

Graph Realizations Corresponding to Optimized Extremal Eigenvalues of the Laplacian

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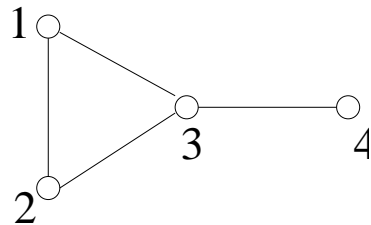
joint work with
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TU Chemnitz

- Introduction: Laplacian and Algebraic Connectivity
- Eigenvalue Optimization and Embedding Problems
- Separators and Optimal Embeddings
- Tree-Width Bound on Minimal Dimensions
- Sharpness of the Bounds
- Some Final Remarks

Introduction

- simple undirected Graph $G = (N, E)$, $N = \{1, \dots, n\}$, $E \subseteq \{ij : i, j \in N, i \neq j\}$
- Laplacian $L(G) = \text{Diag}(Ae) - A$ A ... adjacency matrix, $e = [1, 1, \dots, 1]^T$



$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- weighted Laplacian for $w \geq 0$: $L_w(G) = \sum_{ij \in E} w_{ij} E_{ij}$ $E_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$
- Properties: $L_w \succeq 0$ (sym., pos. semidef.), $\lambda_1(L_w) = 0$ with EV e .

- [\[Fiedler 1973, 1989, 1993\]](#) $\lambda_2(L(G)) > 0$ iff G is connected
proved close ties to edge and node connectivity, “algebraic connectivity”
- EV to $\lambda_2(L)$ used in many partitioning heuristics, “Fiedler vector”
- Laplacian Spectrum in Graph Theory, see e.g. [\[Mohar 1991, 2004\]](#)

Central Question:

What connections exist between eigenvectors of extremal eigenvalues and structural properties of the graph?

Idea: redistribute the weight on the edges of the graph, then hopefully characteristic properties will become more apparent.

[Fiedler 1989] did this for λ_2

$$\max_{w \in \mathcal{W}} \lambda_2(L_w) \quad \mathcal{W} = \{w \in \mathbb{R}_+^E : \sum_{ij \in E} w_{ij} = w^T e \leq 1\}$$

→ *Absolute Algebraic Connectivity*

[Fiedler 1990] exhibits formula for optimal weights in trees.

From Eigenvalues to Embeddings in the Eigenspace

$$\max_{w \in \mathcal{W}} \lambda_2(L_w)$$

$$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$$

as SDP:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq \lambda I \\ & e^T w = 1, w \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \sum_{ij \in E} w_{ij} E_{ij} \preceq \lambda I \\ & e^T w = 1, w \geq 0 \end{aligned}$$

divide by $\lambda_{opt} > 0$

$$\begin{aligned} \min \quad & e^T w \\ \text{s.t.} \quad & \sum_{ij \in E} w_{ij} E_{ij} + \mu e e^T \succeq I \\ & w \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & e^T w \\ \text{s.t.} \quad & \sum_{ij \in E} w_{ij} E_{ij} \preceq I \\ & w \geq 0 \end{aligned}$$

dualize

$$\begin{aligned} \max \quad & \langle I, X \rangle \\ \text{s.t.} \quad & \langle E_{ij}, X \rangle \leq 1, \quad ij \in E \\ & \langle e e^T, X \rangle = 0 \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \langle I, X \rangle \\ \text{s.t.} \quad & \langle E_{ij}, X \rangle \geq 1, \quad ij \in E \\ & X \succeq 0 \end{aligned}$$

embedding: set $X = V^T V$ with $V = [v_1, \dots, v_n]$

$$\begin{aligned} \max \quad & \sum \|v_i\|^2 \\ \text{s.t.} \quad & \|v_i - v_j\|^2 \leq 1, \quad ij \in E \\ & \sum v_i = 0 \\ & v_i \in \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} \min \quad & \sum \|v_i\|^2 \\ \text{s.t.} \quad & \|v_i - v_j\|^2 \geq 1, \quad ij \in E \\ & v_i \in \mathbb{R}^n \end{aligned}$$

($\sum v_i = 0$ holds, as well)

The embedding problem

$\max_{w \in \mathcal{W}} \lambda_2(L_w)$	$\min_{w \in \mathcal{W}} \lambda_{\max}(L_w)$
$\begin{aligned} \max \quad & \sum \ v_i\ ^2 \\ \text{s.t.} \quad & \ v_i - v_j\ ^2 \leq 1, \quad ij \in E \\ & \sum v_i = 0 \\ & v_i \in \mathbb{R}^n \end{aligned}$	$\begin{aligned} \min \quad & \sum \ v_i\ ^2 \\ \text{s.t.} \quad & \ v_i - v_j\ ^2 \geq 1, \quad ij \in E \\ & (\sum v_i = 0) \\ & v_i \in \mathbb{R}^n \end{aligned}$

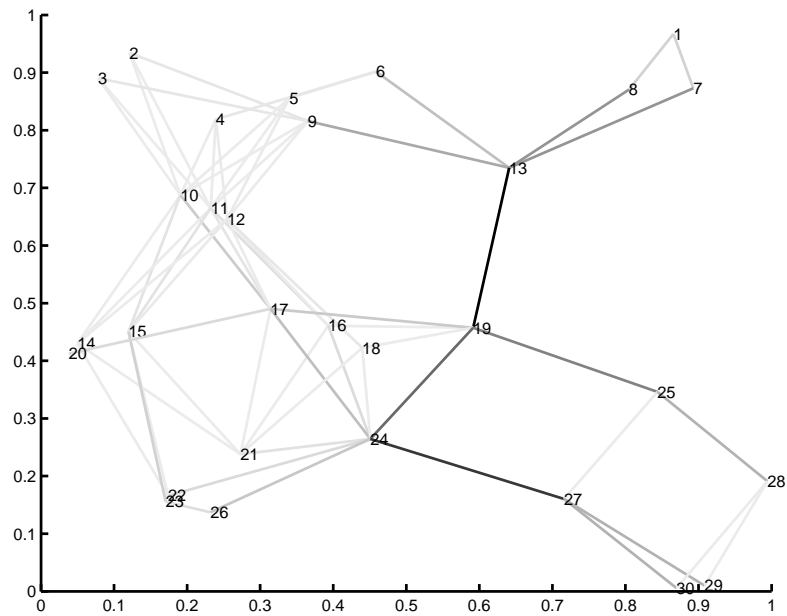
- For any $u \in \mathbb{R}^n$ the projection $V^T u$ yields an eigenvector to λ_{opt} . (by complementarity)
- λ_2 : nodes are pulled outwards and edges are cables of length 1
 λ_{\max} : nodes are pressed inwards and edges are struts of length 1
 w_{ij} is the force acting on the cable/strut $ij \in E$
- Replace the origin by the barycenter $\bar{v} = \frac{1}{n} \sum v_i$

$\begin{aligned} \max \quad & \sum \ v_i - \bar{v}\ ^2 \\ \text{s.t.} \quad & \ v_i - v_j\ ^2 \leq 1, \quad ij \in E \\ & \sum (v_i - \bar{v}) = 0 \\ & v_i \in \mathbb{R}^n \end{aligned}$	$\begin{aligned} \min \quad & \sum \ v_i - \bar{v}\ ^2 \\ \text{s.t.} \quad & \ v_i - v_j\ ^2 \geq 1, \quad ij \in E \\ & \sum (v_i - \bar{v}) = 0 \\ & v_i \in \mathbb{R}^n \end{aligned}$
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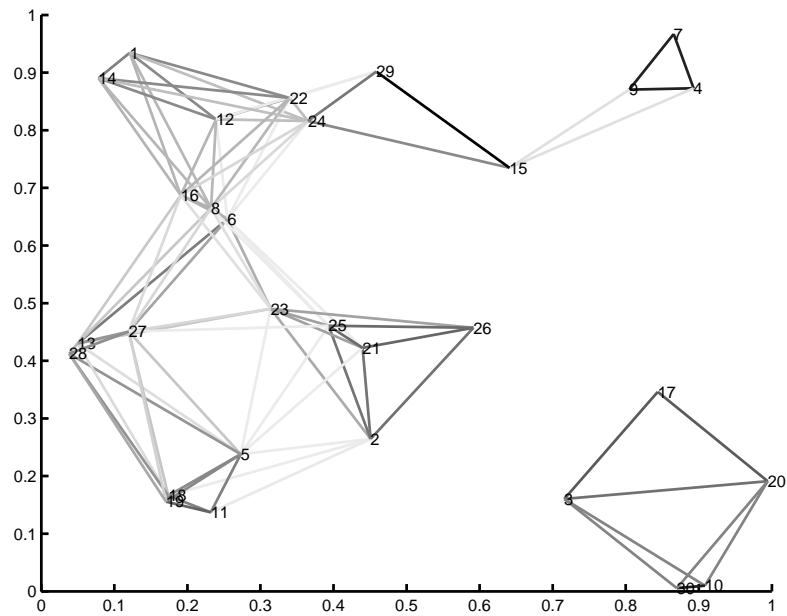
then $\sum \|v_i - \bar{v}\|^2 = \frac{1}{n} \sum_{i,j} \|v_i - v_j\|^2$ is the *variance*.

Equivalent/Related Problems

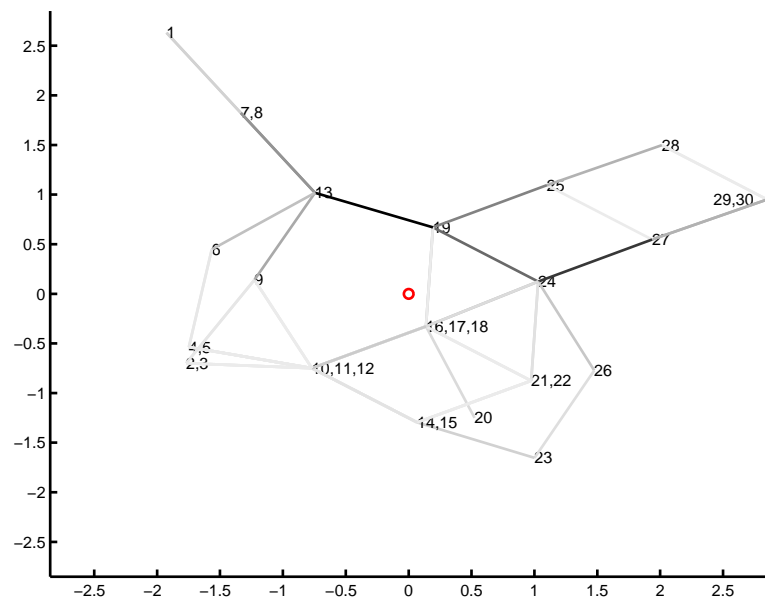
- λ_2 : Fastest Mixing Markov Processes [\[SunBoydXiaoDiaconis2006\]](#)
 λ_2 is the rate of convergence to the stationary distribution
also give an interpretation as maximizing conductance
- Maximum Variance Unfolding (visualization in data mining) [\[WeinbergerSaul2004\]](#)
- Tensegrity Theory [\[Connelly1999\]](#)
the variance corresponds to the potential and L_w to the stress matrix
- Graph Realizations [\[BelkConnelly2007\]](#)
- low dimensional embeddings [\[LinialLondonRabinovich1995\]](#)
- Expanders [\[HooryLinialWigderson2006\]](#)



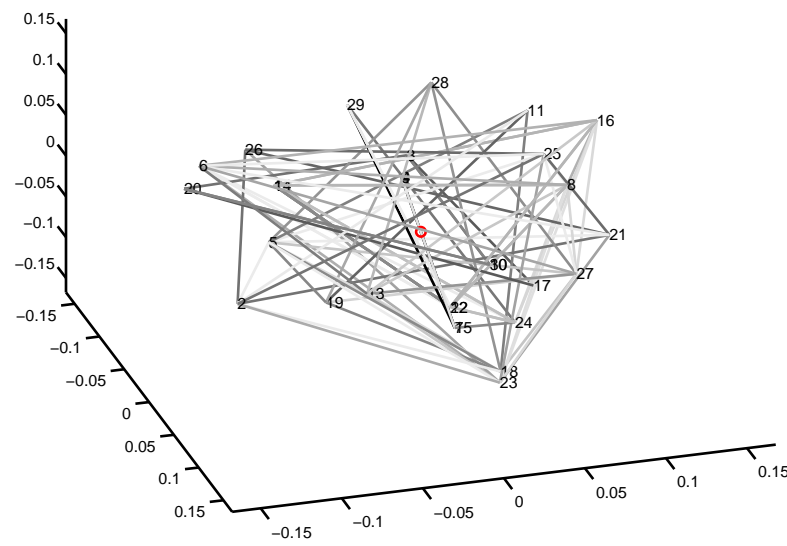
optimal weights for λ_2



optimal weights for λ_{\max}



an optimal λ_2 -embedding (2D!)



an optimal λ_{\max} -embedding (14D?)

Exhibits connections to the (node-)separator structure (at least for λ_2)

For a connected graph $G = (N, E)$ the removal of a (node-)separator $S \subset N$ disconnects the graph into at least two connected components.

Tree-Width

Given $G = (N, E)$, let $T = (\mathcal{N}, \mathcal{E})$ be a tree with $\mathcal{N} \subseteq 2^N$ and $\mathcal{E} \subseteq \binom{\mathcal{N}}{2}$ so that

(i) $N = \bigcup_{U \in \mathcal{N}} U$.

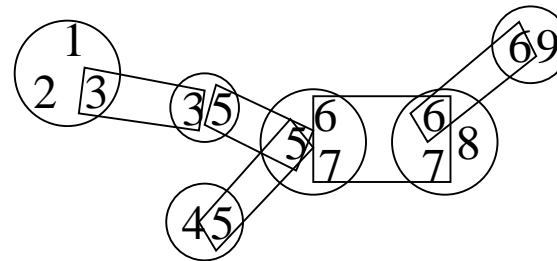
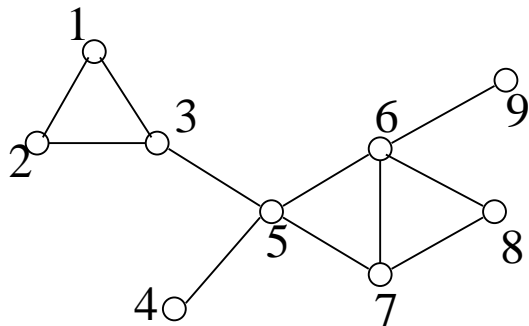
(ii) For every $e \in E$ there is a $U \in \mathcal{N}$ with $e \subseteq U$.

(iii) If $U_1, U_2, U_3 \in \mathcal{N}$ with U_2 on the T -path from U_1 to U_3 , then $U_1 \cap U_3 \subseteq U_2$.

Then T is called a *tree-decomposition* of G .

The *width* of T is the number $\max\{|U| - 1 : U \in \mathcal{N}\}$.

The *tree-width* $tw(G)$ is the least width of any tree-decomposition of G .



Any $U \in \mathcal{N}$ and any $U \cap U'$ with $\{U, U'\} \in \mathcal{E}$ is a separator of G .

Main Result 1: Separators and Optimality of Embeddings

Given optimal $v_i \in \mathbb{R}^n$, $i \in N$, of a connected graph $G = (N, E)$ and a separator $S \subset N$ separating G into node sets C_1, C_2 so that no edges run between C_1 and C_2 , let $\mathcal{S} = \{v_i : i \in S\}$.

$$\lambda_2(L_w)$$

$$\lambda_{\max}(L_w)$$

Separator-Shadow Th.

For at least one $j \in \{1, 2\}$

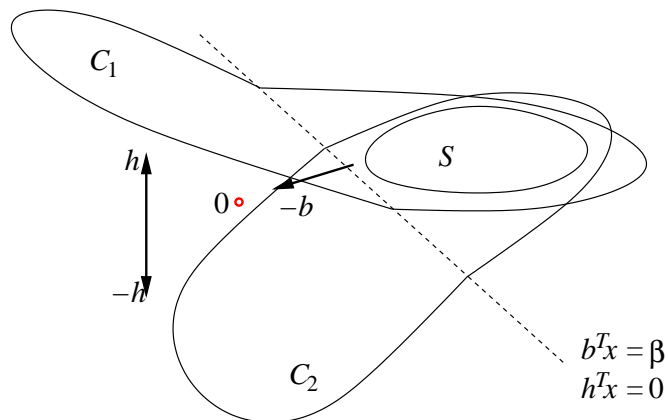
$$[v_i, 0] \cap \text{conv } \mathcal{S} \neq \emptyset \quad \forall i \in C_j$$

Separator's Sunny Side Th.

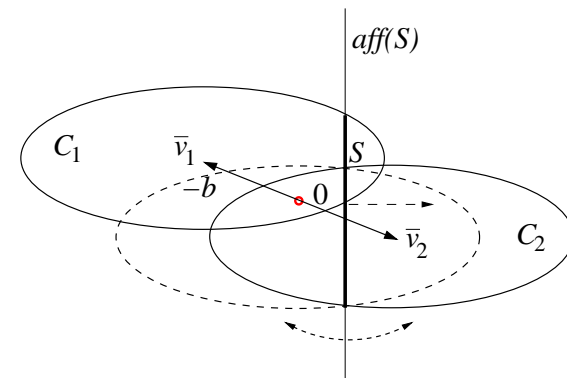
Let $\bar{v}_j = \frac{1}{|C_j|} \sum_{i \in C_j} v_i$, $j \in \{1, 2\}$, be the barycenter of C_j , then

$$\bar{v}_j \in \text{aff}(\mathcal{S}) - \text{cone}(\mathcal{S}) \text{ for } j \in \{1, 2\}$$

geometric proof idea:



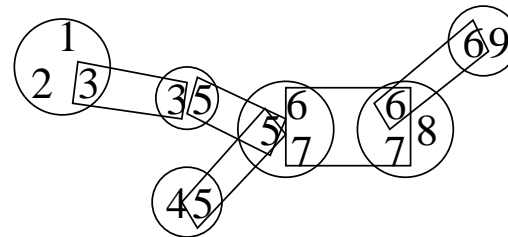
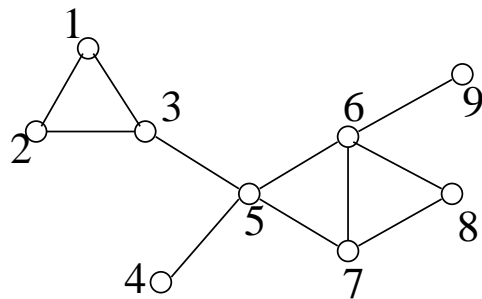
geometric proof idea:



Main Result 2:

Structural bounds on the existence of low dimensional solutions

$\lambda_2(L_w)$	$\lambda_{\max}(L_w)$
<p>Tree-Width Bound</p> <p>There exists an optimal embedding of dimension at most</p> $tw(G) + 1$	<p>Tree-Width Bound</p> <p>There exists an optimal embedding of dimension at most</p> $tw(G) + 1$
<p>needs separator shadow + involved result for separators with $0 \in \text{conv } S$</p> <p>algorithmic proof idea: given a tree decomposition, start at a 0-node, try to flatten all adjacent nodes or move on to the next 0-node</p>	<p>Obs.: in separated sets no forces interact outside separator space.</p> <p>algorithmic proof idea: given a tree decomposition, find a node S with maximal $\dim(\text{lin}(S))$. For adjacent nodes U, rotate basis of U outside $\text{lin}(S \cap U)$ into $\text{lin}(S)$, continue recursively</p>

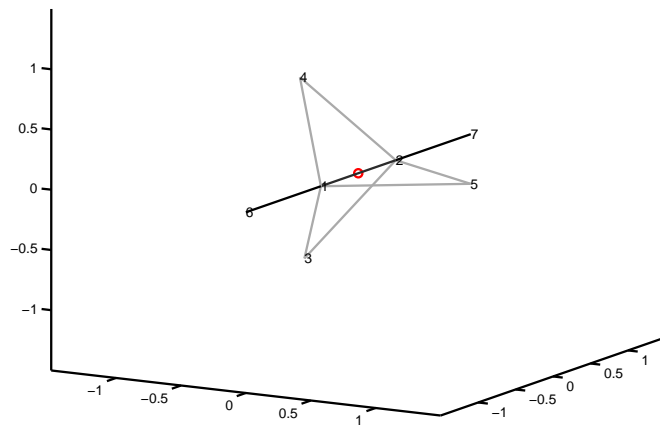
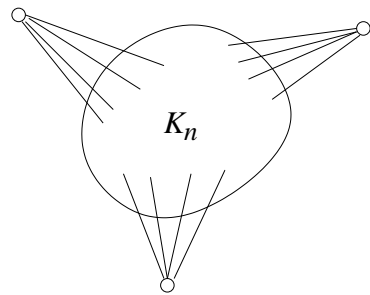


Result 3: Sharpness of Tree-Width Dimension Bounds

$$\lambda_2(L_w)$$

For $n \geq 4$, connect three vertices completely to K_n

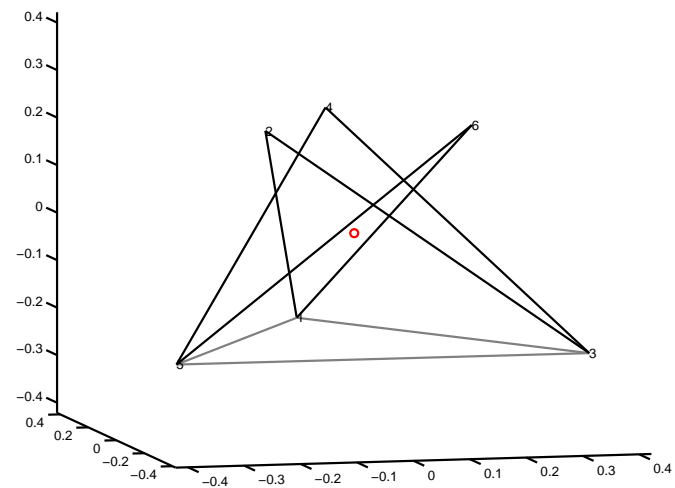
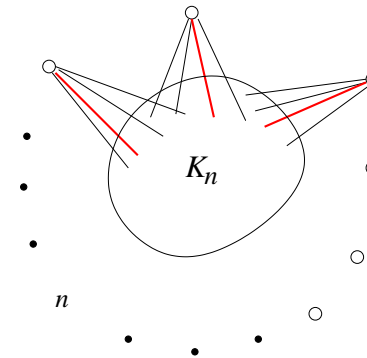
$$tw(G) = n, \quad \dim = n + 1$$



$$\lambda_{\max}(L_w)$$

Connect n vertices completely to K_n , delete a perfect matching

$$tw(G) = n, \quad \dim = n + 1$$



Remarks

- Results show a clear connection between separator structure and structural properties of the extremal eigenvectors
- For many classes of graphs the tree-width bounds are far from optimal:
Theorem for λ_{\max} :
Any bipartite graph has a 1-dim optimal embedding
- Conjecture for λ_2 : planar graphs have 3-dim optimal embeddings

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- Theorem for λ_{\max} : Any optimal embedding is inside the ball of radius 1

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- extensions to minor monotone graph parameters . . .