Reformulations in Mathematical Programming: Symmetry

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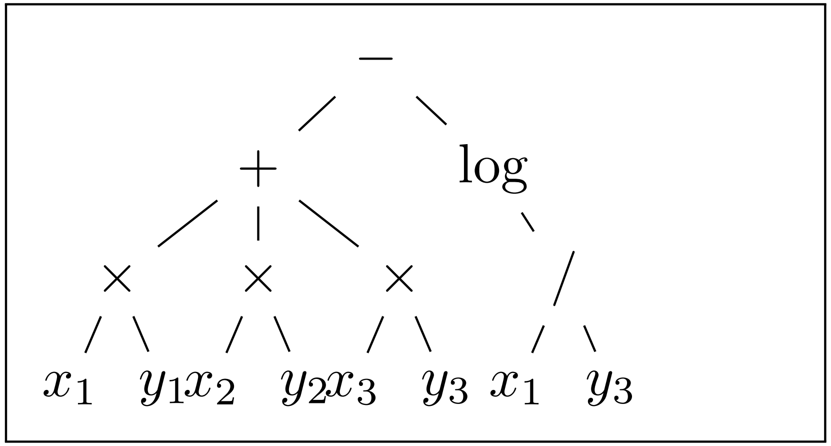
Framework: reformulations

- Given an optimization problem, many different *Mathematical Programming (MP) formulations* can describe its solution set
- The performances of solution algorithms depend on the MP formulation
- **Given an optimization problem and a solution algorithm, what is the MP formulation yielding the best performance?**
- *How do we pass from one formulation to another that keeps some (all) of the mathematical properties of the old formulation?*

Storing MP formulations

- Mathematical expressions as n-ary *expression trees*

$$\sum_{i=1}^{3} x_i y_i - \log(x_1/y_3)$$



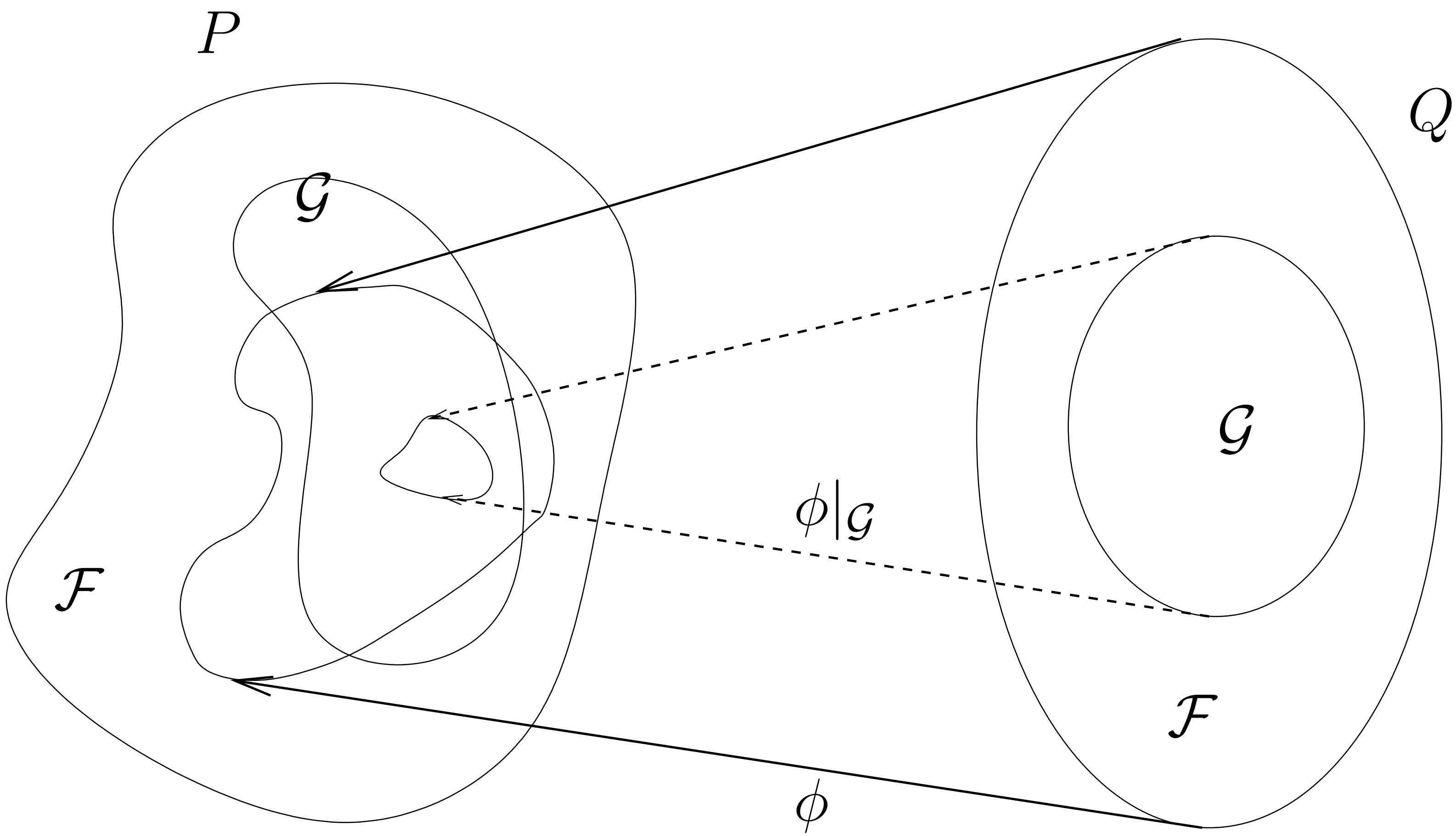
- A *formulation* P is a 7-tuple $(\mathcal{P}, \mathcal{V}, \mathcal{E}, \mathcal{O}, \mathcal{C}, \mathcal{B}, \mathcal{T})$ =(parameters, variables, expression trees, objective functions, constraints, bounds on variables, variable types)
- Objectives are encoded as pairs (d, f) where $d \in \{-1, 1\}$ is the optimization direction and f is the function being optimized
- Constraints are encoded as triplets $c \equiv (e, s, b)$ ($e \in \mathcal{E}$, $s \in \{\leq, \geq, =\}$, $b \in \mathbb{R}$)
- $\mathcal{F}(P)$ = feasible set, $\mathcal{L}(P)$ = local optima, $\mathcal{G}(P)$ = global optima

Auxiliary problems

If problems P, Q are related by a computable function f through the relation $f(P, Q) = 0$, Q is an *auxiliary problem* with respect to P.

- **Opt-reformulations** (or *exact reformulations*): preserve all optimality properties
- **Narrowings**: preserve some optimality properties
- **Relaxations**: provide bounds to an objective function value towards its optimization direction
- **Approximations**: formulation Q depending on a parameter k such that "$\lim_{k\to\infty} Q(k)$" is an opt-reformulation, narrowing or relaxation

Narrowings



Main idea: if we find a global optimum of Q, we can map it back to a global optimum of P. There may be optima of P without a corresponding optimum in Q.

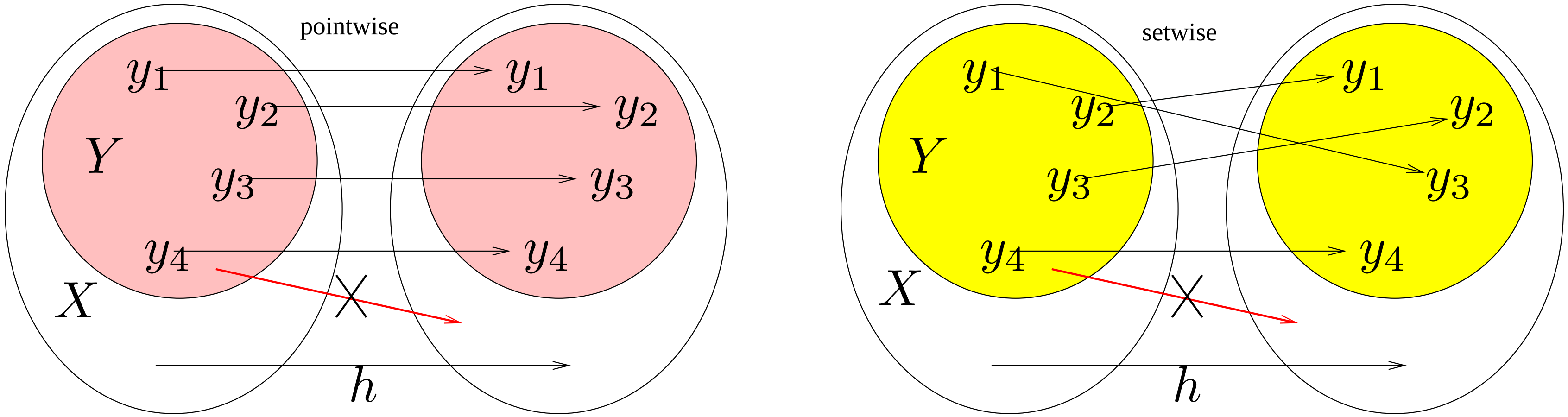
L., *Reformulations in Mathematical Programming: Definitions and Systematics*, RAIRO-RO (accepted)

The setting

- Most common solution algorithm for finding global optima: **Branch-and-Bound** (BB for MILPs, sBB for MINLPs)
- BB (implicit enumeration): provides a certificate of optimality in the linear case, and of ε-approximation in the nonlinear case
- If the problem has symmetries: many BB nodes will contain (symmetric) optimal solutions $\Rightarrow$ *pruning will occur rarely* $\Rightarrow$ BB converges slowly
- Need a reformulation which is guaranteed to *keep at least one global optimum* (but hopefully excludes a lot of symmetric optima): a **narrowing**

Minimal group-theoretical toolbox

- Given a group G acting on a set X and $x \in X$, $Gx = \{gx \mid g \in G\}$ is the *orbit* of x w.r.t. G
- Given $Y \subseteq X$, the *point-wise stabilizer* of Y w.r.t. G is a subgroup $H \leq G$ such that $hy = y$ for all $h \in H, y \in Y$



- The *set-wise stabilizer* of Y w.r.t. G is a subgroup $H \leq G$ such that $HY = Y$ (denote H by $\text{stab}(Y, G)$)
- Let $\pi \in S_n$ with disjoint cycle product $\sigma_1 \cdots \sigma_k$ and $N \subseteq \{1 \ldots, n\}$
- Denote $\pi[N] = \prod_{\substack{j \leq k \\ \sigma_j \in \text{stab}(N, S_n)}} \sigma_j$

Motivating example

- Consider an instance P:

$$
\begin{array}{rrrrrrrcl}
\min & x_{11} & +x_{12} & +x_{13} & +x_{21} & +x_{22} & +x_{23} & & \\
 & x_{11} & +x_{12} & +x_{13} & & & & \geq & 1 \\
 & & & & x_{21} & +x_{22} & +x_{23} & \geq & 1 \\
 & x_{11} & & & +x_{21} & & & \geq & 1 \\
 & & x_{12} & & & +x_{22} & & \geq & 1 \\
 & & & x_{13} & & & +x_{23} & \geq & 1
\end{array}
$$

  of the covering prob. $\boxed{\min \mathbf{1}x : \forall i \sum_j x_{ij} \geq 1 \wedge \forall j \sum_i x_{ij} \geq 1}$

- The set of solutions is $\mathcal{G}(P) =$

$\{(0,1,1,1,0,0),(1,0,0,0,1,1),(0,0,1,1,1,0),(1,1,0,0,0,1),(1,0,1,0,1,0),(0,1,0,1,0,1)\}$

- $G^* = \text{stab}(\mathcal{G}(P), S_n)$ is the *solution group* **(var. permutations keeping $\mathcal{G}(P)$ fixed setwise)**

Symmetries

- For the above instance, G^* is $\langle (2,3)(5,6),(1,2)(4,5),(1,4)(2,5)(3,6)\rangle \cong D_{12}$

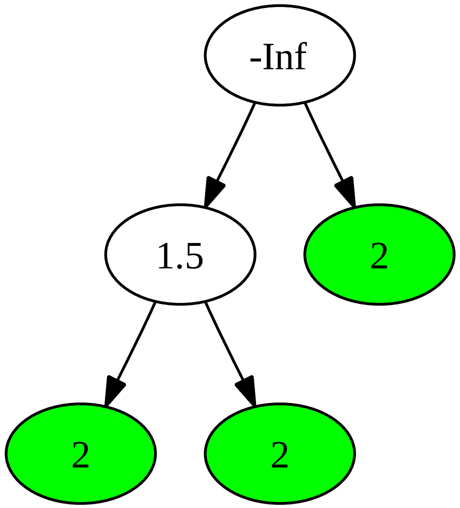
```
gap> S := [[0,1,1,1,0,0],[0,1,0,1,0,1],[0,0,1,1,1,0],
           [1,1,0,0,0,1],[1,0,0,0,1,1],[1,0,1,0,1,0]];
     G:=MatrixAutomorphisms(S); StructureDescription(G);
Group([ (2,3)(5,6), (1,2)(4,5), (1,4)(2,5)(3,6) ]); "D12"
```

- For all $x^* \in \mathcal{G}(P)$, $G^* x^* = \mathcal{G}(P) \Rightarrow \exists$ only 1 orbit $\exists$ only *one* solution in $\mathcal{G}(P)$ (modulo symmetries)

```
gap> Orbit(G,S[1],Permuted);
[[0,1,1,1,0,0],[1,1,0,0,0,1],[1,0,1,0,1,0],
 [1,0,0,0,1,1],[0,0,1,1,1,0],[0,1,0,1,0,1]]
gap> Orbit(G,S[2],Permuted);
[[0,1,0,1,0,1],[1,0,0,0,1,1],[0,0,1,1,1,0],
 [1,0,1,0,1,0],[0,1,1,1,0,0],[1,1,0,0,0,1]]
gap> Orbit(G,S[3],Permuted);
[[0,0,1,1,1,0],[0,1,0,1,0,1],[1,0,0,0,1,1],
 [1,1,0,0,0,1],[1,0,1,0,1,0],[0,1,1,1,0,0]]
...
```

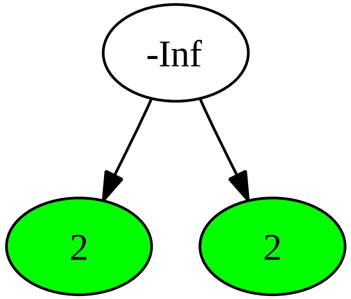
Symmetries

- This is **bad** for Branch-and-Bound techniques: many branches will contain (symmetric) optimal solutions and therefore will not be pruned by bounding $\Rightarrow$ *deep and large BB trees*



$\leftarrow$ BB tree for symmetric 6$\times$9 covering inst. with $G^* = C_2$

BB tree using symmetry breaking constraints $\longrightarrow$



- If we knew G^* in advance, we might add constraints eliminating (some) symmetric solutions out of $\mathcal{G}(P)$
- Can we find G^* (or a subgroup thereof) *a priori*?
- What constraints provide a valid narrowing of P excluding symmetric solutions of $\mathcal{G}(P)$?

Symmetries and formulation

- The cost vector $c^{\mathsf{T}} = (1, 1, 1, 1, 1, 1)$ is fixed by all (column) permutations in S_6
- The vector $b = (1, 1, 1, 1, 1)$ is fixed by all (row) permutations in S_5
- Consider P's constraint matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

- Let $\pi \in S_6$ be a column permutation such that $\exists$ a row permutation $\sigma \in S_5$ with $\sigma(A\pi) = A$
- Then permuting the variables/columns in P according to π does not change the problem formulation

The formulation group

- For a MILP with $c = \mathbf{1}_n$ and $b = \mathbf{1}_m$,

$$G_P = \{\pi \in S_n \mid \exists \sigma \in S_m \ (\sigma A \pi = A)\} \quad (1)$$

  is called the *formulation group* of P

- In the example above, we get $G_P \cong D_{12} \cong G^*$

```
gap> A := [[1,1,1,0,0,0],[0,0,0,1,1,1],
           [1,0,0,1,0,0],[0,1,0,0,1,0],[0,0,1,0,0,1]];
     G:=MatrixAutomorphisms(A); StructureDescription(G);
Group([ (1,4)(2,5)(3,6), (2,3)(5,6), (1,2)(4,5) ]); "D12"
```

  Thm.

  For a covering/packing problem P, $G_P \leq G^*$.

- Result can be extended to all MILPs [Margot: 2002, 2003 (Math. Prog.); 2007 (DO)]

Related results in MILP

- **Isomorphism pruning** [*Margot 02,03*], involves addition of linear inequalities of packing type *locally* to selected nodes of the BB tree (as well as var. fixing)
- **Orbitopes** [*Kaibel et al. 07,08*]: "polytopes modulo symmetries" for C_n and S_n groups only
- **Fundamental domains** [*Friedman 07*]: given a (discrete) domain X and a group G acting on X, a fundamental domain is a subset F of X such that $GF = X$ (determination of smallest FDs w.r.t. given ordering vectors c)
- **Orbital branching** [*Ostrowski et al. 07,08*] branching scheme taking advantage of the problem group (yields fewer branching disjunctions)

Related results in CP

- Much more work in CP than in MILP
- **Definitions**: Cohen et al., *Symmetry Definitions for Constraint Satisfaction Problems*, CP 2005. Relations between constraint and solution groups

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Survey

F. Margot, *Symmetry in Integer Linear Programming*, to appear in “50 Years of Integer Programming”, Springer.

In general, current literature on symmetry in mathematical programming concentrates on MILPs/SDPs and assumes the formulation group is given *a priori*

My contributions

1. **MILPs** (COCOA08 paper):
   - A MILP-based method for finding subgroups of the problem group
   - Some static symmetry-breaking constraints (narrowing reformulation)
2. **MINLPs** (new material):
   - Definition of the problem group
   - Reduction to GRAPH ISOMORPHISM
   - Orbit-based static symmetry-breaking constraints (narrowing reformulation)

Symmetries in MINLPs

- **Consider the following MINLP P:**

$$\left.\begin{array}{rrcl} \min & f(x) & & \\ & g(x) & \leq & 0 \\ & x & \in & X. \end{array}\right\} \qquad (2)$$

where X may contain integrality constraints on x

- For a row permutation $\sigma \in S_m$ and a column permutation $\pi \in S_n$, we define $\sigma P \pi$ as follows:

$$\left.\begin{array}{rrcl} \min & f(x\pi) & & \\ & \sigma g(x\pi) & \leq & 0 \\ & x\pi & \in & X. \end{array}\right\} \qquad (3)$$

- Define $G_P = \{\pi \in S_n \mid \exists \sigma \in S_m \; (\sigma P \pi = P)\}$

Representing $g(x\pi)$

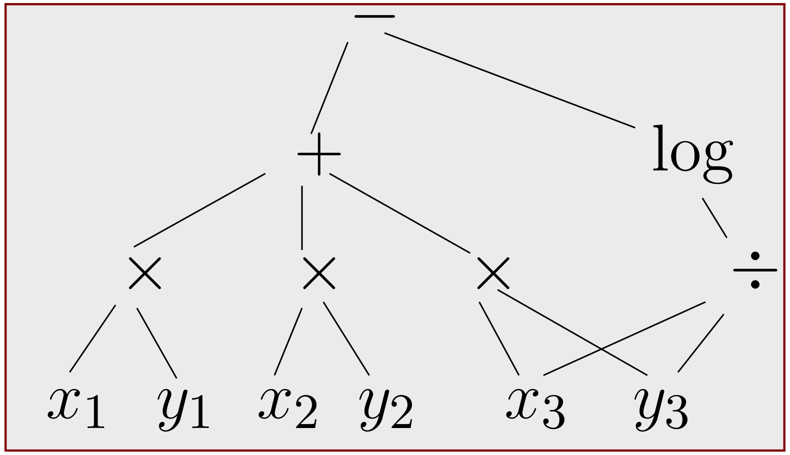
- In the linear case, writing $Ax\pi$ is easy — how do we deal with $g(x\pi)$?

  How do we decide whether $g_i(x) = g_h(x\pi)$ for $i, h \le m$?

- **Answer**: consider the *expression DAG* representation of g

  $$\sum_{i=1}^{3} x_i y_i - \log(x_3/y_3)$$

  List of expressions $\equiv$ expression DAG sharing variable leaf nodes

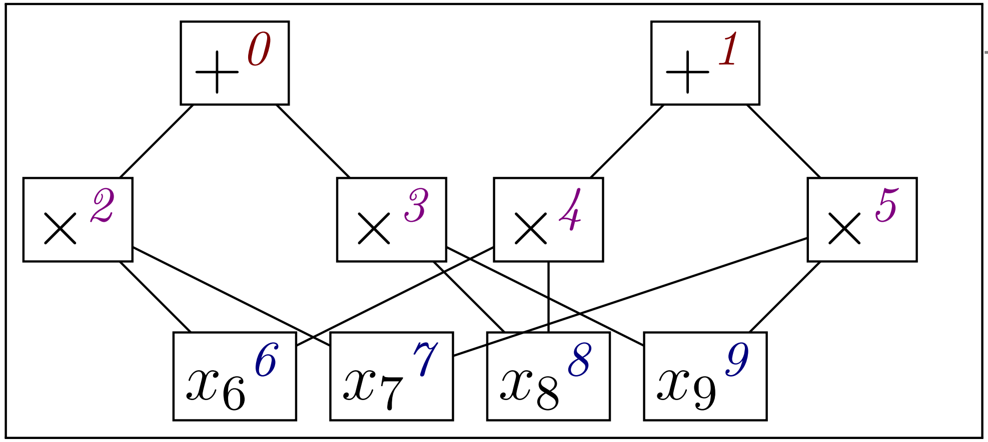


- *Every function $g : \mathbb{R}^n \to \mathbb{R}^m$ is represented by a DAG whose leaf nodes are variables and constants and whose intermediate nodes are mathematical operators*

- Look for relationships between the DAGs representing $g(x)$ and $\sigma g(x\pi)$

Example

$$\begin{aligned} C_0 : x_6x_7 + x_8x_9 &= 1 \\ C_1 : x_6x_8 + x_7x_9 &= 1 \end{aligned}$$



- G_{DAG} = set of automorphisms of expression DAG fixing: (a) root node set having same constr. direction and coeff. (constraint permutations), (b) operators with same label and rank and (c) leaf node set (variable permutations)

```
Dreadnaut version 2.4 (32 bits).
> n=10 g 2 3; 4 5; 6 7; 8 9; 6 8; 7 9.  f=[0:1|2:5|6:9] x
(4 5)(6 7)(8 9)     !variable permutations
(2 3)(6 8)(7 9)     !operator permutations
(0 1)(2 4)(3 5)(7 8) !constraint permutation
```

- G_P is the projection of G_{DAG} to variable indices $\langle(6,7)(8,9),(6,8)(7,9),(7,8)\rangle \cong D_8$

Node colors 1

Colors on the DAG nodes are used to identify those subsets of nodes which can be permuted

1. Root nodes (i.e. constraints) can be permuted if they have the same RHS
2. Operator nodes (including root nodes) can be permuted if they have the same DAG rank and label
3. If an operator node is non-commutative, then the order of the children node must be maintained
4. Constant nodes can be permuted if they have the same DAG rank level and value
5. Variable nodes can be permuted if they have the same bounds and integrality constraints

Node colors 2

- Formalize by equivalence relations on sets: $\mathscr{R}$ =roots, $\mathscr{O}$ =operators, $\mathscr{C}$ =constants, $\mathscr{V}$ =variables
- Let $\mathcal{V}$ be the set of all nodes of the DAG; for all $x, y \in \mathcal{V}$:
  1. $x \sim_R y$ if $x, y \in \mathscr{R} \wedge \text{RHS}(x) = \text{RHS}(y)$ or $x, y \notin \mathscr{R}$
  2. $x \sim_O y$ if $x, y \in \mathscr{O} \wedge \text{level}(x) = \text{level}(y) \wedge \text{label}(x) = \text{label}(y) \wedge (\text{order}(x) = \text{order}(y)$ if x, y noncommutative$)$ or $x, y \notin \mathscr{O}$
  3. $x \sim_C y$ if $x, y \in \mathscr{C} \wedge \text{value}(x) = \text{value}(y) \wedge \text{level}(x) = \text{level}(y)$ or $x, y \notin \mathscr{C}$
  4. $x \sim_V y$ if $x, y \in \mathscr{V} \wedge \text{limits}(x) = \text{limits}(y) \wedge \text{integer}(x) = \text{integer}(y)$ or $x, y \notin \mathscr{V}$
- *Define an integral function* color : $\mathcal{V} \to \mathbb{N}$ *s.t.* $\forall\, x, y \in \mathcal{V}$ $(\text{color}(x) = \text{color}(y)$ iff $x \sim_R y \wedge x \sim_O y \wedge x \sim_C y \wedge x \sim_V y)$
- color *is itself an equivalence relation (call it* $\sim$*) and partitions* $\mathcal{V}$ *in disjoint sets* $V_1, \ldots, V_p$

MINLP problem groups

- Let P be a MINLP and $D = (\mathcal{V}, \mathcal{A})$ be the DAG of P
- Let G_{DAG} be the group of automorphisms of D that fix each class in $\mathcal{V}/\sim$
- Define $\phi : G_{\text{DAG}} \to S_n$ by $\phi(\pi) =$permutation on $\mathscr{V}$ (set of variable nodes) induced by π; then

  Thm.

  ϕ is a group homomorphism and Im$\phi \cong G_P$
- Hence can find G_P by computing Imϕ
- Although the complexity status (**P**/**NP**-complete) of the GRAPH ISOMORPHISM problem is currently unknown, `nauty` is a practically efficient software for computing G_{DAG}
- Also, MILPs are MINLPs! (can apply same methods)

Symmetries in the MIPLib3

Instance	G_P
`air03.mod`	$(C_2)^{13}$
`arki001.mod`	S_{38}
`enigma.mod`	C_2
`gen.mod`	$(C_2)^2 \times D_8 \times S_6$
`fiber.mod`	C_2
`harp2.mod`	C_2
`mas74.mod`	$C_2 \times C_2$
`mas76.mod`	$C_2 \times C_2$
`misc03.mod`	S_3
`misc06.mod`	$(C_2)^2 \times (S_3)^2 \times S_4$
`misc07.mod`	S_3
`mitre.mod`	$(C_2)^7$
`noswot.mod`	C_2
`nw04.mod`	C_2
`p0201.mod`	C_2
`p0282.mod`	$(C_2)^3 \times (S_3)^3$
`p0548.mod`	$(C_2)^7$
`p2756.mod`	$(C_2)^{32}$
`qiu.mod`	$C_2 \times S_4$
`rgn.mod`	S_5
`rout.mod`	S_5
`stein27.mod`	$((C_3)^3 \ltimes PSL(3,3)) \ltimes C_2$
`swath.mod`	S_{79}
`vpm1.mod`	S_{48}
`vpm2.mod`	$(S_3)^2 \times S_4 \times S_5$

Instance	*Error*
`mkc.mod`	RAM (36 gens)
`seymour.mod`	RAM (78 gens)

All others: $G_P = \{e\}$

All instances have been pre-solved by AMPL

Symmetries in the MIPLib2003

Instance	G_P
arki001.mod	S_{48}
fiber.mod	C_2
glass4.mod	C_2
mas74.mod	$C_2 \times C_2$
mas76.mod	$C_2 \times C_2$
misc07.mod	S_3
mzzv11.mod	$(C_2)^{155}$
mzzv42z.mod	$(C_2)^{110}$
noswot.mod	C_2
opt1217.mod	C_2
p2756.mod	$(C_2)^{32}$
protfold.mod	$(C_2)^2$
qiu.mod	$C_2 \times S_4$
rout.mod	S_5
timtab1.mod	C_2
timtab2.mod	C_2

Instance	*Error*
mkc.mod	RAM (36 gens)
seymour.mod	RAM (78 gens)
swath.mod	RAM (922 gens)
atlanta-ip	CPU time
dano3mip	CPU time
mod011.mod	CPU time
sp97ar.mod	CPU time
t1717.mod	CPU time

All others: $G_P = \{e\}$

AMPL presolver disabled

Symmetries in the MINLPLib

Instance	G_P
cecil_13.mod	$(C_2)^9$
elf.mod	S_3
gastrans.mod	C_2
gear.mod	D_8
gear2.mod	D_8
gear3.mod	D_8
gear4.mod	D_8
hmittelman.mod	C_2
lop97icx.mod	$(C_2)^7 \times S_{762}$
nuclear14.mod	S_6
nuclear24.mod	S_6
nuclear25.mod	S_5
nuclear49.mod	S_7
nuclearva.mod	S_3
nuclearvb.mod	S_3
nuclearvc.mod	S_3
nuclearvd.mod	S_3
nuclearve.mod	S_3
nuclearvf.mod	S_3
nvs09.mod	S_{10}
product.mod	S_{50}
risk2b.mod	$(C_2 \times S_3 \times S_6 \times S_{13})^3$
super2.mod	$(C_2)^9 \times (S_3)^2$
super3.mod	$(C_2)^9 \times (S_3)^2$
synheat.mod	S_4

Instance	*Error*
qap.mod	CPU time
qapw.mod	CPU time

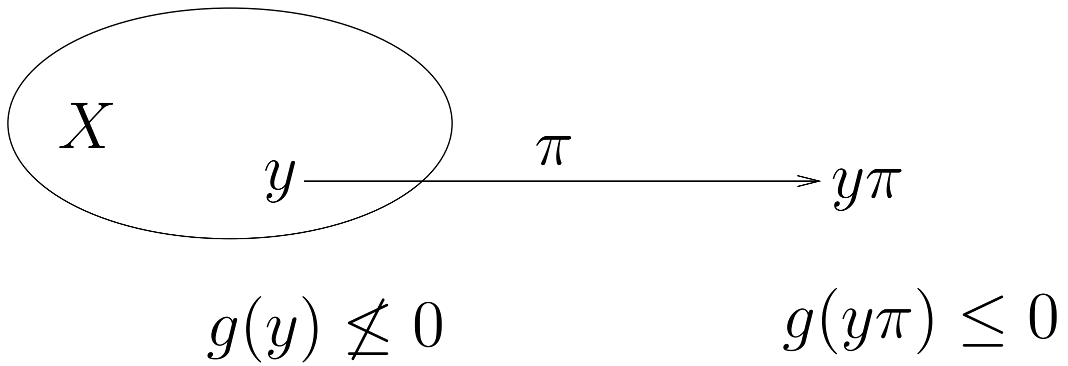
All others: $G_P = \{e\}$

All instances have been pre-solved by AMPL

Breaking symmetries

Defn.

Given a permutation $\pi \leq S_n$ acting on the component indices of the vectors in a given set $X \subseteq \mathbb{R}^n$, the constraints $g(x) \leq 0$ (that is, $\{g_1(x) \leq 0, \ldots, g_q(x) \leq 0\}$) are *symmetry breaking constraints* (SBCs) with respect to π and X if there is $y \in X$ such that $g(y\pi) \leq 0$.



Usually $y\pi$ is an optimum, but not all optima satisfy the SBCs

Defn.

Given a group G, $g(x) \leq 0$ are SBCs w.r.t G and X if there is $y \in XG$ such that $g(y) \leq 0$.

SBCs and narrowings

Adjoining SBCs to an MP formulation provides a valid narrowing

Thm.

If $g(x) \leq 0$ are SBCs for any subgroup G of G_P and $\mathcal{G}(P)$, then the problem Q obtained by adjoining $g(x) \leq 0$ to the constraints of P is a narrowing of P.

Notation: $g[B](x) \leq 0$ *if* $g(x)$ *only involve variable indices in* B

Conditions allowing adjunctions of many SBCs

Thm.

Let $\omega, \theta \subseteq \{1, \ldots, n\}$ be such that $\omega \cap \theta = \emptyset$. Consider $\rho, \sigma \in G_P$, and let $g[\omega](x) \leq 0$ be SBCs w.r.t. $\rho, \mathcal{G}(P)$ and $h[\theta](x) \leq 0$ be SBCs w.r.t. $\sigma, \mathcal{G}(P)$. If $\rho[\omega], \sigma[\theta] \in G_P[\omega \cup \theta]$ then the system of constraints $c(x) \leq 0$ consisting of $g[\omega](x) \leq 0$ and $h[\theta](x) \leq 0$ is an SBC system for $\rho\sigma$.

SBCs from orbits

Let Ω be the set of nontrivial orbits of the regular action of G_P on $\{1,\ldots,n\}$

Thm.

Let $\omega \in \Omega$. The constraints

$$\forall j \in \omega \smallsetminus \{\min \omega\} \quad x_{\min \omega} \le x_j. \tag{4}$$

are SBCs with respect to G_P.

Lemma: $G[\omega]$ *is the transitive constituent of* G *on its orbit* ω

Thm.

Let $\omega^- = \omega \smallsetminus \{\max \omega\}$ and for all $j \in \omega^- (j^+ = \min\{h \in \omega \mid h > j\})$. Provided $G_P[\omega] = \text{Sym}(\omega)$, the following constraints:

$$\forall\, j \in \omega^- \quad x_j \le x_{j^+} \tag{5}$$

are SBCs with respect to G_P.

Automatic SBC generation

1. Transform MINLP from AMPL input format into a DAG representation (`ROSE`)
2. Compute node colors according to relation $\sim$ defined above (`ROSE`)
3. Compute G_{DAG} (`nauty`)
4. Compute Imϕ (`gap`)
5. Compute nontrivial orbits Ω (`gap`)
6. Generate SBCs (4) or (5) according to the structure of $G_P[\omega]$, where ω is the longest orbit in Ω (`gap`)
7. If conditions hold, try to generate compatible SBCs from other orbits (`gap`)

`ROSE`=Reformulation/Optimization Software Engine; `nauty`=Graph Isomorphism software; `gap`=Group Theory software; data flow provided by Unix scripts

Tests

- Computed group structures for 669 instances in MIPLib3 ∪ MIPLib2003 ∪ GlobalLib ∪ MINLPLib
- Out of 18% instances with nontrivial groups, 74 could be solved by BB algorithms (CPLEX for MILPs; Couenne, BARON for (MI)NLPs)
- *Narrowing1*: only use (4) for longest orbit
- *Narrowing2*: also use (5) and attempt to combine SBCs if possible
- Test 1: over all instances
- Test 2: over a selection of 6 difficult instances with long BB runs

	Original problem			*Narrowing1*			*Narrowing2*		
T.	*CPU*	*Best gap*	*Nodes*	*CPU*	*Best gap*	*Nodes*	*CPU*	*Best gap*	*Nodes*
1	157263	69 2.26E4%	21.44M	152338	70 2.26E4%	14.23M	153470	72 2.26E4%	15.72M
2	815018	5 242.88%	12.26M	888089	5 219.14%	14.63M	786406	5 217.05%	11.28M

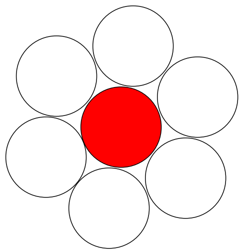
Test 2

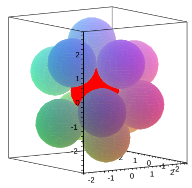
		Original problem			Narrowing1			Narrowing2			
Instance	*Slv*	*CPU*	f^* *gap*	*nodes* *tree*	*CPU*	f^* *gap*	*nodes* *tree*	*CPU*	f^* *gap*	*nodes* *tree*	*R.t.*
MILPLib(s)											
`mkc`[a]		146850	-563.846 0.13%	1945500 1479080	133924	-563.846 0.13%	**2104500** **1449867**	-	-	-	2712.33
`protfold`[b]		300000	-26 30.51%	592000 458813	300000	**-29** **16.54%**	**536100** **353823**	-	-	-	592.14
`seymour`[a]		262817	423 0.9%	3992700 3026077	283311	423 **0.83%**	**4343500** **3038821**	233643	423 1.0%	3960700 3064665	5.95
GlobalLib											
`ex5_2_5`[a]	C	19805	-3500 28.14%	5452500 1259853	82320	-3500 18.5%	**7373700** **747262**	18151	-3500 **17.41%**	4425400 1076927	1.40
`maxmin`[a]	B	58643	-0.366 145%	237100 150803	57762	-0.366 **144%**	**238000** **150355**	-	-	-	1.29
MINLPLib											
`lop97icx`[a]	B	26903	**4391.1** 38.2%	**44858** **27824**	30772	4493.5 39.14%	43948 27189	42926	4412.9 **37.97%**	23416 14708	24.96

The KNP group

- Kissing Number Problem (decision version): Given integers $D, N > 1$, can N unit spheres be adjacent to a given unit sphere in $\mathbb{R}^d$?
- Formulation:

$$\begin{array}{ll} \max_{x,\alpha} & \alpha \\ \forall i \leq N & \|x_i\|^2 = 1 \\ \forall i < j \leq N & \|x_i - x_j\|^2 \geq \alpha \\ \alpha \in [0,1], \forall i \leq N \; x_i \in [-1,1]^D & \end{array}$$





- If $\alpha \geq 1$, answer YES, otherwise NO
- *The group Aut$(\mathcal{G}(P))$ has infinite (uncountable) cardinality: each feasible solution can be rotated by any angle in $\mathbb{R}^D$*; however, the **problem group** G_P is finite (permutations of spheres and/or dimensions)
- Conjecture (formulated by software): $G_P \cong S_D$
- Rewrite constraint: $\|x_i - x_j\|^2 = \sum_{k \leq D} (x_{ik} - x_{jk})^2 = \sum_{k \leq D} (x_{ik}^2 + x_{jk}^2 - 2x_{ik}x_{jk}) = 2(D - \sum_{k \leq D} x_{ik}x_{jk})$
- **Conjecture becomes**: $G_P \cong S_D \times S_N$ (eventually proved correct)

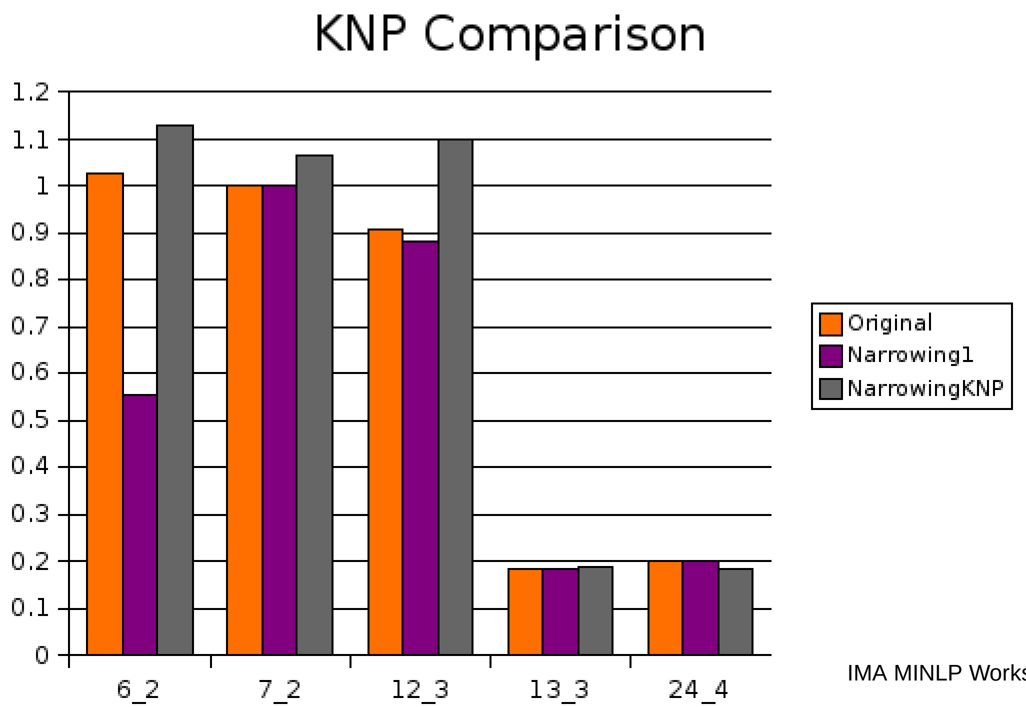
Some KNP results

		Original problem			Narrowing1			NarrowingKNP			
Instance	*Slv*	*CPU*	f^* *gap*	*nodes* *tree*	*CPU*	f^* *gap*	*nodes* *tree*	*CPU*	f^* *gap*	*nodes* *tree*	*R.t.*
knp-6_2[a]	B	8.66	-1 **0%**	1118 0	36000	-1 20%	131 16	**1.91**	-1 **0%**	**186** **0**	1.43
knp-7_2	B	147.21	-0.753 0%	13729 0	154.9	-0.753 0%	14182 0	**3.86**	-0.753 0%	**260** **0**	1.47
knp-12_3	B	36000	-1.105 8.55%	299241 12840	36000	-1.105 8.55%	241361 12431	36000	-1.105 8.55%	**273923** **5356**	3.39
knp-13_3	B	36000	-0.914 118%	102150 64499	36000	-0.914 118%	97709 61545	36000	-0.914 118%	**68248** **33013**	3.38
knp-24_4	B	36000	**-0.966** **107%**	10156 7487	36000	**-0.966** **107%**	**11394** **8340**	36000	-0.92 117%	4059 2985	5.62

Indicator:

$$\frac{1}{4}\left(\frac{1}{\text{CPU}} + \frac{1}{\text{GAP}+1} + \frac{\text{nodes}}{\text{tree}}\right)$$

"bigger is better"



The end

Thank you