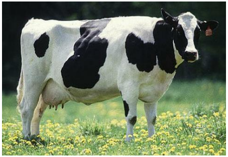
Triangles, Envelopes, and Nonconvex QP

Jeff Linderoth

Dept. of Industrial and Systems Engineering
Univ. of Wisconsin-Madison
linderoth@wisc.edu



IMA Hot Topics Workshop on MINLP   Nov 18, 2008

Nonconvex Quadratic Programm

$$
(\text{QCQP}) \left\{ \begin{array}{lll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & q_0(x) & \\ \text{subject to} & q_k(x) \geq b_k & \forall k \in \mathcal{I} \\ & q_k(x) = b_k & \forall k \in \mathcal{E} \\ & l \leq x \leq u & \end{array} \right.
$$

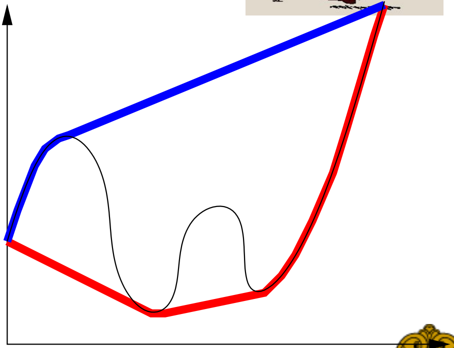
- $q_k(x) = (c^k)^T x + x^T Q^k x \qquad \forall k \in \{0 \cup \mathcal{I} \cup \mathcal{E}\}$
- $q_k(x)$ could be convex, concave, or nonconvex
- l and u are finite

Caution!

- Work is > 4 years old at this point...
- I'm certainly not an expert in this area. (Go See Sam's Poster...)

Envelopes

- Popular (best?) method to solve (QCQP) is to use convex and concave envelopes. $f : \Omega \to \mathbb{R}$
- Convex Envelope ($\text{vex}_\Omega(f)$): Pointwise supremum of convex underestimators of f over Ω.
- Concave Envelope ($\text{cav}_\Omega(f)$): Pointwise infimum of concave overestimators of f over Ω.



Solving QCQP

- Consider term $x_i x_j$, for $(x_i, x_j) \in R \stackrel{\text{def}}{=} [l_i, u_i] \times [l_j, u_j]$.

$$
\begin{aligned}
x_i x_j &\geq \max\{l_i x_j + l_j x_i - l_i l_j, u_i x_j + u_j x_i - u_i u_j\} \\
x_i x_j &\leq \min\{l_i x_j + u_j x_i - l_i u_j, u_i x_j + l_j x_i - u_i l_j\}
\end{aligned}
$$

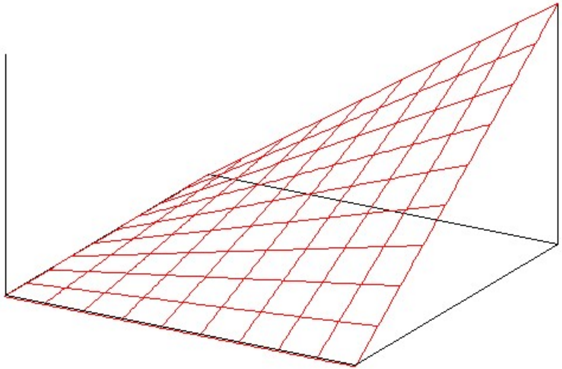
- **Thm:** (McCormick '76, Al-Khayyal and Falk, '83)

$$
\begin{aligned}
\text{vex}_R(x_i x_j) &= \max\{l_i x_j + l_j x_i - l_i l_j, u_i x_j + u_j x_i - u_i u_j\} \\
\text{cav}_R(x_i x_j) &= \min\{l_i x_j + u_j x_i - l_i u_j, u_i x_j + l_j x_i - u_i l_j\}
\end{aligned}
$$

- Integer Programming Lingo: These are "facets" of the "convex hull"

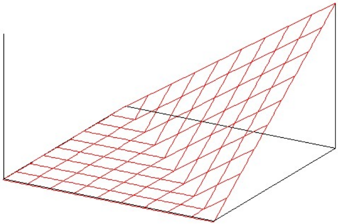
Worth 1000 Words?

$$x_i x_j$$

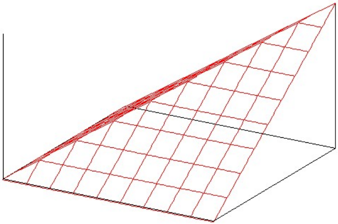


Convex and Concave Envelopes

$\text{vex}_R(x_i x_j)$



$\text{cav}_R(x_i x_j)$



(LP) Relaxation of QCQP

$$z_{LP} = \min \sum_{i=1}^{n} c_i^0 x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}^0 z_{ij}$$

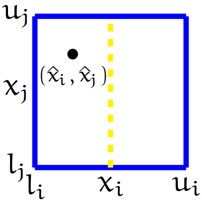
subject to

$$\begin{array}{rcll}
\sum_{i=1}^{n} c_i^k x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}^k z_{ij} & \geq & b_k & \forall k \in \mathcal{I} \\
\sum_{i=1}^{n} c_i^k x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}^k z_{ij} & = & b_k & \forall k \in \mathcal{E} \\
z_{ij} - l_i x_j - l_j x_i + l_i l_j & \geq & 0 & \forall i = 1, \ldots, n, j = 1, \ldots, n \\
z_{ij} - u_i x_j - u_j x_i + u_i u_j & \geq & 0 & \forall i = 1, \ldots, n, j = 1, \ldots, n \\
z_{ij} - l_i x_j - u_j x_i + l_i u_j & \leq & 0 & \forall i = 1, \ldots, n, j = 1, \ldots, n \\
z_{ij} - u_i x_j - l_j x_i + u_i l_j & \leq & 0 & \forall i = 1, \ldots, n, j = 1, \ldots, n \\
x_i & \in & [l_i, u_i] & \forall i = 1, \ldots, n
\end{array}$$

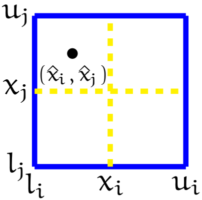
Branching

- In LP relaxation, $z_{ij} = x_i x_j \ \ \forall x_i, x_j \in \partial\Omega$.
- If $z_{ij} \neq x_i x_j$, we branch.
- Two suggested branching schemes

Two Rectangles

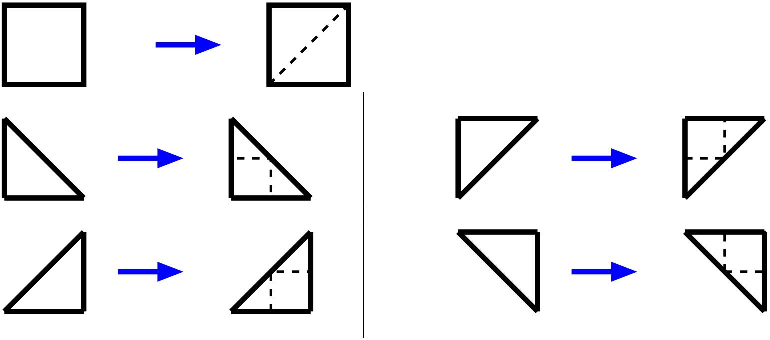


Four Rectangles



Why Rectangles?

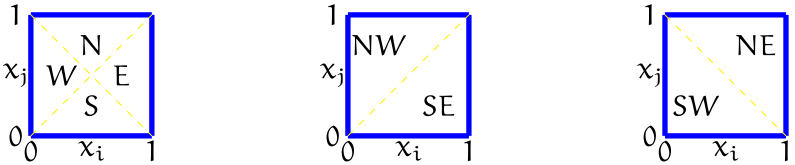
- Because rectangles "beget" rectangles in a branching scheme?
- Why not triangles?



Envelopes

One Problem

You need to know the formulae for convex and concave envelopes for bilinear functions over triangles

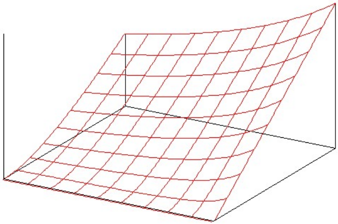


$$\text{vex}_{SE} = \frac{x_j^2}{1 + x_j - x_i} \qquad \text{cav}_{SW} = \frac{x_i x_j}{x_i + x_j}$$

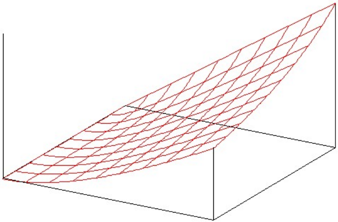
$$\text{vex}_{NW} = \frac{x_i^2}{1 + x_i - x_j} \qquad \text{cav}_{NE} = \frac{x_i^2 + x_i x_j + x_j^2 - 2x_i - 2x_j + 1}{x_i + x_j - 2}$$

Envelopes over Triangles

$\text{vex}_{SE}(x_i x_j)$

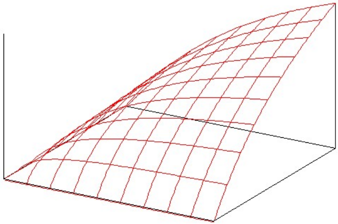


$\text{vex}_{NW}(x_i x_j)$

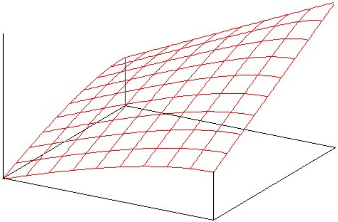


Envelopes Over Triangles

$\mathrm{cav}_{SW}(x_i x_j)$



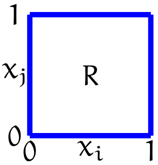
$\mathrm{cav}_{NE}(x_i x_j)$

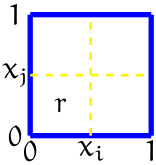


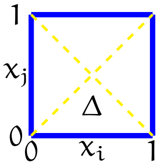
Tight Relaxations

- Total Error over region Γ: $\eta_\Gamma \stackrel{\text{def}}{=} \int_\Gamma (\text{cav}_\Gamma(x_i x_j) - \text{vex}_\Gamma(x_i x_j)) dx_i dx_j$.
- Max Error over region Γ:

$$\phi(\Gamma) \stackrel{\text{def}}{=} \max_{(x_i, x_j) \in \Gamma} (\text{cavxy}_\Gamma(x_i, x_j) - \text{vexxy}_\Gamma(x_i, x_j))$$







$$\eta(R) = 1/6 \qquad \phi(R) = 1/2$$

$$\eta(r) = 1/24 \qquad \phi(r) = 1/8$$

$$\eta(\Delta) = 1/72 \qquad \phi(\Delta) = 3/2 - \sqrt{2}$$

SOC Representation

- The nonlinear inequalities for $\text{vex}_\Delta(x_i x_j)$ and $\text{cav}_\Delta(x_i x_j)$ all have a second order cone representation
- Restrict $(x_i, x_j) \in W \stackrel{\text{def}}{=} \{(x_i, x_j) | x_i \leq x_j, x_i + x_j \leq 1\}$:

$$z_{ij} \geq \frac{x_i^2}{x_i - x_j + 1}, z_{ij} \leq \frac{x_i x_j}{x_i + x_j}$$

$$z_{ij} \geq \frac{x_i^2}{x_i - x_j + 1} \quad \Leftrightarrow \quad \begin{bmatrix} z_{ij} + 1 - x_j + x_i \\ 2x_i \\ z_{ij} - 1 + x_j - x_i \end{bmatrix} \in \mathcal{K}_q^3$$

$$z_{ij} \leq \frac{x_i x_j}{x_i + x_j} \quad \Leftrightarrow \quad \begin{bmatrix} 2x_i + x_j - z_{ij} \\ 2x_i \\ -x_j - z_{ij} \end{bmatrix} \in \mathcal{K}_q^3$$

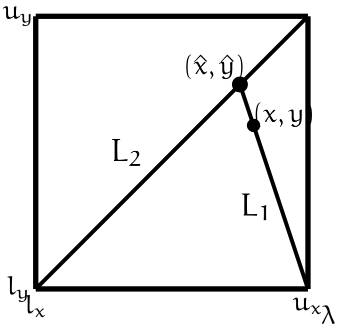
Semi-Infinite approximation

- Didn't know how to embed SOC solver into branch and bound code
- Did a "linear relaxation"

$$\mathrm{vex}_{SE}(x,y) = \min_{\xi,\lambda} \sum_{(\hat{x}_i,\hat{y}_i \in L_2)} \hat{x}_i \hat{y}_i \xi_i + u_x l_y \lambda$$

subject to

$$\begin{aligned} \sum_{(\hat{x}_i,\hat{y}_i)\in L_2} \hat{x}_i \xi_i + u_x \lambda &= x, \\ \sum_{(\hat{x}_i,\hat{y}_i)\in L_2} \hat{y}_i \xi_i + l_y \lambda &= y, \\ \sum_{(\hat{x}_i,\hat{y}_i)\in L_2} \xi_i + \lambda &= 1 \end{aligned}$$



Performance Profile: Number of nodes