Nonlinear Discrete Optimization

Shmuel Onn

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Based on several papers joint with several co-authors including
Berstein, De Loera, Hemmecke, Lee, Rothblum, Weismantel

Outline

0. The Setup

1. Convex Discrete Maximization

2. Nonlinear Integer Programming

3. Nonlinear Combinatorial Optimization

Setup for Nonlinear Discrete Optimization

The **problem** is:

$$\min/\max \{ f(Wx) : x \text{ in } S \}$$

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Setup for Nonlinear Discrete Optimization

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This setup enables:

Determination of broad classes of triples **S**, **W**, **f** where we can solve efficiently (deterministically, randomly, or approximately)

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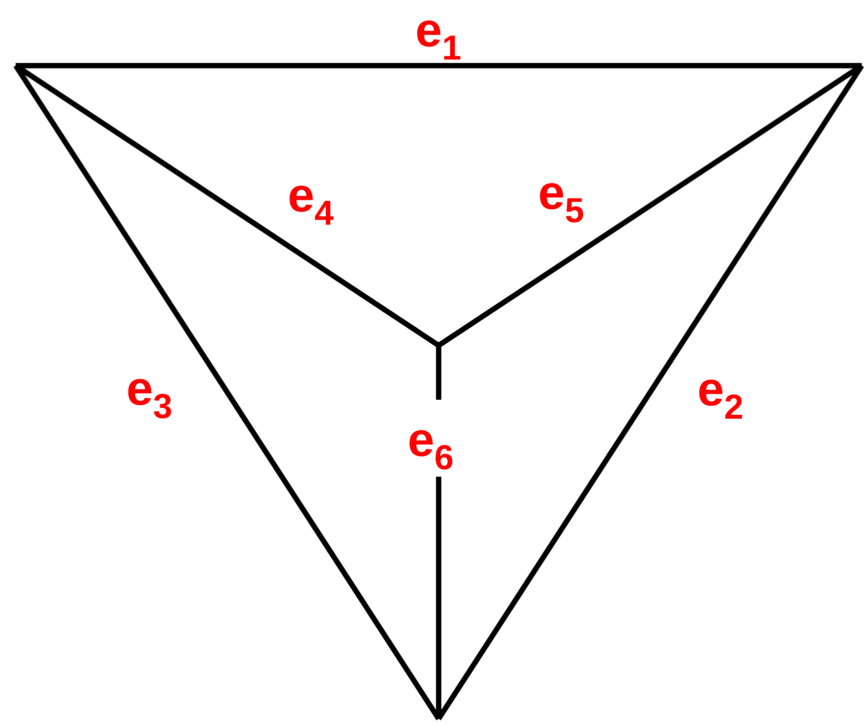
This setup enables:

Interpretation as multi-objective optimization with objective $f(Wx) = f(W_1x, \ldots, W_dx)$ balancing criteria W_ix of d players

Small Example

Small Example

S in $\{0,1\}^6$ consists of all spanning trees in the graph K_4



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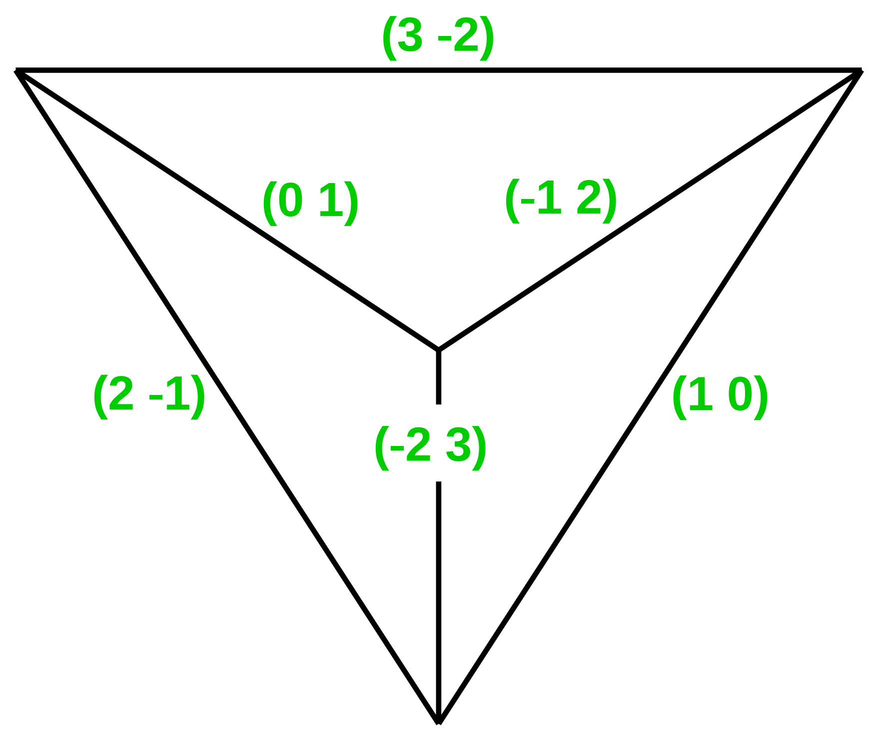
W =

3	1	2	0	-1	-2	criterion/player 1
-2	0	-1	1	2	3	criterion/player 2



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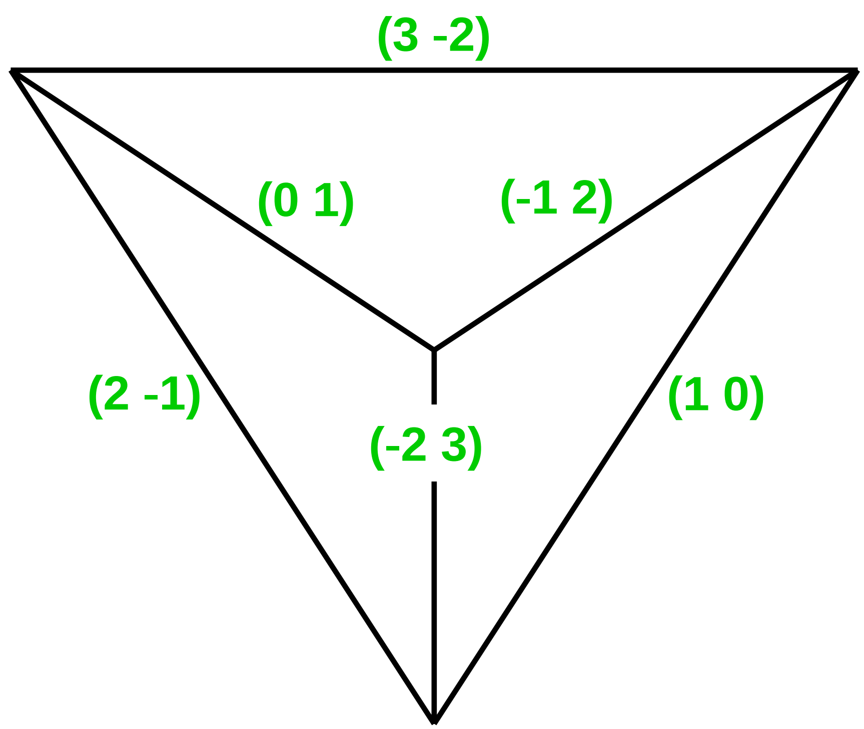
f is Euclidean norm $f(y) = |y|^2 = y_1^2 + y_2^2$ balancing criteria



Small Example

S in $\{0,1\}^6$ consists of all spanning trees in the graph K_4

$$W = \begin{bmatrix} 3 & 1 & 2 & 0 & -1 & -2 \\ -2 & 0 & -1 & 1 & 2 & 3 \end{bmatrix}$$

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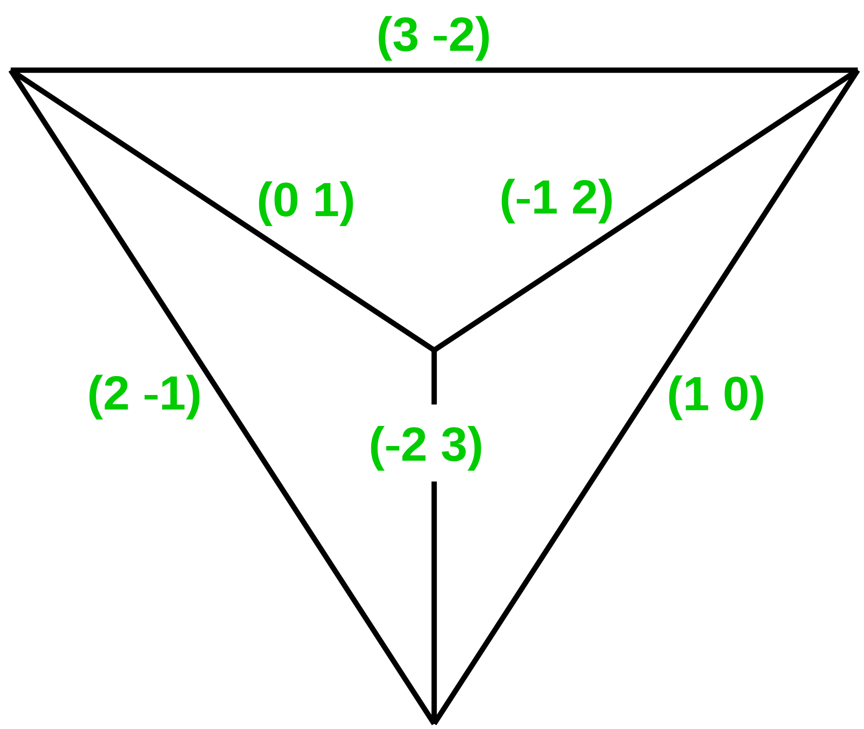

The nonlinear problem is:

$\max \{ f(Wx) : x \text{ in } \{0,1\}^6 \text{ spaning tree} \}$

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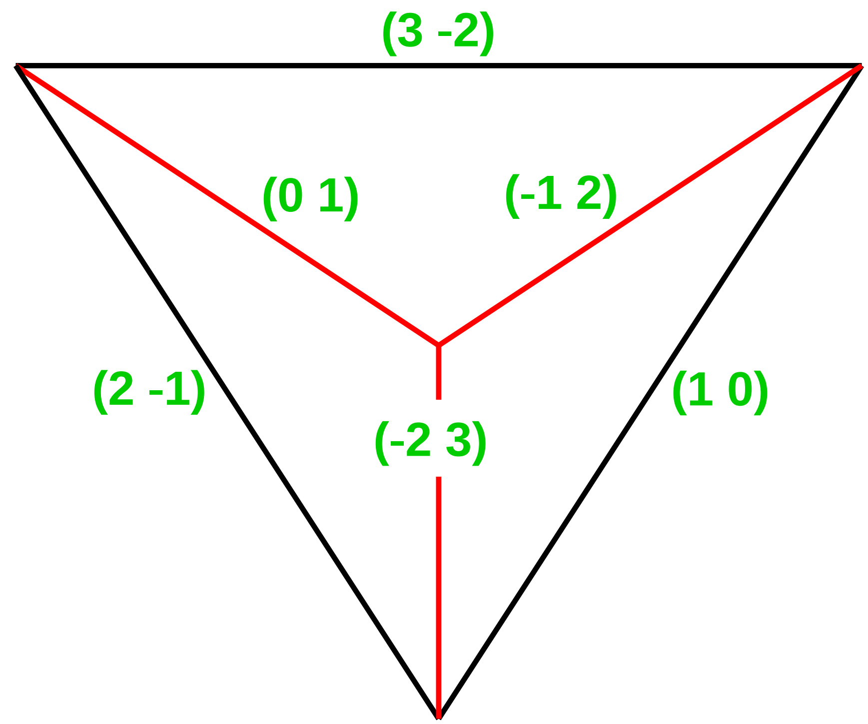

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The optimal tree is $x = (0\ 0\ 0\ 1\ 1\ 1)$

with $Wx = (-3\ \ 6)$ and $f(Wx) = 45$

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Some Possible Assumptions on the Data S,W,f

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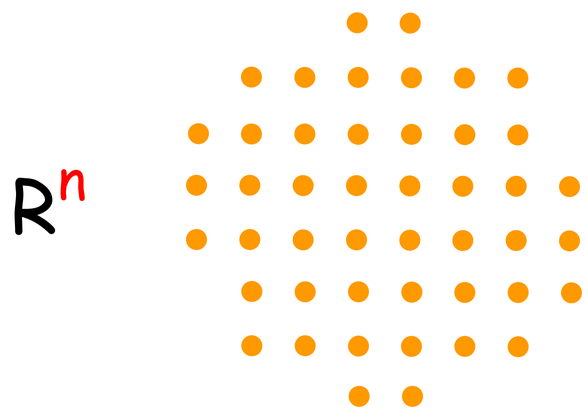
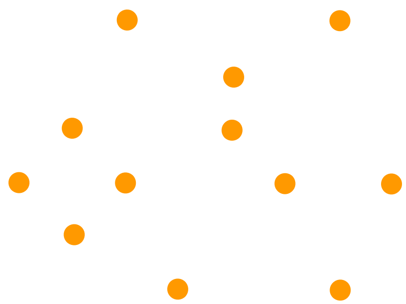


- Feasible set S in Z^n presented by

Some Possible Assumptions on the Data S,W,f

R^n



- Feasible set S in Z^n presented by
  - membership oracle: very little information

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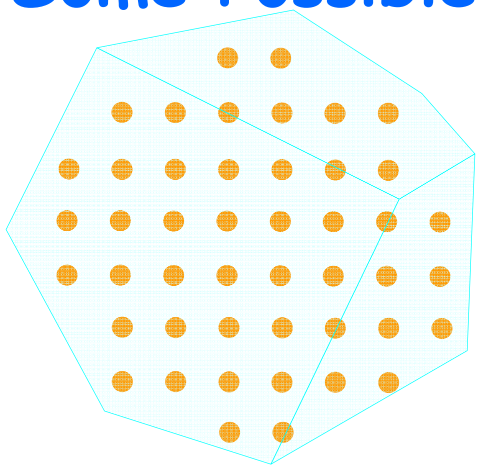

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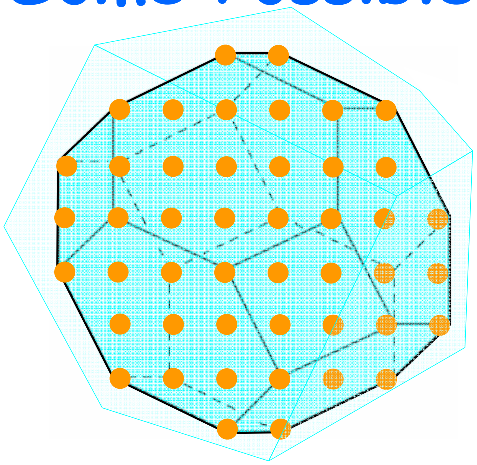

- Feasible set S in Z^n presented by
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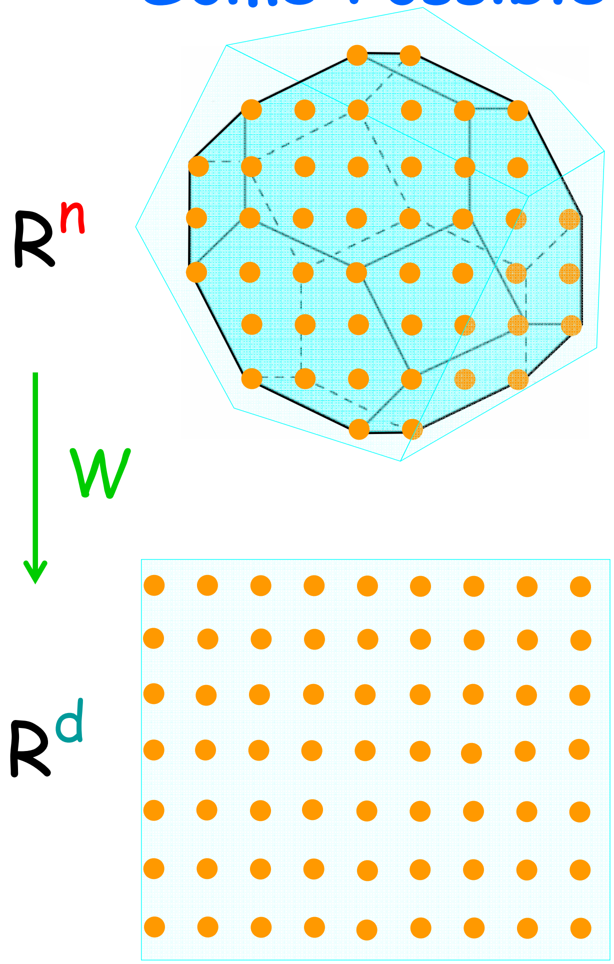

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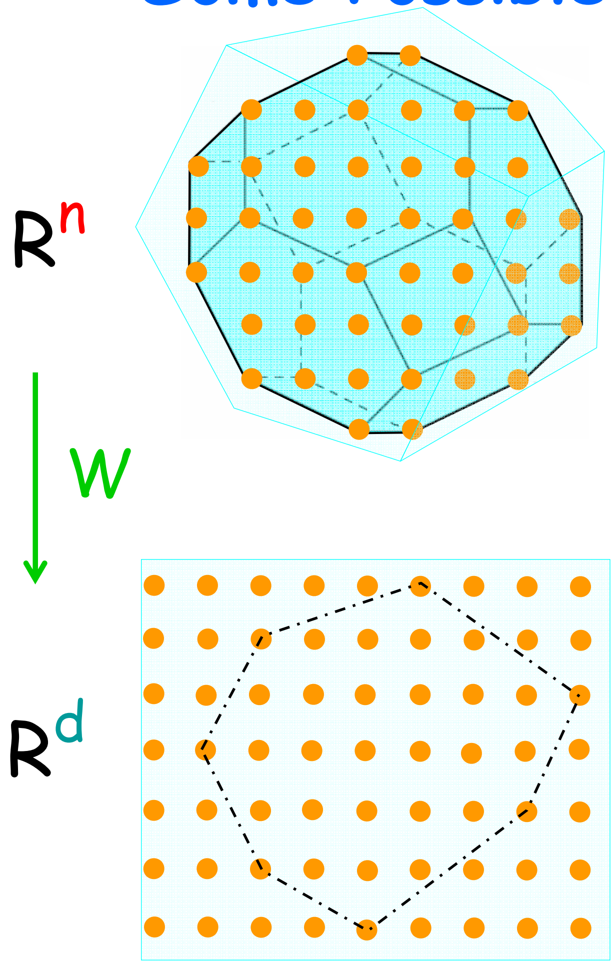

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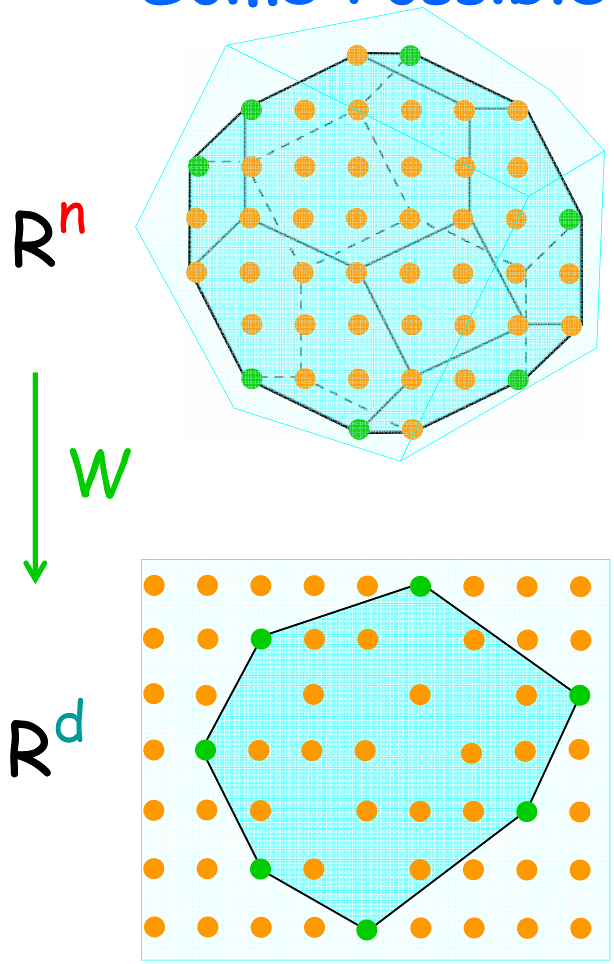

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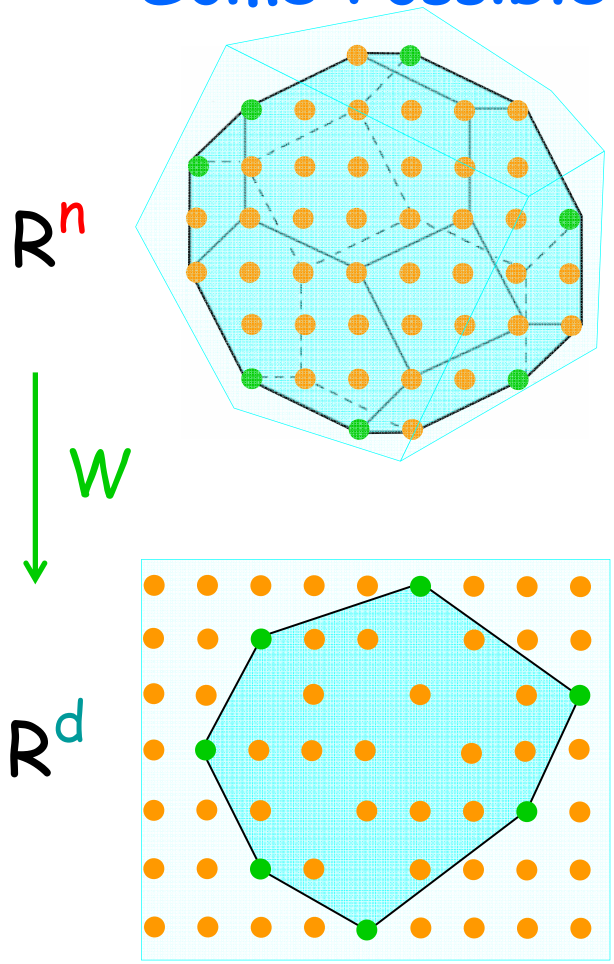

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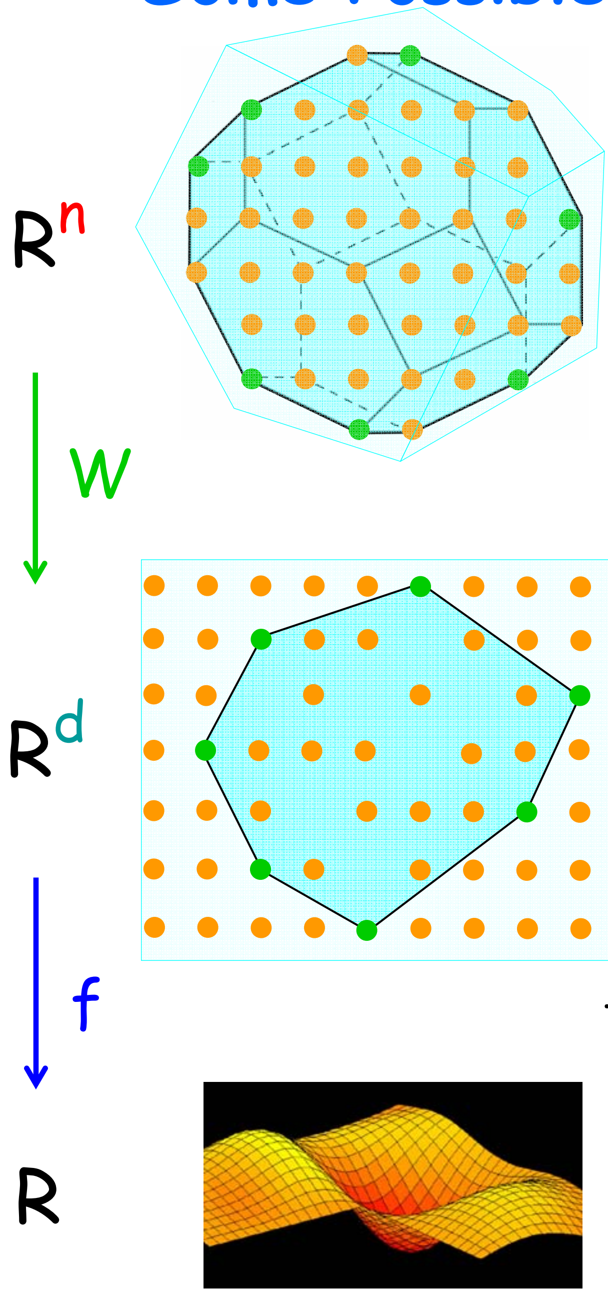

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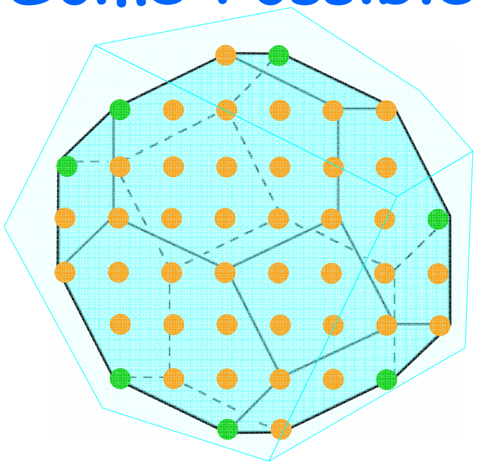
- Function f on R^d presented by comparison oracle

Some Possible Assumptions on the Data S,W,f

R^n



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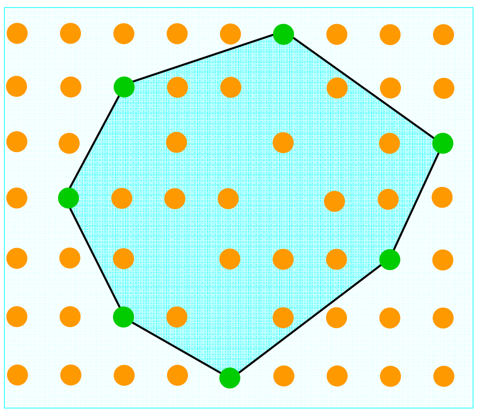
R^d



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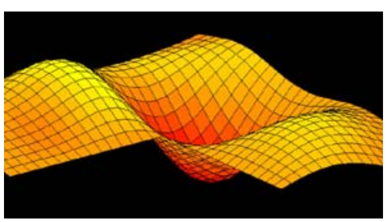
f

R



- Function f on R^d presented by comparison oracle
  - arbitrary nonlinear
  - separable
  - convex

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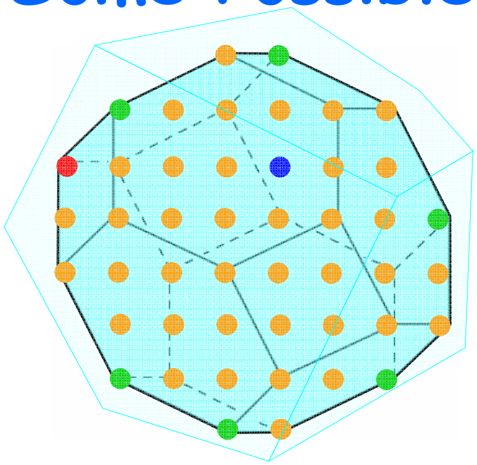
Some Possible Assumptions on the Data S,W,f

R^n



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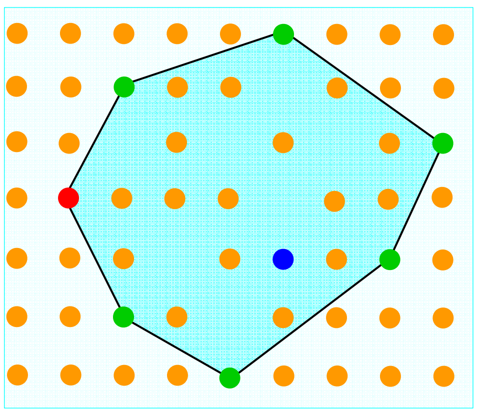
$\downarrow W$

R^d



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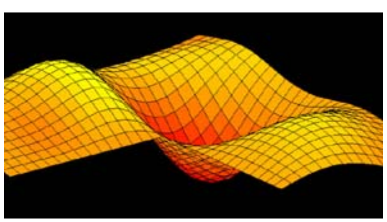
$\downarrow f$

R



- Function f on R^d presented by comparison oracle
  - arbitrary nonlinear
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min/max

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1. Convex Discrete Maximization

Data:

- S given by linear-optimization oracle and edge-behaved
- W binary encoded
- f convex to be maximized given by comparison oracle

1. Convert Discrete Maximization

Data:

- S given by linear-optimization oracle and edge-behaved
- W binary encoded
- f convex to be maximized given by comparison oracle

Methods: **mostly geometric - zonotopes**

References: **Onn, Rothblum, Schulman**

- Partition problems with convex objectives (Math. OR)
- Convex matroid optimization (SIAM Disc. Math.)
- Convex combinatorial optimization (Disc. Comp. Geom.)

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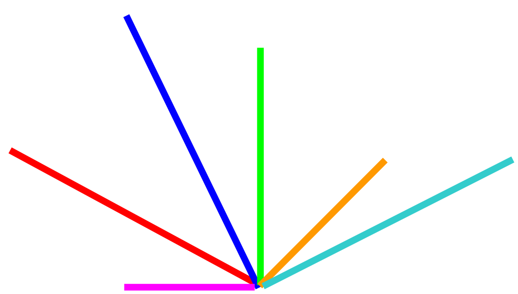
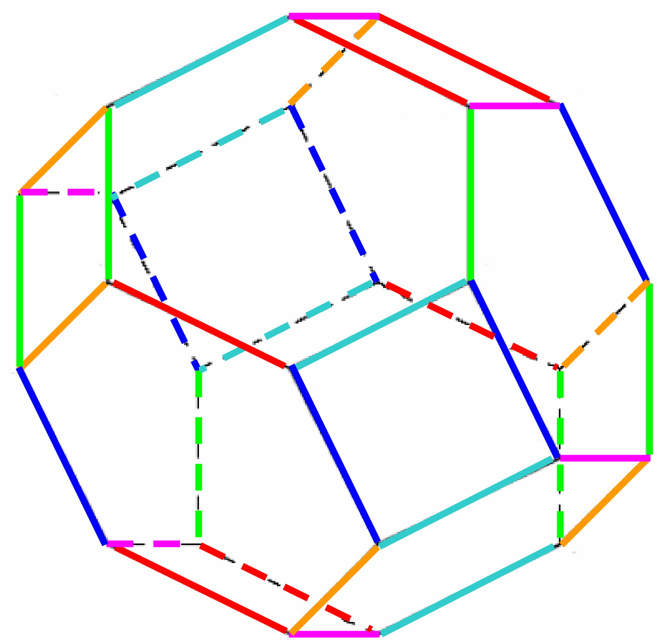
Convex Discrete Maximization

If we have a set E covering all edge-directions of conv{S}





then we can reduce convex maximization over S to
polynomially many linear optimization counterparts.

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Convex Discrete Maximization

Theorem 1.1: Fix any d. Then can solve in strongly polynomial-time

$$\max \{ f(Wx) \ : \ x \text{ in } S \}$$

for S in Z^n given by linear-optimization oracle and endowed with set E covering edge-directions of conv$\{S\}$, any W, any convex f.

Convex Combinatorial Maximization:

membership oracle suffices

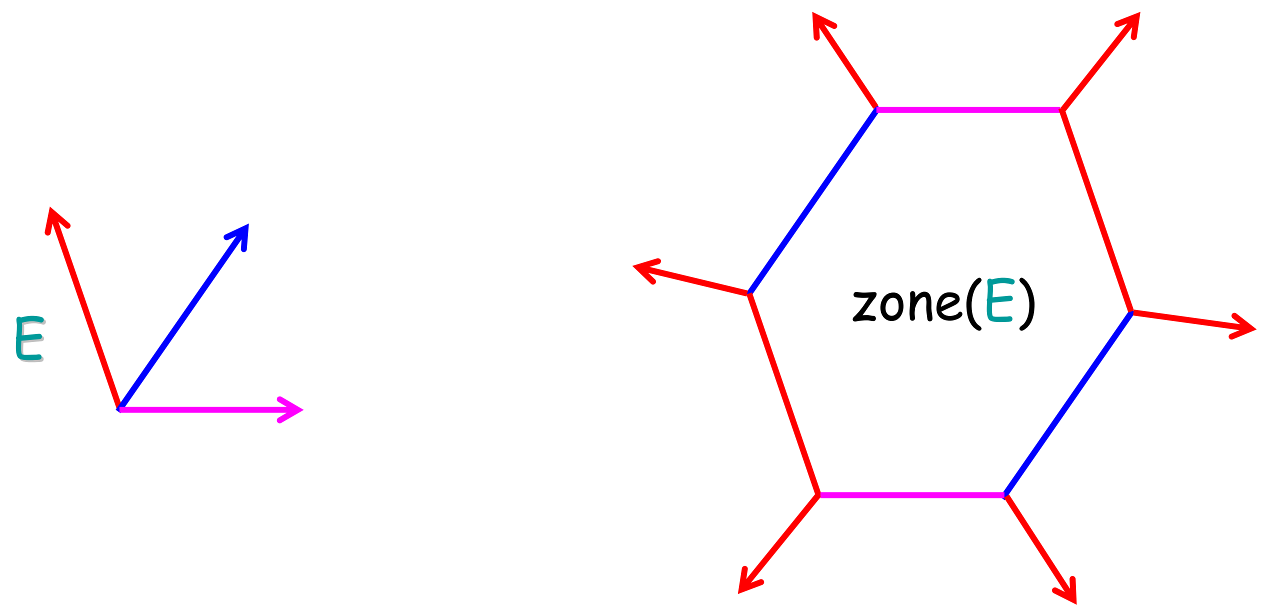
Theorem 1.2: Fix any d. Then can solve in strongly polynomial-time

$$\max \{ f(Wx) \ : \ x \text{ in } S \}$$

for S in $\{0,1\}^n$ given by membership oracle and endowed with set E covering edge-directions of conv$\{S\}$, any W, any convex f.

Preliminaries on Zonotopes

The zonotope of E is the convex hull zone(E) of all subset sums of E



Lemma: For fixed d can compute zone(E) and normals in polynomial-time

(Edelsbrunner, Gritzmann, Orourk, Seidel, Sharir)

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Proof of Theorem 1.1: Convex Discrete Maximization

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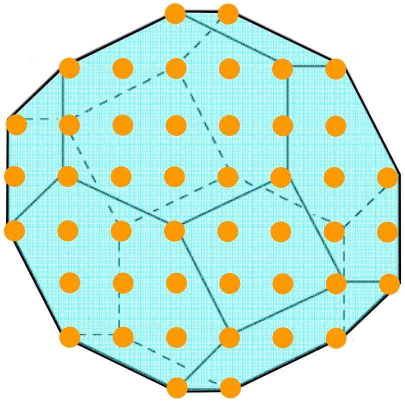
Proof of Theorem 1.1: Convex Discrete Maximization

R^n



- Set S in Z^n given by linear-optimization oracle

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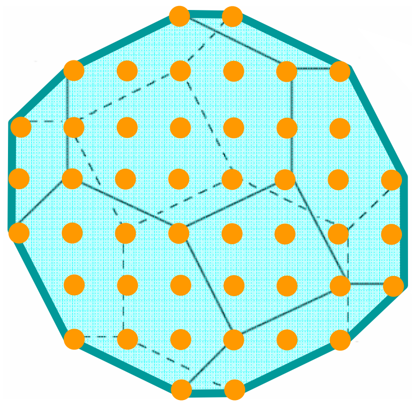
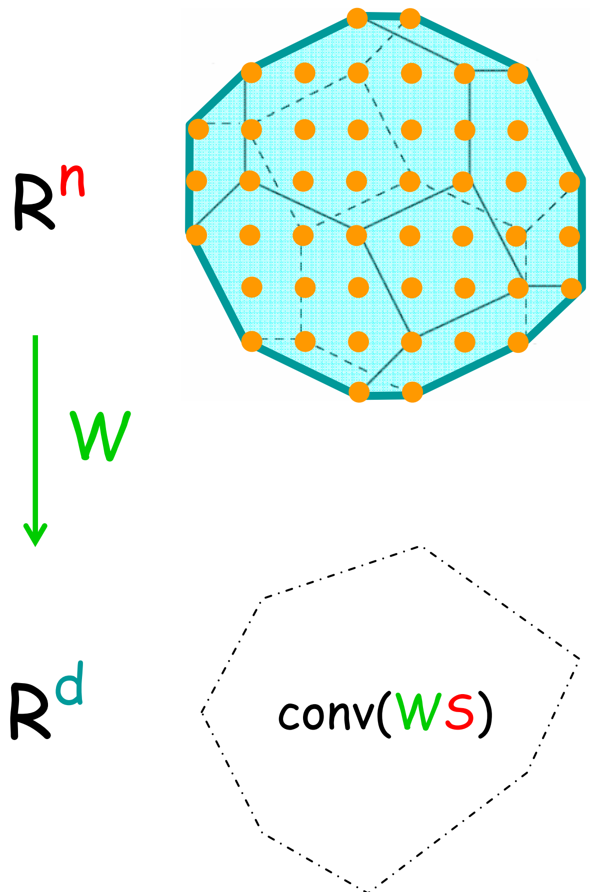
Proof of Theorem 1.1: Convex Discrete Maximization

R^n



- Set S in Z^n given by linear-optimization oracle
    - set E covering edge-directions of conv(S)

Proof of Theorem 1.1: Convex Discrete Maximization



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- Binary encoded $d \times n$ matrix W

Proof of Theorem 1.1: Convex Discrete Maximization

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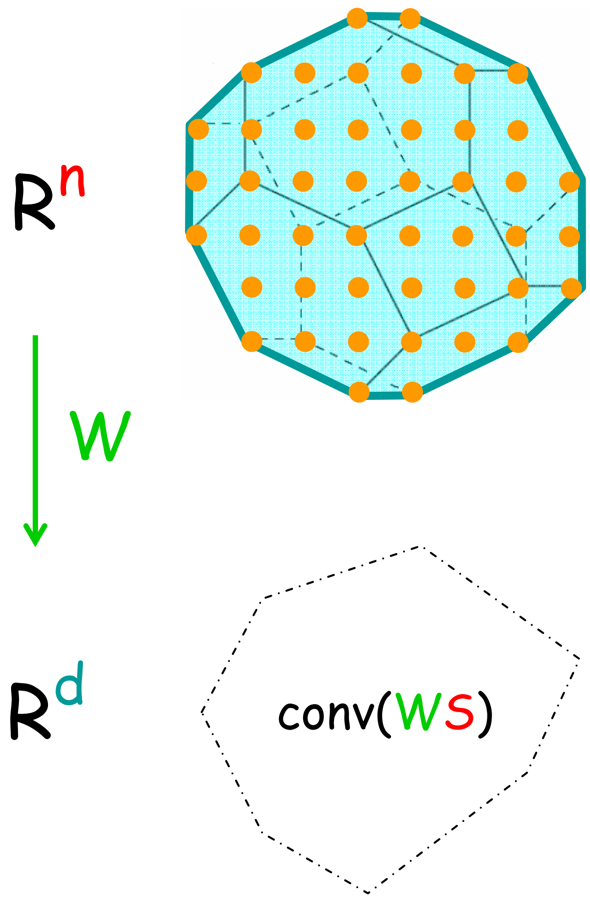
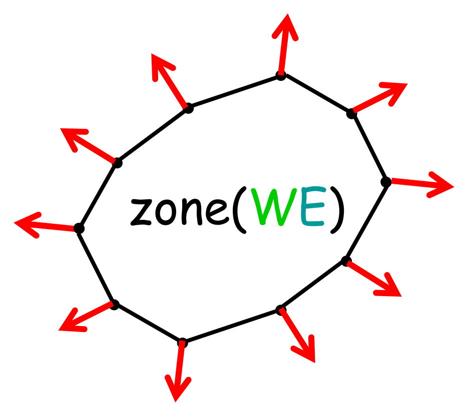


Construct zone(WE)



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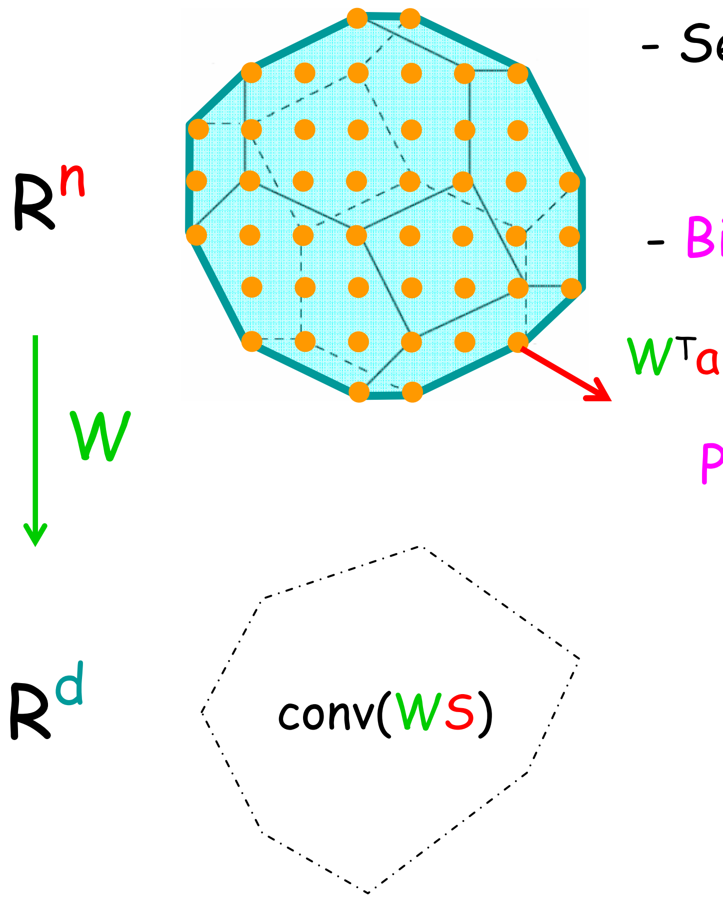
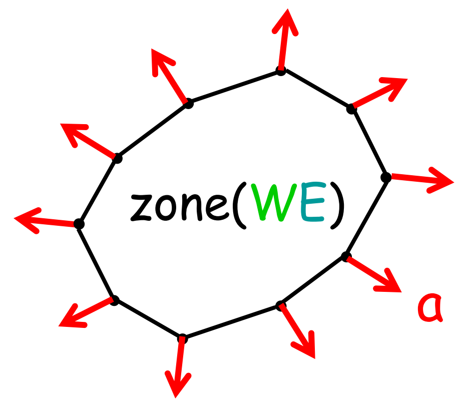


Pick a and Optimize W^Ta over S



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Proof of Theorem 1.1: Convex Discrete Maximization

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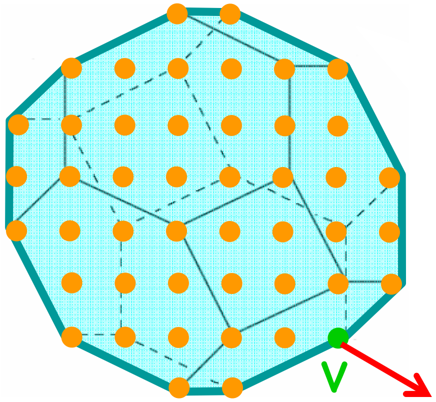
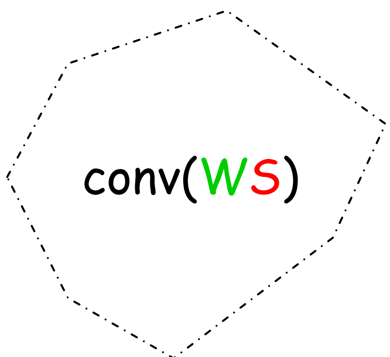
R^n

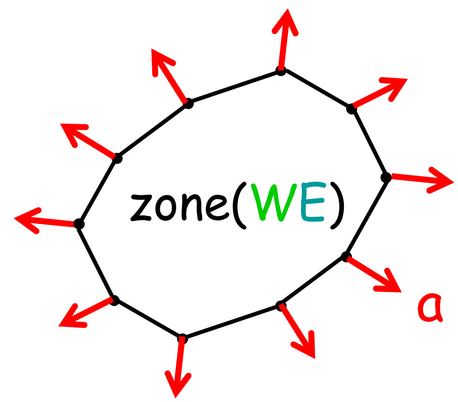


Get vertex v of conv(S)

W

R^d





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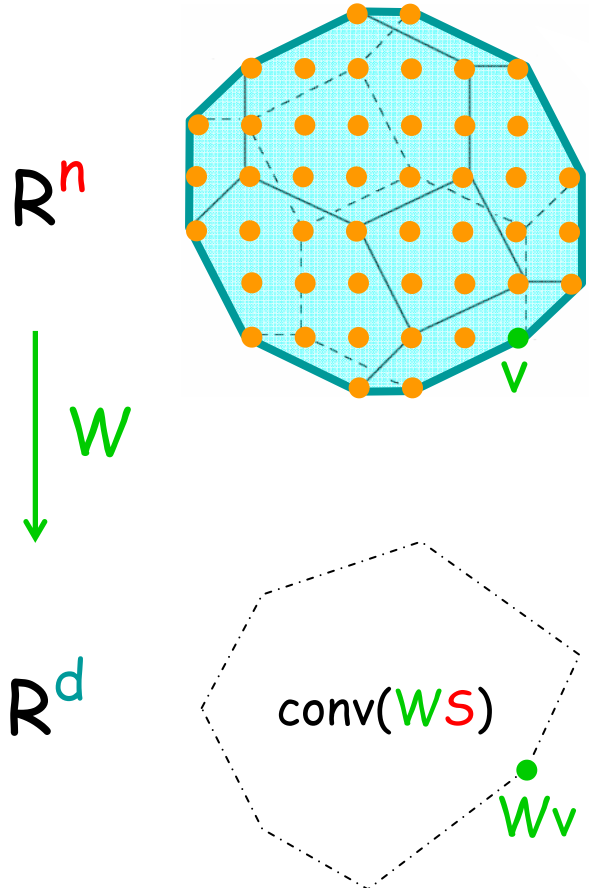

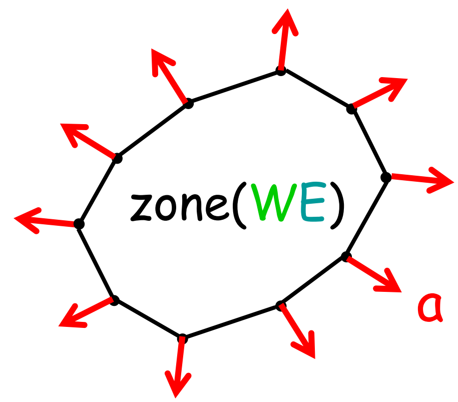

Get point Wv in conv(WS)

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Proof of Theorem 1.1: Convex Discrete Maximization

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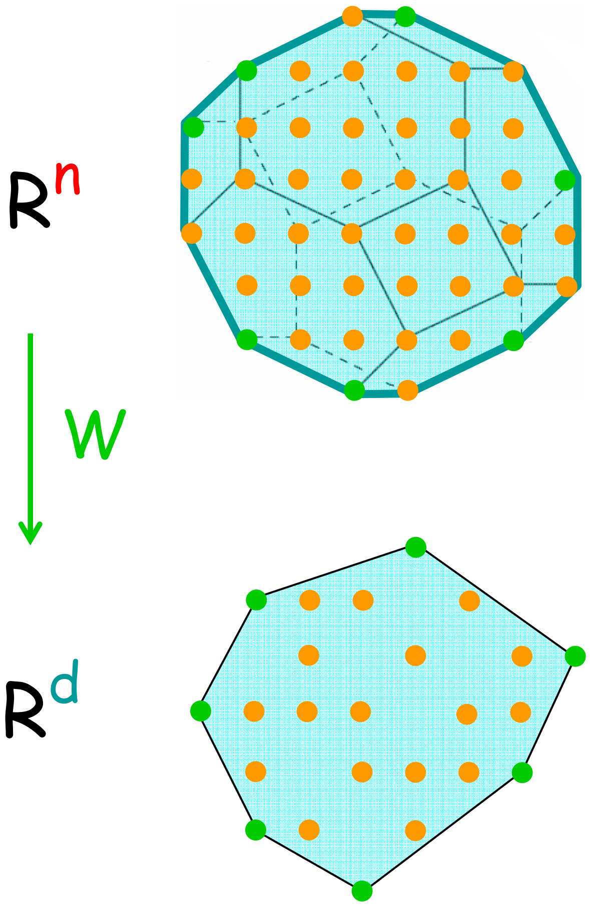

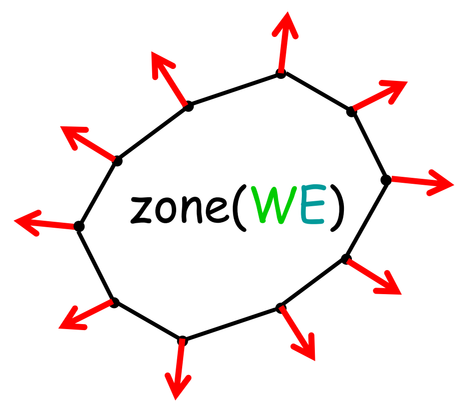

Repeat for all normals and Get vertices of conv(WS) and preimages in S

Since zone(WE) refines conv(WS) assured to get all vertices

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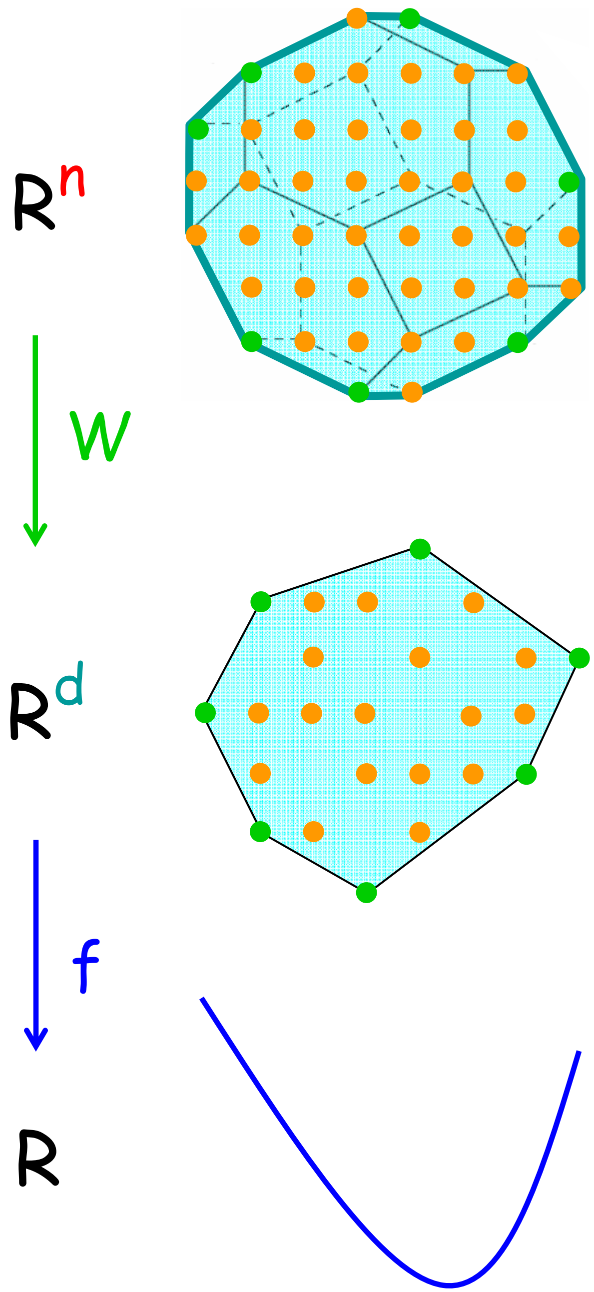
Proof of Theorem 1.1: Convex Discrete Maximization



- Set S in Z^n given by linear-optimization oracle
    - set E covering edge-directions of conv(S)
- Binary encoded $d \times n$ matrix W

- Convex function f on R^d to be maximized

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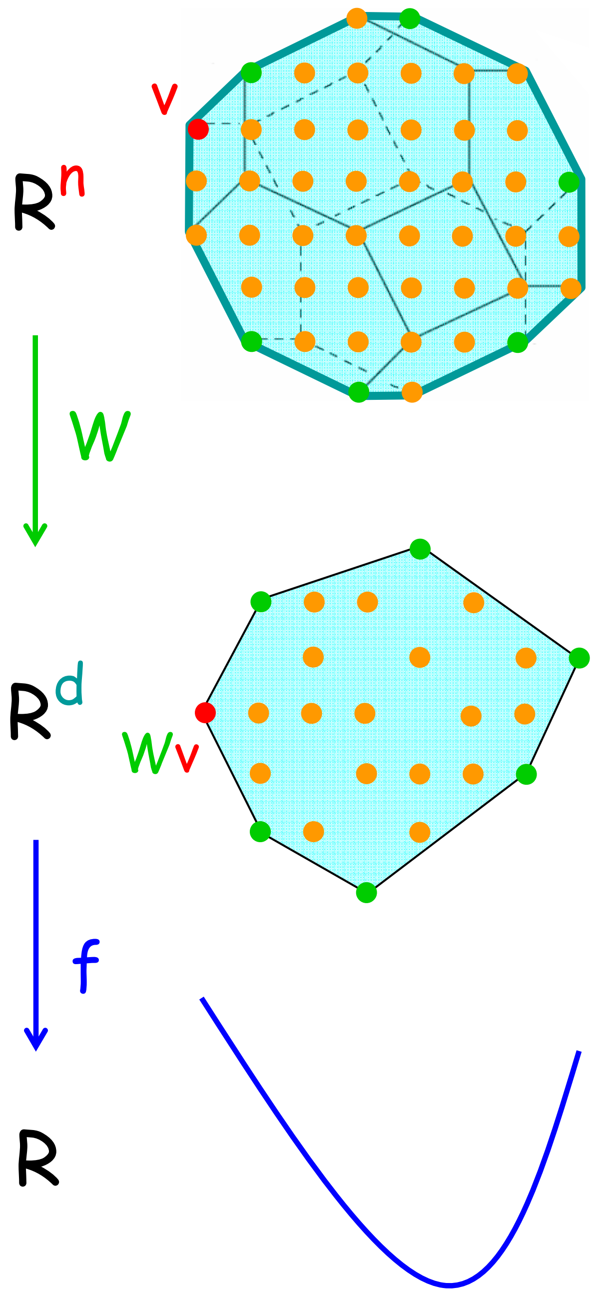

- Set S in Z^n given by linear-optimization oracle
  - set E covering edge-directions of conv(S)

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- Convex function f on R^d to be maximized

Output any preimage v in S maximizing $f(Wv)$

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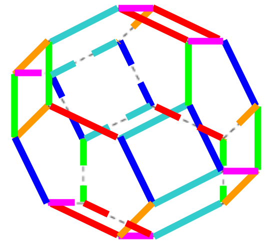
Convex Discrete Maximization - Some Applications



Some edge-behaved polytopes	And their applications
Cubes: unit vectors 1_i	e.g. 0-1 quadratic programming
Matroid polytopes: pairs $1_i - 1_j$ Also poly-matroids	e.g. spanning trees, experimental design
Multiway $m_1 \times \cdots \times m_k \times n$ transportation polytopes:	e.g. partitioning, clustering, multicommodity flows

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2. Nonlinear Integer Programming

Data:

- $S = \{x \text{ in } Z^N : A^{(n)}x = b,\ l \le x \le u\}$ given by inequalities
- W binary encoded
- f seperable (min), convex (max), given by comparison oracle

2. Nonlinear Integer Programming

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- f seperable (min), convex (max), given by comparison oracle

Methods: **mostly algebraic – Graver bases**

References: **De Loera, Hemmecke, Onn, Weismantel**

- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Convex integer maximization via Graver bases (J. Pure App. Algebra)
- Convex integer minimization (submitted)

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N-Fold Systems

The n-fold product of $(r+s) \times t$ matrix A is the following $(r+ns) \times nt$ matrix:

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} \quad .$$

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We have four theorems on n-fold integer programming:

Theorem 2.1: For fixed A, **linear integer programming** over n-fold products of A can be done in polynomial-time:

$$\max \{ wx : A^{(n)}x = b,\ l \le x \le u,\ x \text{ integer}\}$$

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N-Fold Systems

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We have four theorems on n-fold integer programming:

Theorem 2.2: For fixed A, **separable convex integer minimization** over n-fold products of A can be done in polynomial-time:

$$\min \{ \sum f_i(x_i) \; : \; A^{(n)}x = b, \; l \leq x \leq u, \; x \text{ integer} \}$$

N-Fold Systems

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We have four theorems on n-fold integer programming:

Theorem 2.3: For fixed A, **integer point l_p-nearest to a given x** over n-fold products of A can be found in polynomial-time:

$$\min \{ \, |x - x|_p \; : \; A^{(n)}x = b, \; l \le x \le u, \; x \text{ integer} \}$$

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N-Fold Systems

The n-fold product of $(r+s) \times t$ matrix A is the following $(r+ns) \times nt$ matrix:

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We have four theorems on n-fold integer programming:

Theorem 2.4: For fixed d and A, **convex integer maximization** over n-fold products of A can be done in polynomial-time:

$$\max \{ f(Wx) \ : \ A^{(n)}x = b, \ l \leq x \leq u, \ x \text{ integer} \}$$

Preliminaries on Graver Bases

The Graver basis of an integer matrix A is the finite set $G(A)$ of conformal-minimal nonzero integer vectors x satisfying $Ax = 0$.

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Lemma: For fixed A, can compute in polytime the Graver basis $G(A^{(n)})$ of

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(Use stabilization of $G(A^{(n)})$ due to Hosten, Santos, Sturmfels, Sullivant)

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Proof of Theorems 2.1 - 2.3
(linear and convex n-fold minimization)

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Proof of Theorems 2.1 - 2.3
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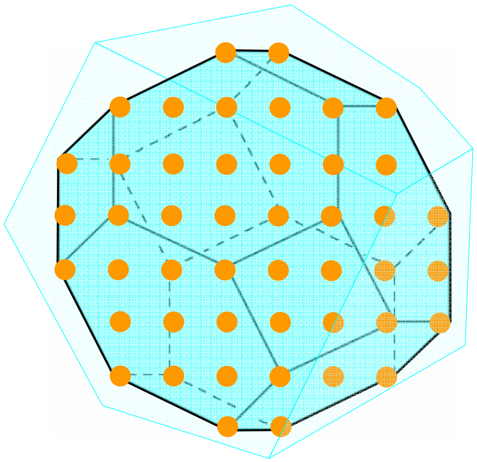
R^N



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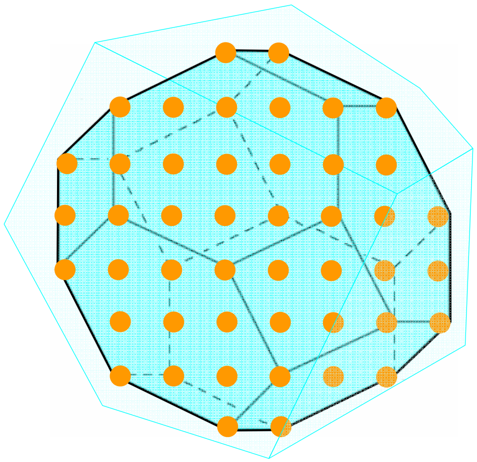

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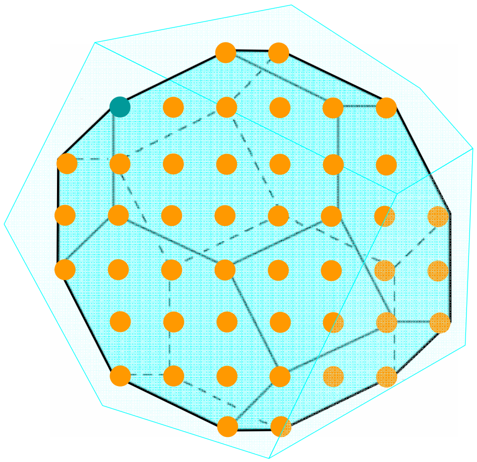

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  Find initial point by auxiliary n-fold program

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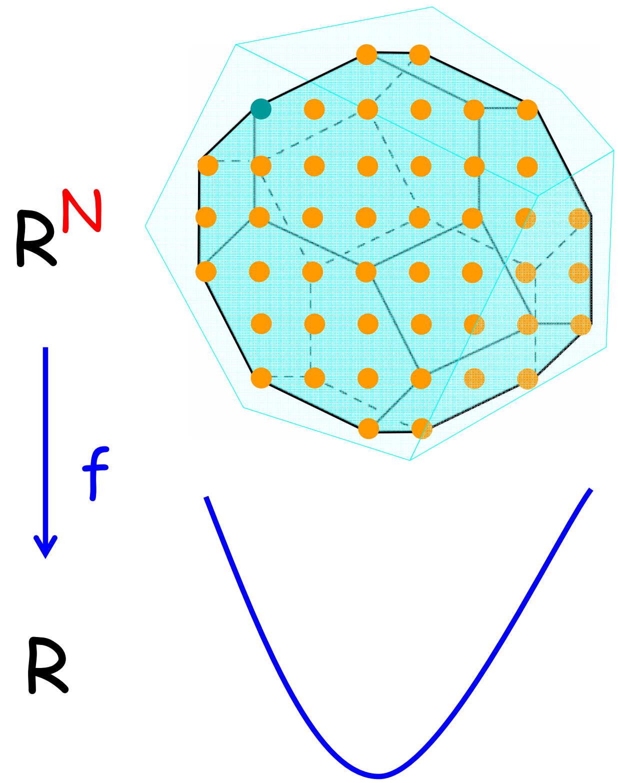

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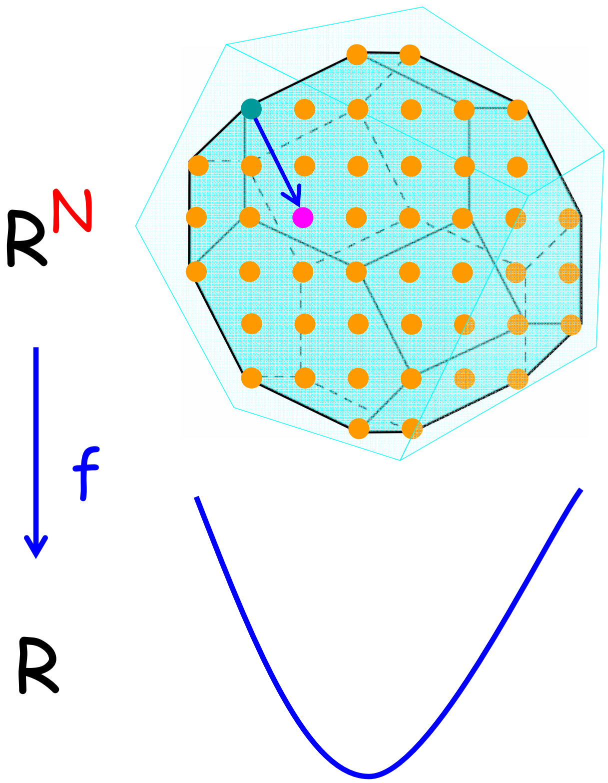

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  Iteratively greedily augment initial point
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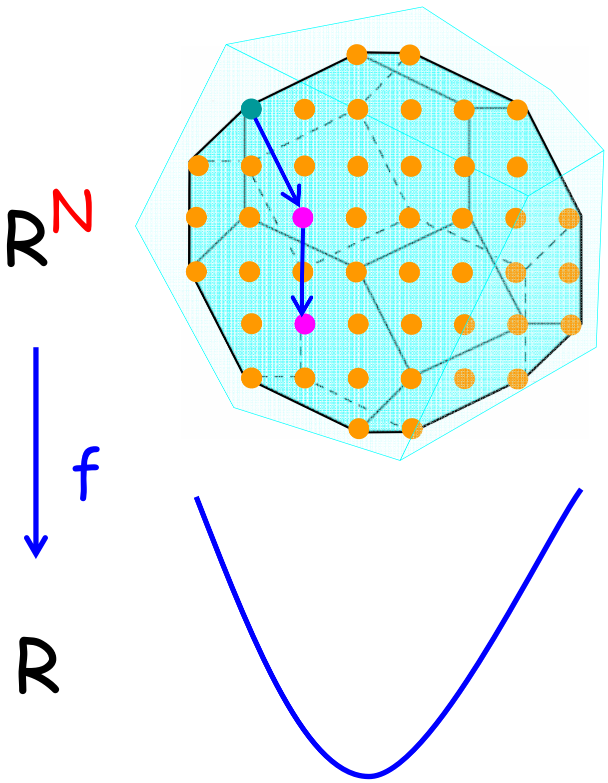

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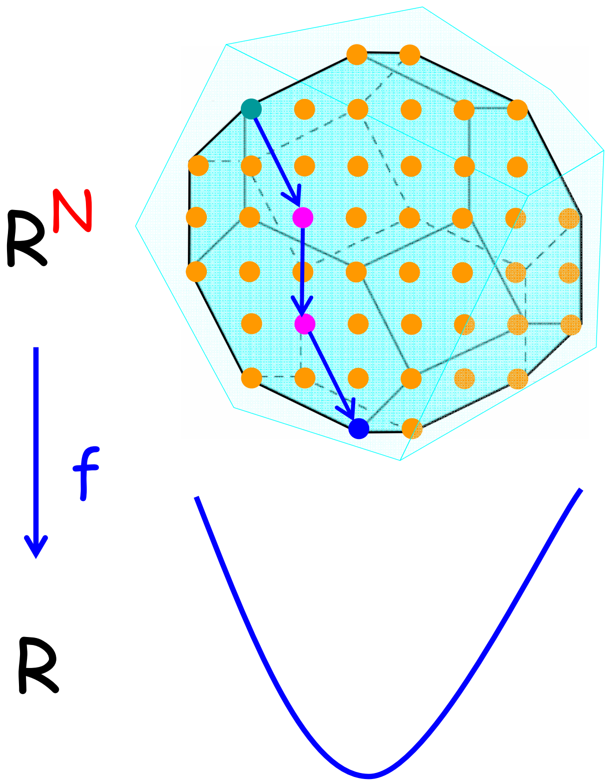

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Integer Caratheodory Theorem assures polynomial convergence

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Proof of Theorem 2.4
(convex n-fold maximization)

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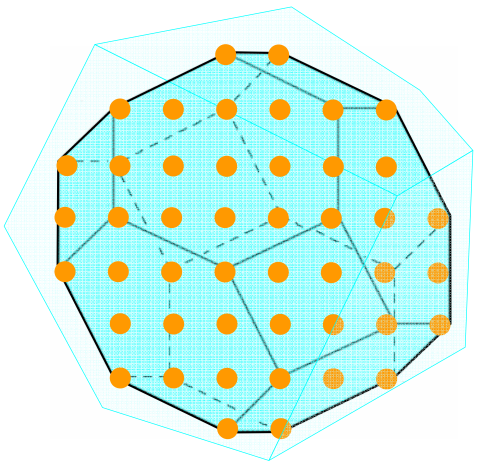
R^N



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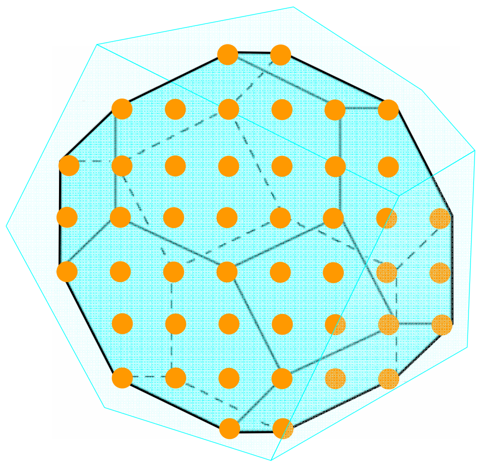

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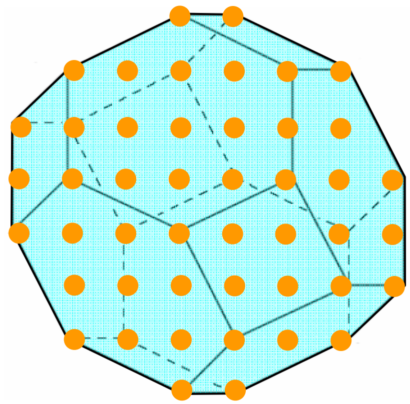

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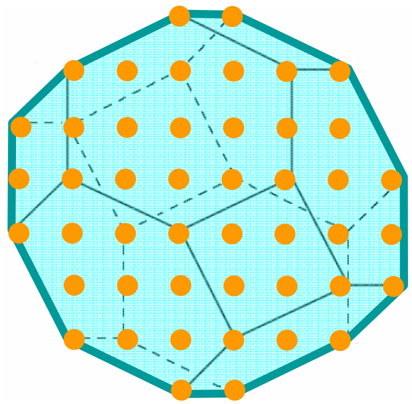

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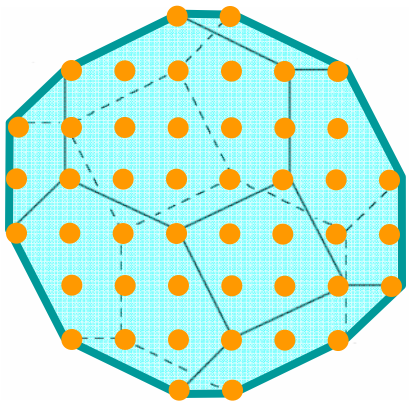

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Simulate linear-optimization oracle over S using Theorem 2.1

The Graver basis covers all edge-directions of conv(S)

Apply Theorem 1.1 on convex discrete maximization

Applications & Universality of N-Folds: Multiway Tables

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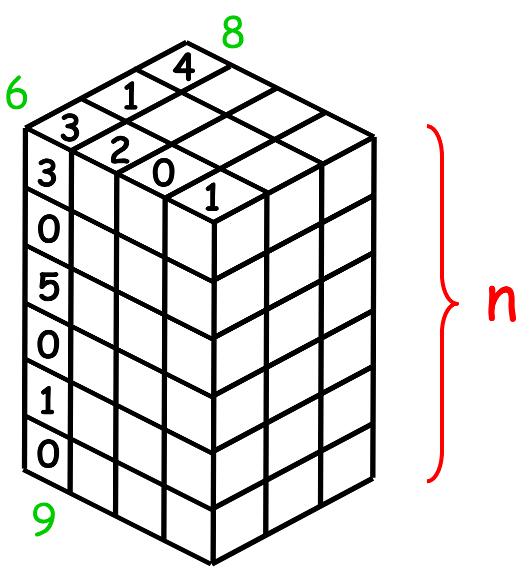
Applications & Universality of N-Folds: Multiway Tables

The $m_1 \times \cdots \times m_k \times n$ tables with given line-sums form an n-fold system

$S = \{x : A^{(n)}x = b,\ x \geq 0,\ x \text{ integer}\}$ for suitable A depending on $m_1, \ldots, m_k$



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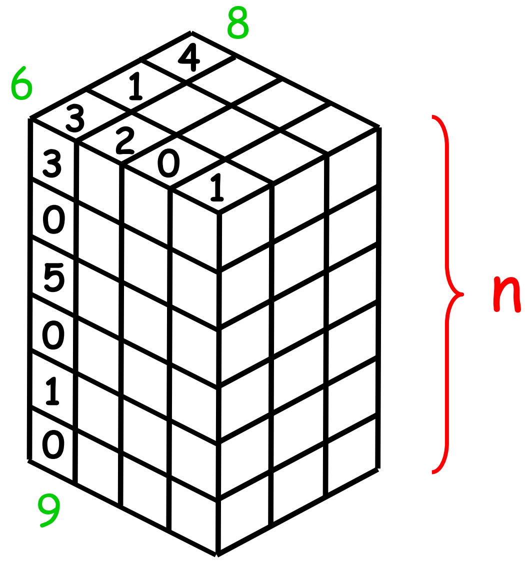

Corollary of Theorems 2.1 – 2.4: nonlinear discrete optimization over $m_1 \times \cdots \times m_k \times n$ tables with given line-sums can be done in polynomial-time

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Applications & Universality of N-Folds: Multiway Tables

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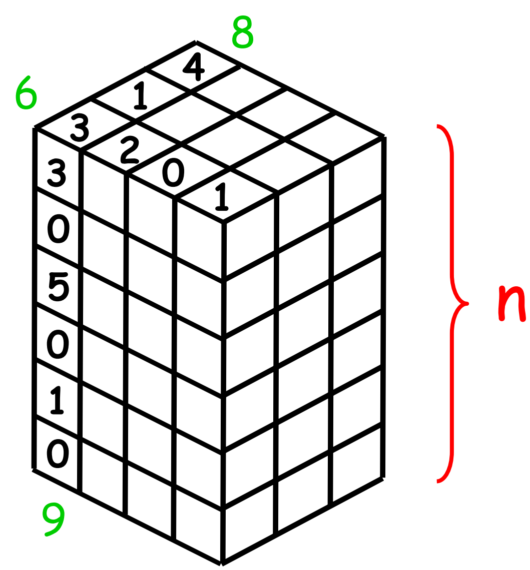

Corollary of Theorems 2.1 – 2.4: nonlinear discrete optimization over $m_1 \times \cdots \times m_k \times n$ tables with given line-sums can be done in polynomial-time

Universality Theorem for N-fold Systems (De Loera - Onn, SIAM Opt.)**:**
Every integer program is one over $3 \times m \times n$ tables with given line-sums

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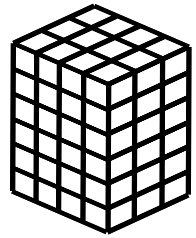
N-fold Systems and Multiway Tables – Some Applications



Some Applications	References
Privacy in statistical data-bases	Onn (Lect. Notes Comp. Sci.)
Multicommodity flows	Berstein - Onn (Annals Combin.)
Congestion-avoiding transportation	Hemmecke - Onn - Weismantel
Error-correcting codes	- || -
Universality - scheme for arbitrary nonlinear integer programming	De Loera - Onn (SIAM Opt.)

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3. Nonlinear Combinatorial Optimization

Data:

- S {0,1}-valued with combinatorial structure
  - e.g. matchings, matroid intersections, independence systems
- W unary encoded
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- S {0,1}-valued with combinatorial structure
  e.g. matchings, matroid intersections, independence systems
- W unary encoded
- f arbitrary given by comparison oracle

Methods: mostly algebraic - polynomial identities and interpolation, number theory (Frobenius)

References: **Berstein, Lee, Onn, Weismantel, Wynn**

- Nonlinear bipartite matching (Disc. Opt.)
- Nonlinear matroid optimization and experimental design (SIAM Disc. Math.)
- Nonlinear optimization for matroid intersection and extensions (submitted)
- Nonlinear optimization over a weighted independence system (submitted)

Nonlinear Combinatorial Optimization

Theorem 3.1: Fix any d. Then can solve in randomized polynomial-time

$$\min/\max \{ f(Wx) \; : \; x \text{ in } S \}$$

for S in $\{0,1\}^n$ bipartite matchings or matroid intersection, unary W, any f.

Proof of Theorem 3.1: Nonlinear Combinatorial Optimization

(very rough outline)

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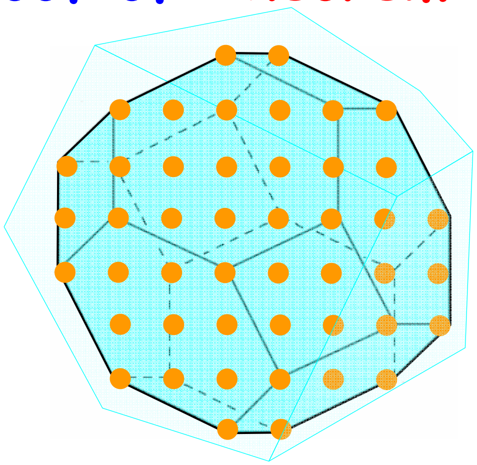
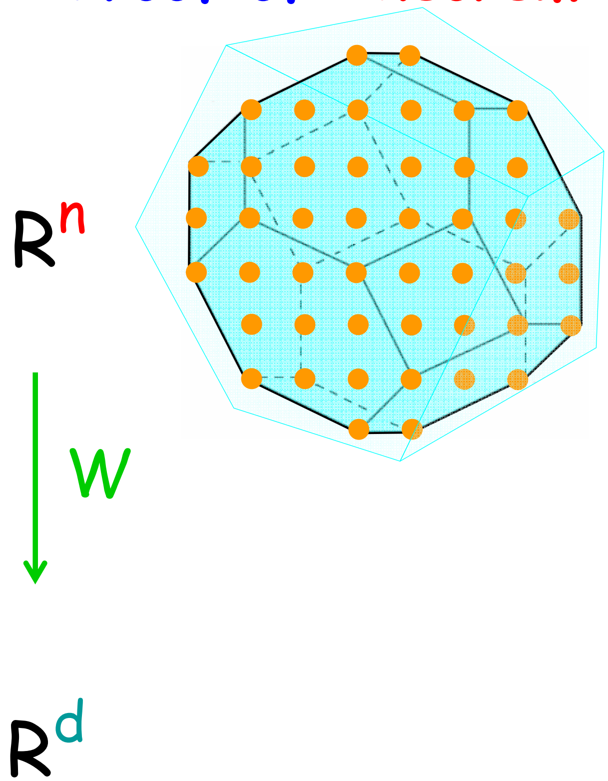
Proof of Theorem 3.1: Nonlinear Combinatorial Optimization

R^n



- Set S in $\{0,1\}^n$ - bipartite matchings in graph or intersection of matroids of two matrices

Proof of Theorem 3.1: Nonlinear Combinatorial Optimization



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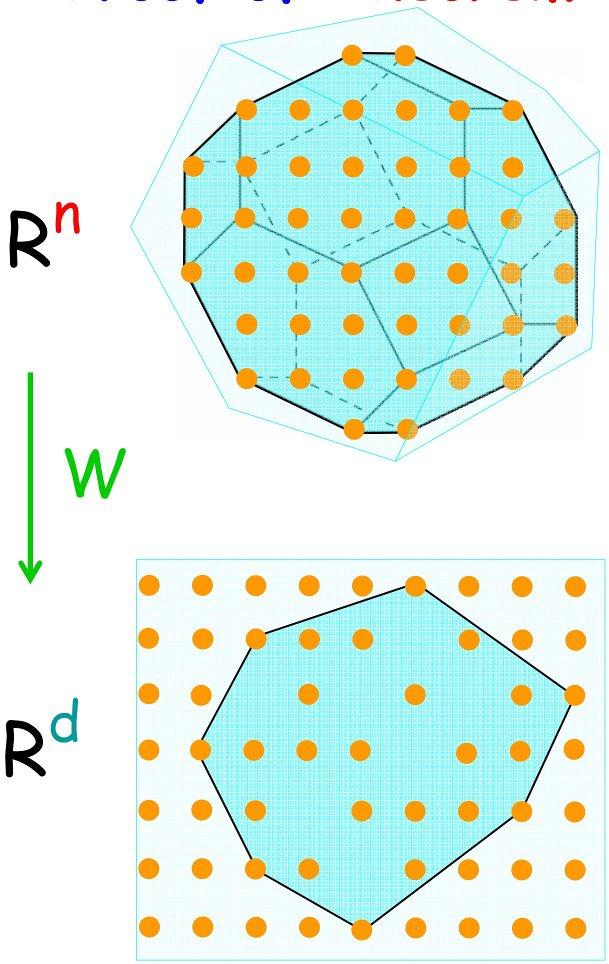
Proof of Theorem 3.1: Nonlinear Combinatorial Optimization

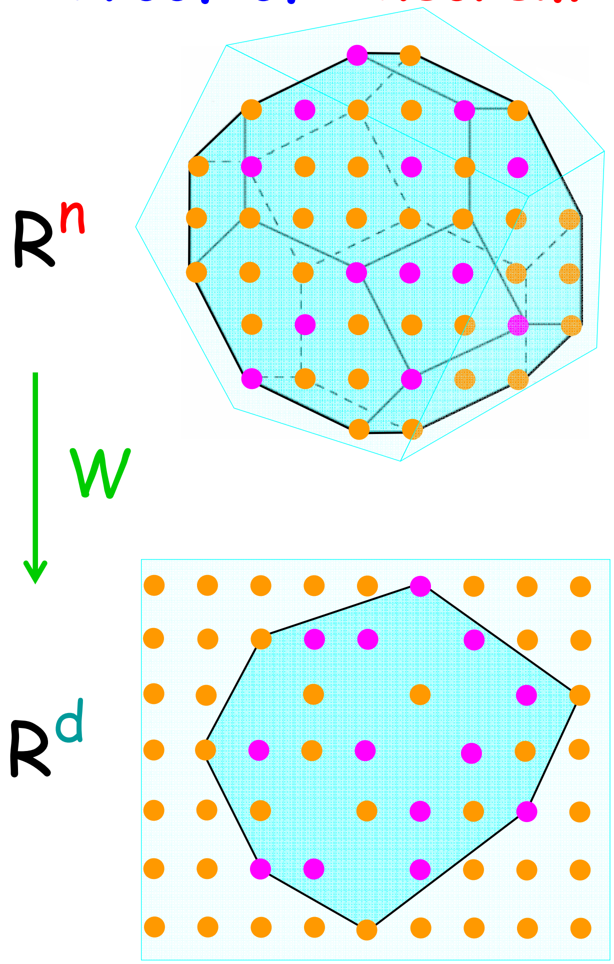


- Set S in $\{0,1\}^n$ - bipartite matchings in graph or intersection of matroids of two matrices

- Unary encoded $d \times n$ matrix W

Difficulty :
cannot check locally if any point is in WS

Proof of Theorem 3.1: Nonlinear Combinatorial Optimization

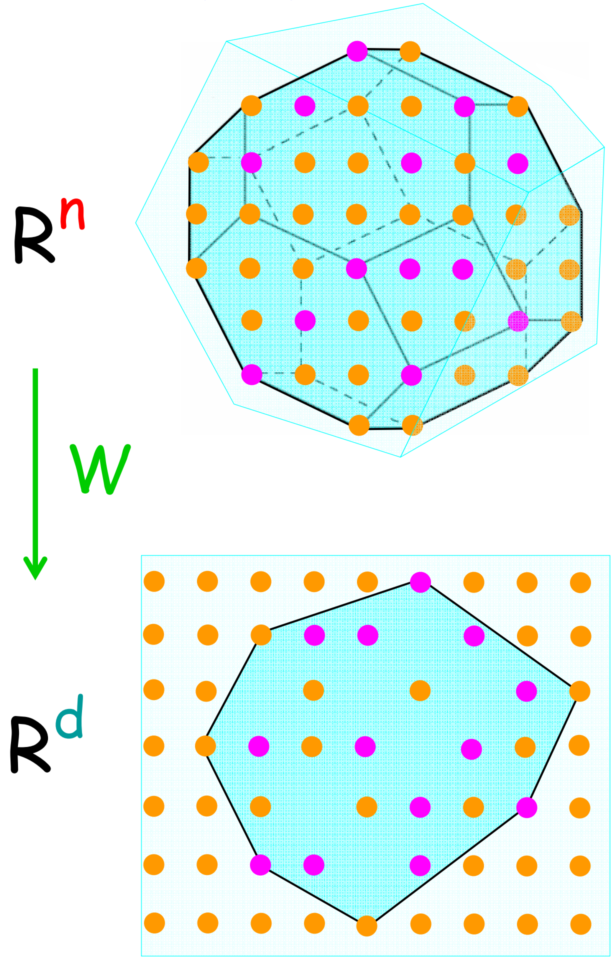


- Set S in $\{0,1\}^n$ - bipartite matchings in graph or intersection of matroids of two matrices

- Unary encoded $d \times n$ matrix W

Instead:
construct globally random subset of WS containing each point with high probability along with suitable random preimages in S

Proof of Theorem 3.1: Nonlinear Combinatorial Optimization



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Instead:

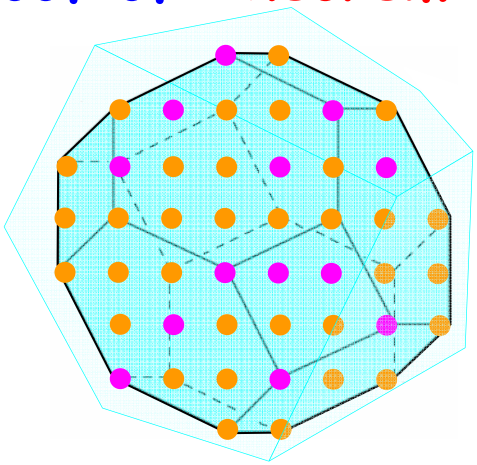
construct globally random subset of WS containing each point with high probability along with suitable random preimages in S

Use:

- Schwartz polynomial-identity testing
- Binet-Cauchy identity
- interpolation

Proof of Theorem 3.1: Nonlinear Combinatorial Optimization

R^n



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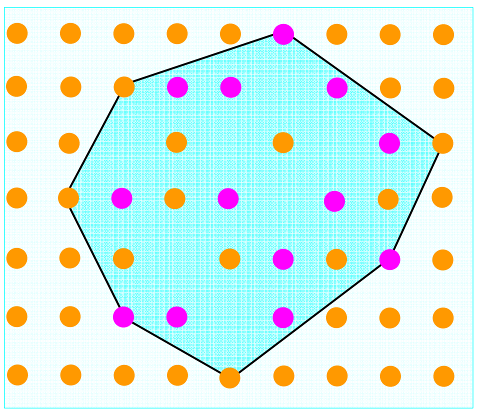
$\downarrow W$

R^d



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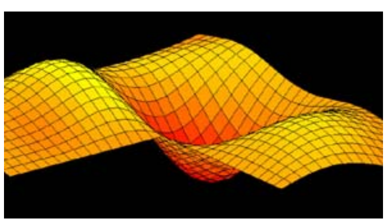
$\downarrow f$

R



- function f on R^d to be minimized

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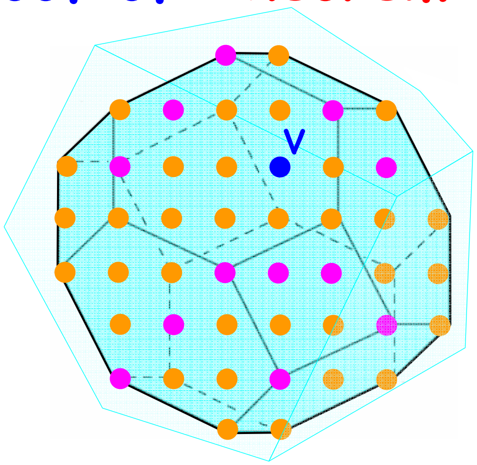
Proof of Theorem 3.1: Nonlinear Combinatorial Optimization

R^n



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$\downarrow W$

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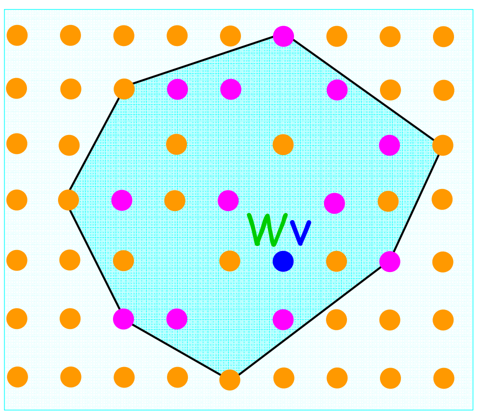
R^d



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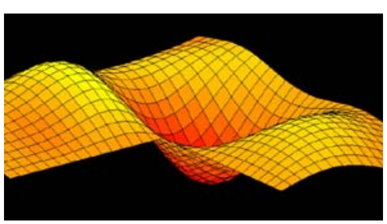
R



- function f on R^d to be minimized

Output v in S minimizing $f(Wv)$ on random subset

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Arbitrary Independence Systems: finding r-best solution

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Arbitrary Independence Systems: finding r-best solution

Theorem 3.2: For $d=1$ we can find in polynomial-time r-best solution of

$$\min/\max \{ f(Wx) \ : \ x \text{ in } S \}$$

for S independence system given by lin-opt oracle, W in $\{a_1, \ldots, a_p\}^n$, any f

Arbitrary Independence Systems: finding r-best solution

Theorem 3.2: For $d=1$ we can find in polynomial-time r-best solution of

$$\min/\max \{ f(Wx) \ : \ x \text{ in } S \}$$

for S independence system given by lin-opt oracle, W in $\{a_1, \ldots, a_p\}^n$, any f

Note:

A solution is r-best if there are at most r better objective values.

$r = r(a_1, \ldots, a_p)$ is constant, with $r(a_1, a_2) = F(a_1, a_2)$ the Frobenius number.

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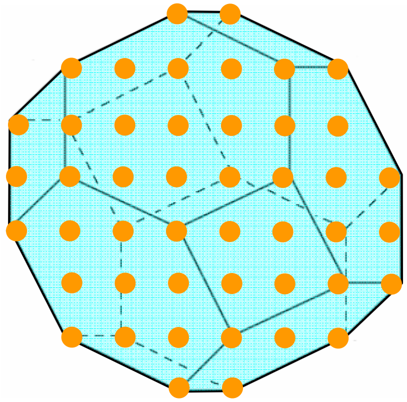
So for W in $\{2,3\}^n$ can efficiently find a 1-best solution. But amazingly:

Theorem 3.3: With data as above and W in $\{2,3\}^n$, finding an optimal (0-best) solution requires exponential-time.

Finding 1-best versus Optimal (0-best) Solution

Finding 1-best versus Optimal (0-best) Solution

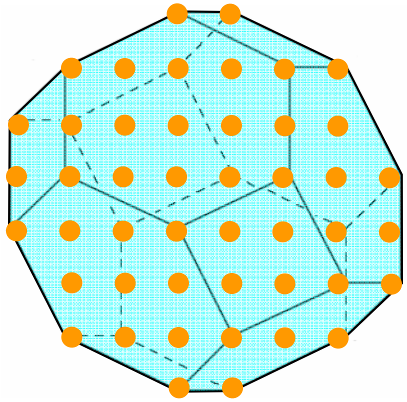
R^n



- Set S in $\{0,1\}^n$ independence system
  given by linear-optimization oracle

Finding 1-best versus Optimal (0-best) Solution

R^n



$\downarrow W$

R

- Set S in $\{0,1\}^n$ independence system
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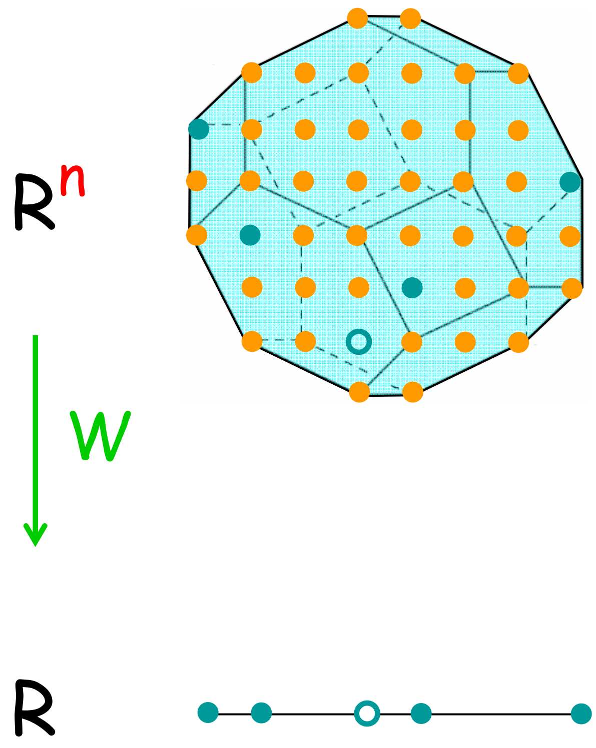
- Weight vector W in $\{2,3\}^n$

Finding 1-best versus Optimal (0-best) Solution



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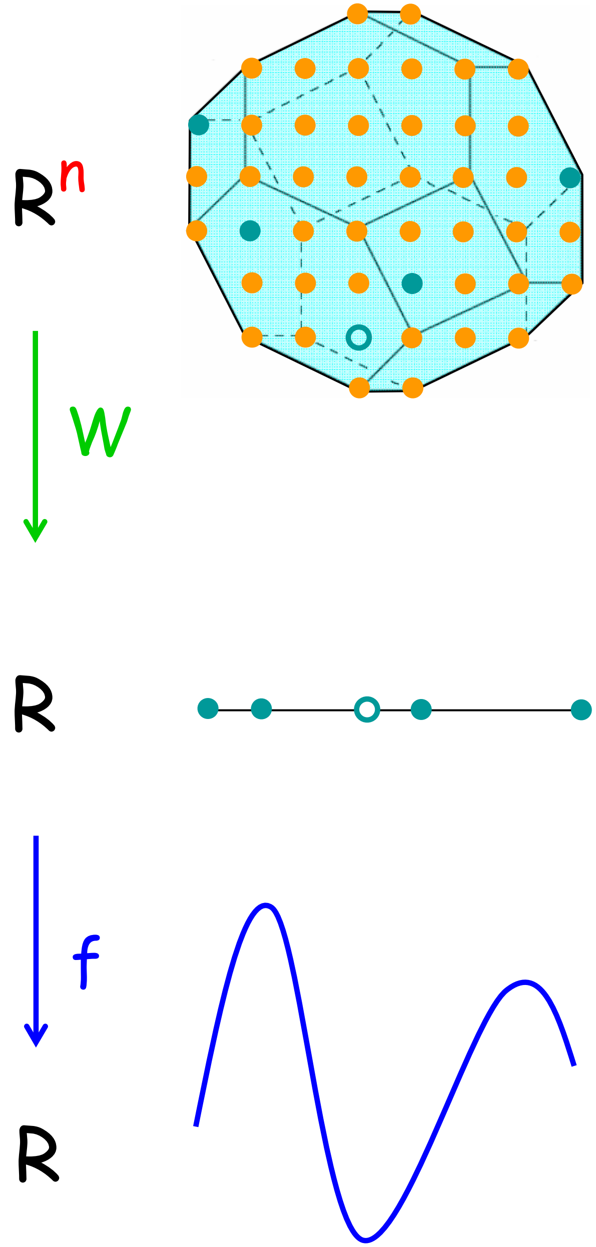
Can compute in polynomial-time
the image WS in R up to a point
But need exponential-time
to compute WS exactly

Finding 1-best versus Optimal (0-best) Solution



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- Weight vector W in $\{2,3\}^n$

Can compute in polynomial-time the image WS in R up to a point
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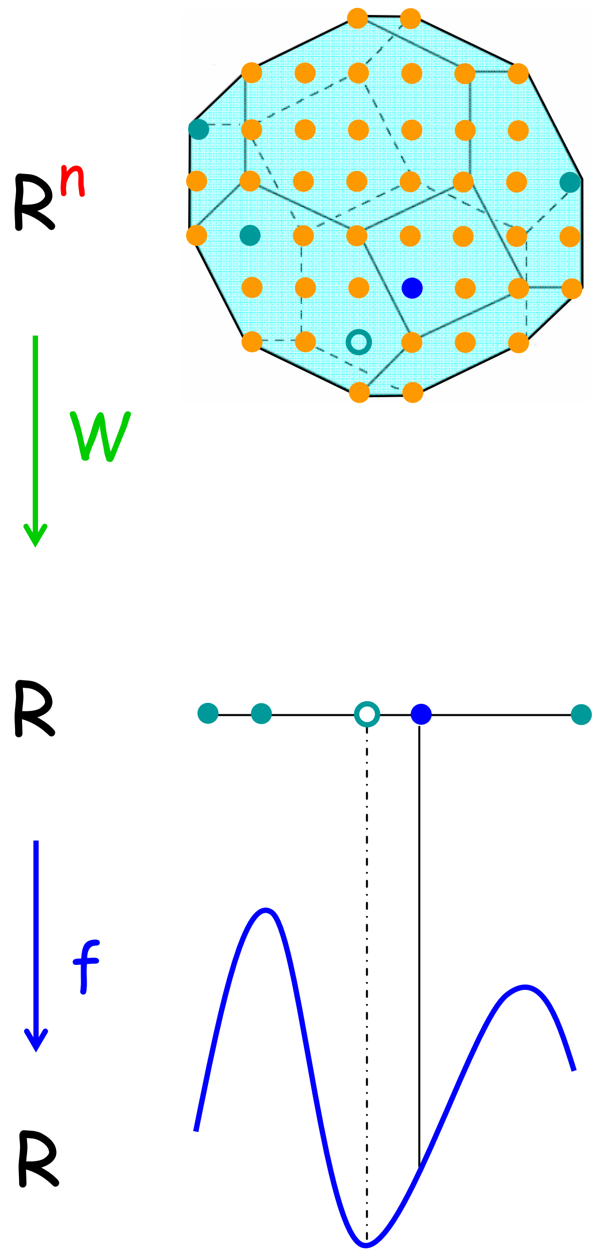
- Function f on R to be minimized

Shmuel Onn

Finding 1-best versus Optimal (0-best) Solution



- Set S in $\{0,1\}^n$ independence system given by linear-optimization oracle

- Weight vector W in $\{2,3\}^n$

  Can compute in polynomial-time the image WS in R up to a point
  But need exponential-time to compute WS exactly

- Function f on R to be minimized

Can get 1-best but not optimal solution

Shmuel Onn

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