \\ & -Q \succeq_{sd} 0 \end{aligned}$$





Branch-and-Cut for mixed 0-1 problems

Relaxation hierarchies

Cut generation

Algorithm

Outer Approximation based Branch-and-Bound

Subdifferential

Second Order Cone Subproblems

Linear Outer Approximations

Algorithm

Computational Results

Initialize root with $\min c^T x$, s.t. $x \in C$, $CUB := \infty$

- 1) choose subproblem
- 2) solve subproblem with solution \bar{x}
- 3) cutting loop: until
(solution binary/infeasible/iterations exceeded)
do (add valid cuts to the subproblem and solve again)
- 4) if(subproblem infeasible): go to 1)
- 5) if (solution binary): update upper bound CUB , go to 1)
- 6) if ($c^T \bar{x} < CUB$) : do branching
→ 2 subproblems with $x_j = 0$ resp. $x_j = 1$, go to 1)
... as long as there are unsolved subproblems



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Mixed Integer Second Order Cone Problem

$$\begin{aligned} \text{(MISOCP)} \quad & \min \quad c^T x \\ & \text{s.t.} \quad Ax = b \\ & \quad g_i(x) \leq 0, \quad (i = 1, \dots, \text{noc}), \\ & \quad x_J \in \mathbb{Z}^{|J|} \cap [l, u], \end{aligned}$$

where $g_i(x) := -(x_i)_1 + \|(x_i)_{2:k_i}\|_2$. Remember $x = \begin{pmatrix} x_1 \\ \vdots \\ x_{\text{noc}} \end{pmatrix}$, $x_i \in \mathbb{R}^{k_i}$, $\sum_{i=1}^{\text{noc}} k_i = n$.

Assumptions:

A1 The feasible set is bounded

A2 Slater CQ holds at every second order cone subproblem $\text{NLP}(x_j^k)$ or $\text{F}(x_j^k)$.



The function $g_i : \mathbb{R}^{k_i} \mapsto \mathbb{R}$ is

- ▶ **differentiable** on $\mathbb{R}^{k_i} \setminus \{x_i : \|(x_i)_{2:k_i}\|_2 = 0\}$, with $\nabla g_i(x_i) = \begin{pmatrix} -1 \\ \frac{(x_i)_{2:k_i}}{\|(x_i)_{2:k_i}\|} \end{pmatrix}$,
- ▶ **subdifferentiable** on $\{x \in \mathbb{R}^{k_i} : \|(x_i)_{2:k_i}\|_2 = 0\}$, with $\partial g_i(x_i) = \left\{ \begin{pmatrix} -1 \\ \eta \end{pmatrix}, \eta \in \mathbb{R}^{k_i-1}, \|\eta\|_2 \leq 1 \right\}$.

Notation in \mathbb{R}^n :

$$\partial g_i(x) = \{ \xi \in \mathbb{R}^n : (\xi)_i \in \partial g_i(x_i) \in \mathbb{R}^{k_i}, (\xi)_j = 0 \in \mathbb{R}^{k_j}, j \neq i \}$$

and

$$\nabla g_i(x) = \{ \xi \in \mathbb{R}^n : (\xi)_i = \nabla g_i(x_i) \in \mathbb{R}^{k_i}, (\xi)_j = 0 \in \mathbb{R}^{k_j}, j \neq i \}.$$



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Let x_j^k be an integer assignment.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ \text{NLP}(x_j^k) & g_i(x) \leq 0, \quad (i = 1, \dots, \text{noc}), \\ & x_j = x_j^k. \end{array}$$

Let x_j^k be an integer assignment.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ \text{NLP}(x_j^k) & g_i(x) \leq 0, \quad (i = 1, \dots, \text{noc}), \\ & x_j = x_j^k. \end{array}$$

Let \bar{x} solve $\text{NLP}(x_j^k)$. A2 \Rightarrow There exist $\mu, \lambda \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$:

$$\text{(KKT1)} \quad c + (A^T, I_J^T)\mu + \sum_{i: g_i(\bar{x})=0} \lambda_i \xi_i = 0,$$

where $I_J x = x_J$.

If $NLP(x_j^k)$ is infeasible, then we solve the **Feasibility Problem**

$$\begin{array}{ll} \min & u \\ \text{s. t.} & Ax = b \\ F(x_j^k) & g_i(x) \leq u, \quad (i = 1, \dots, \text{noc}), \\ & x_j = x_j^k, \\ & u \geq 0. \end{array}$$

If $NLP(x_j^k)$ is infeasible, then we solve the **Feasibility Problem**

$$\begin{aligned} \min \quad & u \\ \text{s.t.} \quad & Ax = b \\ F(x_j^k) \quad & g_i(x) \leq u, \quad (i = 1, \dots, \text{noc}), \\ & x_j = x_j^k, \\ & u \geq 0. \end{aligned}$$

Let (\bar{u}, \bar{x}) solve $F(x_j^k)$ with $\bar{u} > 0$. \Rightarrow There exist $\mu, \lambda \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$:

$$\begin{aligned} \text{(KKT2)} \quad & (A^T, I_J^T)\mu + \sum_{i: g_i(\bar{x}) = \bar{u}} \lambda_i \xi_i = 0, \\ & \sum_{i: g_i(\bar{x}) = \bar{u}} \lambda_i = 1. \end{aligned}$$

(Lagrange multipliers of $u \geq 0$ vanish due to $\bar{u} > 0$ and complementarity)



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Since $g_i : \mathbb{R}^n \mapsto \mathbb{R}$ is convex,

$$(SE) \quad g_i(\bar{x}) + \xi^T(x - \bar{x}) \leq g_i(x)$$

holds for all $\xi \in \partial g_i(\bar{x})$, $\bar{x} \in \mathbb{R}^n$.

Idea:

- ▶ Identify subgradients of active constraints in the solutions of the nonlinear problems $NLP(x_j^k)$ / $F(x_j^k)$, that satisfy KKT1 / KKT2.
- ▶ Use these subgradients to create linear outer approximations based on (SE).

Lemma

Assume **A1** and **A2**. Let $(\bar{x}, \bar{s}, \bar{y})$ be the primal-dual solution of $NLP(x_j^k)$. Then there exist Lagrange multipliers $\bar{\mu} = -\bar{y}$ and $\bar{\lambda}_i = (\bar{s}_i)_1 \geq 0$, for $i \in A_{NLP} := \{i : g_i(\bar{x}) = 0\}$, that solve the KKT conditions (KKT1) in \bar{x} with subgradients $\xi_i(\bar{x}) = (0, \dots, \bar{\xi}_i^T, \dots, 0)^T$, where

$$\begin{array}{ll} \text{SG(NLP)} & \begin{array}{l} \bar{\xi}_i = \nabla g_i(\bar{x}) \\ \bar{\xi}_i = -\frac{1}{(\bar{s}_i)_1} \bar{s}_i, \quad \text{if } (\bar{s}_i)_1 > 0 \\ \bar{\xi}_i = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \text{if } (\bar{s}_i)_1 = 0 \end{array} \end{array} \quad \begin{array}{l} \text{for } i \in A_{NLP} : \|(\bar{x}_i)_{2:k_i}\| > 0, \\ \text{for } i \in A_{NLP} : \|(\bar{x}_i)_{2:k_i}\| = 0. \end{array}$$

Lemma

Assume **A1, A2**. Let (\bar{x}, \bar{u}) solve $F(x_j^k)$ with $\bar{u} > 0$ and let (\bar{s}, \bar{y}) be the solution of its dual program. Then there exist Lagrange multipliers $\bar{\mu} = -\bar{y}$ and $\bar{\lambda}_i = (\bar{s}_i)_1 \geq 0$ for $i \in A_F := \{i : g_i(\bar{x}) = \bar{u}\}$, that solve the KKT conditions (KKT2) in (\bar{x}, \bar{u}) with subgradients $\xi_i(\bar{x}) = (0, \dots, \bar{\xi}_i^T, \dots, 0)^T$, where

$$\begin{array}{ll} \text{SG}(F) & \begin{array}{ll} \bar{\xi}_i = \nabla g_i(\bar{x}_i) & \text{for } i \in A_F : \|(\bar{x}_i)_{2:k_i}\| > 0, \\ \bar{\xi}_i = -\frac{1}{(\bar{s}_i)_1} \bar{s}_i, & \text{if } (\bar{s}_i)_1 > 0 \\ \bar{\xi}_i = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & \text{if } (\bar{s}_i)_1 = 0 \end{array} \end{array} \quad \text{for } i \in A_F : \|(\bar{x}_i)_{2:k_i}\| = 0.$$

T : set contains solutions of problems $NLP(x_j^k)$,

S : set contains solutions of problems $F(x_j^k)$.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \end{array}$$

OA(T,S)

$$\xi_i(\bar{x})^T (x - \bar{x}) \leq 0,$$

$$\xi_i(\bar{x})^T (x - \bar{x}) \leq 0,$$

$$c^T x < c^T \bar{x},$$

$$x_j \in \mathbb{Z}^{|J|} \cap [l, u].$$

$$\forall \xi_i(\bar{x}) \text{ from } SG(NLP), \forall \bar{x} \in T,$$

$$\forall \xi_i(\bar{x}) \text{ from } SG(F), \forall \bar{x} \in S,$$

$$\forall \bar{x} \in T, \bar{x}_j \in \mathbb{Z}^{|J|},$$

with continuous relaxation $OA^k(T,S)$.



$CUB := \infty, Nodes := \{N^0 = (lb, ub) = (l, u)\},$

solve $NLP([l, u])$: if $\bar{x}_j \in \mathbb{Z}^{|J|}$: STOP,
else $S = \emptyset, T = \{\bar{x}\}.$

while $Nodes \neq \emptyset$: Select node $N^k, Nodes := Nodes \setminus N^k,$

1. Solve LP $OA^k(T, S)$: solution $x^k.$

while $(x_j^k \text{ integer}) \ \& \ (OA^k(T, S) \text{ feasible})$

if $(NLP(x_j^k) \text{ feasible}) \ T := T \cup \{\bar{x}\}, \ CUB = \min\{CUB, c^T \bar{x}\}$

else compute solution \bar{x} of $F(x_j^k), \ S := S \cup \{\bar{x}\}$

compute solution x^k of updated $OA^k(T, S)$

2. if $(OA^k(T, S) \text{ feasible})$ branch on $x_j^k \notin \mathbb{Z}$:

Create 2 nodes with bounds $ub_j = \lfloor x_j^k \rfloor \ / \ lb_j = \lceil x_j^k \rceil.$

Theorem

Assume A1 and A2. Then the outer approximation algorithm terminates in a finite number of steps at an optimal solution of (MISOC) or with the indication, that it is infeasible.

Proof:

- ▶ A1, A2 \Rightarrow KKT and primal-dual optimality in solutions of $\text{NLP}(x_j^k)$ and $F(x_j^k)$ exist!
- ▶ $F(x_j^k)$: $\bar{u} > 0$, subgradients satisfy (KKT2)
 $\Rightarrow x_j = x_j^k$ is infeasible in $\xi_i(\bar{x})^T(x - \bar{x}) \leq 0, \forall \xi_i(\bar{x})$ from $\text{SG}(F)$.
- ▶ $\text{NLP}(x_j^k)$: subgradients satisfy (KKT1) in solution \bar{x} of $\text{NLP}(x_j^k)$
 $\Rightarrow x_j = x_j^k$ is infeasible in $\xi(\bar{x})^T(x - \bar{x}) \leq 0, \forall \xi_i(\bar{x})$ from $\text{SG}(\text{NLP})$ and condition $c^T x < c^T \bar{x}$. (Farkas' Lemma)

\Rightarrow No integer assignment is generated twice. \square



SOCP Solver:[Ulbrich, Hess, Drewes] (C-implementation)

- ▶ Extension of a primal-dual interior point approach (Tsuchiya, 98) to guarantee convergence for *infeasible* interior starting points
→ easy choice of starting points
- ▶ Feasibility test, that uses an always solvable auxiliary problem
→ reliable information about the existence of feasible and interior points
- ▶ Penalty approach for problems without interior points
- ▶ Semidefinite cut problems are solved with sedumi.

B&C [Hess, Drewes]/ OA -Framework [Drewes]: (C++-implementation)

- ▶ **Branching:** first / most fractional / pseudocost
- ▶ **Node selection:** depth first search / best bound first

LP/MIP-Solver: CPLEX 10.01

OA(Hybrid version): solve NLP-relaxation every 10 nodes

- ▶ Different problems with: $n = 8 \dots 25$, $|J| = 1 \dots 8$,
- ▶ Each tested with all combinations of **node selections** and **branching rules**
- ▶ B&C: Tightness of the dual cutting problems $|B| = 2$ and 1 cut per node

	B&B	B&C (Linear)	B&C (SOC Quad)	B&C (SDP Quad)	B&C (Subgrad)	OA-B&B	OA-B&C (Lin)	OA-B&C (Quad)	
NLP-Nodes		628	507	415	448	536	137	127	138
LP-Nodes	-	-	-	-	-	-	445	421	424
Nodes		628	507	415	448	536	582	548	562
<i>Reduction Nodes to</i>		80.73%	66.08%	71.34%	85.35%	92.68%	87.26%	89.49%	
<i>Best Reduction to</i>		20.00%	19.40%	12.00%	44.00%	-	64.16%	55.61%	
<i>Reduction NLP Nodes to</i>		80.73%	66.08%	71.34%	85.35%	21.82%	20.22%	21.97%	

TestDrz07GF n=84,p=12,m=75
#combinations (ns/br): 6

	B&B	B&C (SOC Linear)	B&C (SOC Quad)	B&C (SDP Quad)	OA-B&B
NLP-Nodes	448	340	340	448	26
LP-Nodes	-	0	0	0	275
Nodes	448	340	340	448	301
Reduction Nodes to	-	75.89%	75.89%	100.00%	67.19%
Best Reduction to	-	55.73%	59.29%	100.00%	35.88%
Reduction NLP Nodes to	-	75.89%	75.89%	100.00%	5.80%

TestDrz54GF N=369,p=12,m=357
#combinations (ns/br): 2

	B&B	B&C (SOC Linear)	B&C (SOC Quad)	B&C (SDP Quad)	OA-B&B
NLP-Nodes	264	264	262	-	20
LP-Nodes	-	0	0	-	60
Nodes	264	264	262	-	80
Reduction Nodes to	-	100.00%	99.24%	-	30.30%
Best Reduction to	-	100.00%	95.56%	-	16.44%
Reduction NLP Nodes to	-	100.00%	99.24%	-	7.58%

TestDrz07_lowb n=212,p=56,m=145
#combinations (ns/br): 1

	B&B	B&C (SOC Linear)	B&C (SOC Quad)	B&C (SDP Quad)	OA-B&B
NLP-Nodes	964	962	956	964	490
LP-Nodes	-	0	0	0	5412
Nodes	964	962	956	964	5902
Reduction Nodes to	-	99.79%	99.17%	100.00%	612.24%
Reduction NLP Nodes to	-	99.79%	99.17%	100.00%	50.83%



- ▶ Cuts reduce the number of B&B nodes to 85-90% in average (down to 12% in the best case).
- ▶ But: Solving cut problems is *very expensive* → runtime is increased by cuts!
- ▶ OA based B& B reduces the number of NLP problems to 17.5% in average (down to 6% in best cases) → best running time



- ▶ Consider different problem types
- ▶ Cut application in OA based B& B
- ▶ Simplify cut problems
- ▶ Apply conic Gomory cuts
- ▶ ...



Thank you for your attention!