Linear Programs with Complementarity Constraints: Algorithms and Applications[1]

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November 18, 2008

[1]Supported by AFOSR.

Linear Programs with Complementarity Constraints

Find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ to **globally** solve the linear program with complementarity constraints (LPCC):

$$\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x + d^T y \\ \text{subject to} & Ax + By \geq f \\ \text{and} & 0 \leq y \perp q + Nx + My \geq 0, \end{array}$$

Fundamental importance

Novel paradigms in mathematical programming

- hierarchical optimization
- inverse optimization
- parameter identification/model validation in optimization
- optimizing Value-at-Risk and order statistics

Key formulations for

- B-stationary conditions of MPECs
  - verification and computation without MPEC-constraint qualification
- global resolution of nonconvex quadratic programs

Preliminary observations

An LPCC is equivalent to 2^m linear programs, each called a piece and derived from a subset $\alpha \subseteq \{1, \cdots, m\}$ with complement $\bar{\alpha}$:

$$\text{LP}(\alpha): \quad \begin{array}{lll} \underset{(x,y)}{\text{minimize}} & c^T x + d^T y & \\ \text{subject to} & Ax + By \geq f & \\ & (q + Nx + My)_\alpha \geq 0 = y_\alpha & \\ \text{and} & (q + Nx + My)_{\bar{\alpha}} = 0 \leq y_{\bar{\alpha}} & \end{array}$$

Thus, there are 3 states of an LPCC in general:

- infeasibility–all pieces are infeasible
- unboundedness–one piece is feasible and unbounded below
- global solvability–one piece is feasible and all feasible pieces are bounded below.

Equivalent Integer Program

Given a sufficiently large parameter θ and denoting the vector of ones by **1**, get an equivalent mixed integer problem:

$$\begin{array}{llcl} \underset{(x,y,z)}{\text{minimize}} & c^T x + d^T y & & \\ \text{subject to} & Ax + By & \geq & f \\ & \theta z \geq w := q + Nx + My & \geq & 0 \\ & \theta(\mathbf{1} - z) \geq y & \geq & 0 \\ \text{and} & z & \in & \{0, 1\}^m \end{array}$$

If good bounds can be found on all components of w and y then this can be solved as an MIP.

Of interest: (some of) the variables are unbounded.

Removing θ for fixed $\bar{z}$

For fixed $\bar{z}$, get linear program.
The limiting dual problem for large θ can be expressed:

$$D(\bar{z}) \qquad \begin{array}{ll} \underset{(\lambda, u^{\pm}, v)}{\text{maximize}} & f^T \lambda + q^T (u^+ - u^-) \\ \text{subject to} & A^T \lambda - N^T (u^+ - u^-) = c \\ & B^T \lambda - M^T (u^+ - u^-) - v \leq d \\ & \bar{z}^T u^+ + (\mathbf{1} - \bar{z})^T v = 0 \\ \text{and} & (\lambda, u^{\pm}, v) \geq 0, \end{array}$$

If $\bar{z}_i = 1$ then $u_i^+ = 0$.
If $\bar{z}_i = 0$ then $v_i = 0$.

Logical Benders Decomposition algorithm

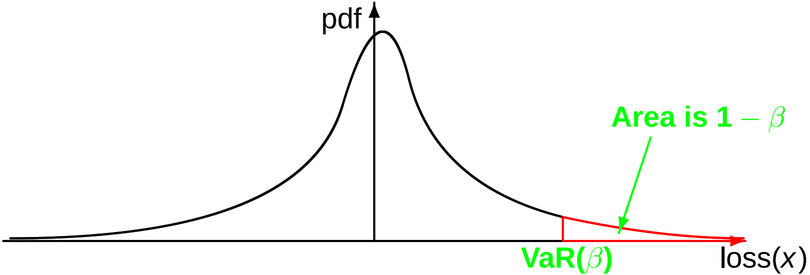
Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem using Satisfiability cuts generated using the optimal solution to the subproblem. These satisfiability cuts are sparsified to make them as strong as possible.
7. Return to Step 2.

(See Hooker, and also Codato and Fischetti.)

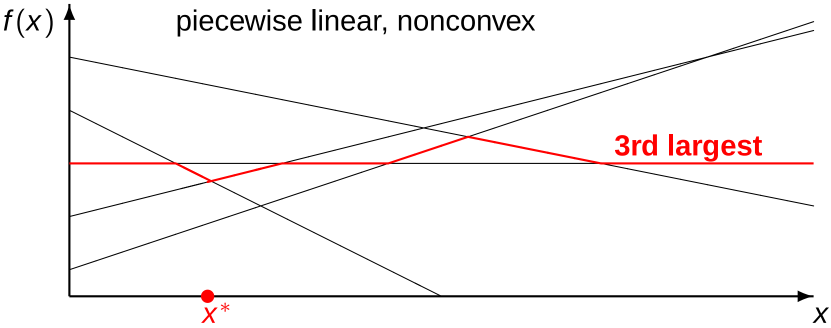
Optimizing Value-at-Risk (VaR)

Choose *x* to minimize VaR(β) for some β.



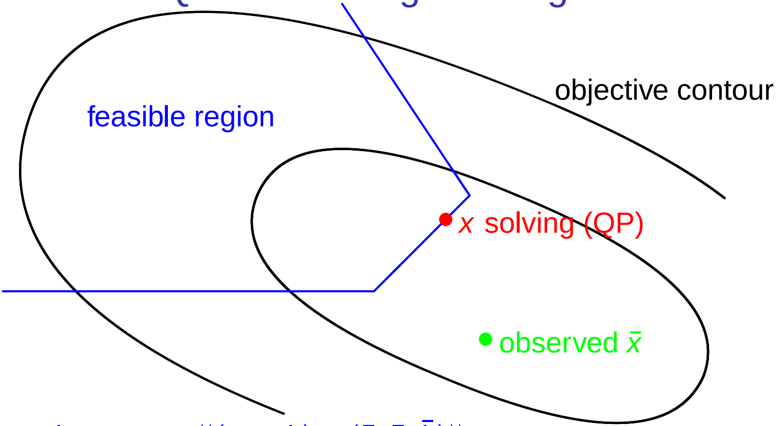
Rockafellar and Uryasev: VaR is argmin of a CVaR problem.
(See also Pang and Leyffer; Luedtke, Ahmed and Shapiro,...)
Get LPEC formulation when have finite number of scenarios.

Minimizing the kth largest



$$\begin{array}{lll} \min_{\alpha,\beta,x,s} & \alpha & \\ \text{subject to} & \alpha + \beta_i \geq f_i(x) & i = 1, \ldots, m \\ & 0 \leq \beta \perp s \geq 0 & \\ & \mathbf{1}^T s = m - k + 1 & \\ & 0 \leq s \leq 1, \quad x \in P & \end{array}$$

Inverse Convex Quadratic Programming Problems



$$\min_{x,c,b,y,w} \quad ||(x,c,b)-(\bar{x},\bar{c},\bar{b})||_1$$

$$\text{subject to} \quad 0 \le x \perp w := c + Qx + A^T y \ge 0$$

$$0 \le b - Ax \perp y \ge 0$$

$$(c,b) \in P$$

General LPCCs with logical Benders decomposition

LPCCs with $B = 0$, $A \in \mathbb{R}^{200 \times 300}$, and 300 complementarities.

# IPs	Time	LPCC_{min}	# LPs	rx-cnt	rx-dual	rx-master
2	18.61	2478.2256	140	122	1	17
3	67.02	3270.1844	513	413	16	84
2	30.84	3660.5412	237	205	13	19
3	54.27	3176.4108	506	427	10	69
2	5.06	2959.9495	23	20	2	1
4	38.25	2672.5709	383	334	3	46
0	0.23	2617.2638	0	0	0	0
2	16.53	2771.2372	134	131	1	2
2	25.28	2847.6926	197	188	1	8
3	46.94	3230.9896	436	361	2	73

Solved on a Dell Core Duo CPU 2.33 GHz 1.95 GB of RAM.
rx-cnt: sparsification LPs. rx-dual: LPs to find ray cuts. rx-master: LPs to find point cuts.

Conclusions

The LPCC has broad applicability. We have two approaches for finding global solutions:

> *When good bounds are available, formulate as a fixed charge mixed integer program and solve directly.*
> *When it is not easy to find good bounds, use a logical Benders decomposition approach.*

The complementarities can be further exploited by **lift-and-project** or **RLT**. This is very helpful in the logical Benders approach. In the direct MIP approach, it reduces the size of the tree, but (to date) we have not found it to reduce the runtime.

Further work: Extend the theory and computational work to more general classes of mathematical programs with equilibrium constraints.