Nonlinear Discrete Optimization

Robert Weismantel

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The many aspects of nonlinear discrete optimization

To begin with

$$\begin{array}{rl} \text{max/min} & f(x_1,\dots,x_n) \\ \text{subject to} & (x_1,\dots,x_n) \in P \cap \mathbf{Z}^n. \end{array}$$

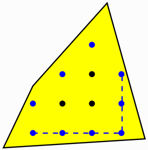
Parametric non-linear optimization

A borderline case from the point of view of computational complexity

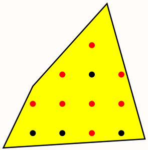
$$\begin{array}{rl} \max & f(Wx) \\ \text{s.t.} & x \in P \cap \mathbf{Z}^n \end{array}$$

with $W \in \mathbf{Z}^{m \times n}$ with fixed m. "Maps high-dimensional problems to fixed dimension." [Onn, Rothblum '05]

Convex maximization



Polynomial optimization



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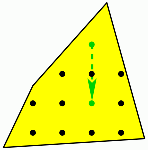

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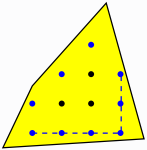

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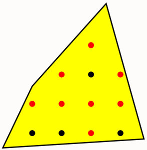

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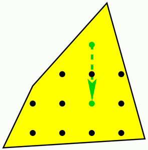

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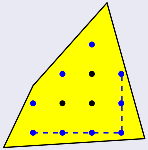

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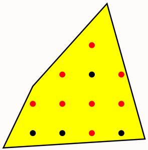

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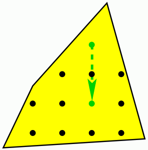

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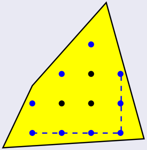

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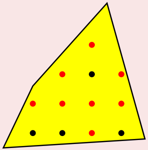

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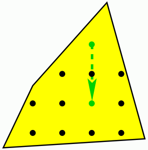

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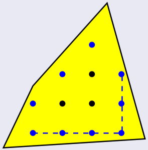

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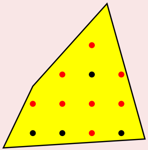

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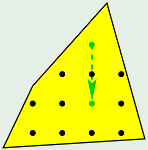


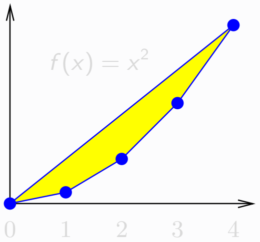
To get started: Convex integer maximization

Problem type

$$\begin{array}{rl} \max & f(x_1, \ldots, x_d) \\ \text{subject to} & (x_1, \ldots, x_d) \in P \cap \mathbf{Z}^d, \end{array}$$

where

- P is a polytope with $P \subseteq B$ (box),
- $f : \mathbf{Z}^d \rightarrow \mathbf{R}$ is convex



Observation

Let

$$G(f) = \text{conv}\{ (y, f(y)) : y \in B \cap \mathbf{Z}^d \}.$$

f is convex over B
$\iff$ every $(x, \pi) \in G(f)$ satisfies $f(x) \leq \pi$.

Reduction Scheme

$$\max_{x \in P \cap \mathbf{Z}^d} f(x) = \min_{\substack{x \in P \cap \mathbf{Z}^d \\ f(x) \leq \pi}} \pi = \min_{\substack{x \in P \cap \mathbf{Z}^d \\ (x,\pi) \in G(f)}} \pi$$

$G(f)$ is a polyhedron describing the nonlinearities!

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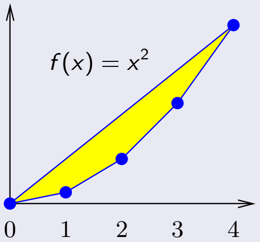

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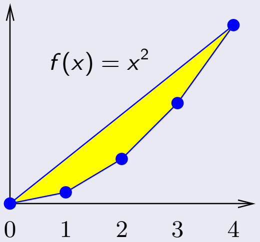

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Grey areas of computational complexity

Objective function	Variables and constraints: Fixed dimension	Combinatorial	Arbitrary
Convex max	Cook, Hartmann, Kannan, McDiarmid '89		Reduction technique to ILP: Michaels '07
Convex min	Khachiyan, Porkolab '00		Outer approximation: Grossmann '86; Fletcher, Leyffer '94
Parametric			
Polynomial			SOS programming [Shor '87; Parrilo '03; Laurent '01], Positivstellensatz [Putinar '93]
Arbitrary			

The setting

$$\begin{aligned} \max \quad & f(Wx) \\ \text{s.t.} \quad & x \in P \cap \mathbf{Z}^n \end{aligned}$$

with $W \in \mathbf{Z}^{m \times n}$ with fixed m.

Would like to work with

$$\begin{aligned} \max \quad & f(w) \\ \text{s.t.} \quad & w \in WP \cap \mathbf{Z}^m; \end{aligned}$$

Observation

No control over the holes if W is arbitrary and encoded in binary

(Reduction to Subset Sum)

Parametric nonlinear optimization

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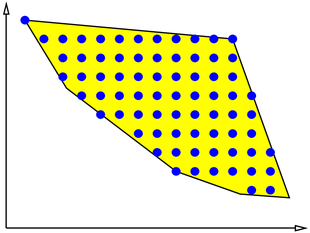

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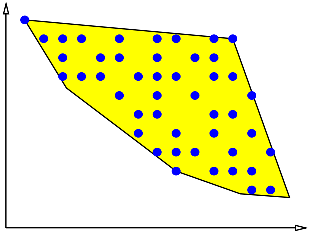

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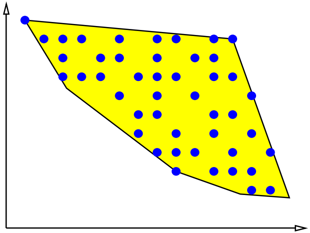

Independence system

$S \subseteq 2^N$ closed under taking subsets

endowed with an oracle for optimizing linear functions over S.

Problem

Let $m = 1$, $w \in \{a_1, \ldots, a_p\}^n$.

Consider

$$\begin{array}{ll} \max & f(w^\top x) \\ \text{s.t.} & x \in S \end{array}$$

Algorithmic Framework

Step 1: **Define** appropriate upper bounds λ_i for $i = 1, \ldots, p$.

Step 2: **For all** $S_1 \subseteq N_1, \ldots, S_p \subseteq N_p$ such that $|S_i| \leq \lambda_i$:

2.1 **Determine**

$$\max w^\top x \text{ s.t. } \quad x \in S, \quad x_j = 1 \text{ for all } j \in S_i \text{ and } i = 1, \ldots, p.$$

Let $x(S_1, \ldots, S_p)$ denote the optimal solution.

2.2 **Determine** a sequence of at most n vectors

$$x^0 = x(S_1, \ldots, S_p) \geq x^1 \geq \ldots \geq x^t \in \mathbf{Z}^n_+$$

and select the one, $x'(S_1, \ldots, S_p)$ say, attaining

$$\max\{f(wx^i) \mid i = 0, \ldots, t\}.$$

Step 3: **Return** x' attaining

$$\begin{array}{ll} \max & f\left(wx'(S_1, \ldots, S_p)\right) \\ \text{s.t.} & S_1 \subseteq N_1, \ldots, S_p \subseteq N_p, \ |S_i| \leq \lambda_i. \end{array}$$

Let S be a matroid

$m = 1,\ p = 2,\ a_1 = 0,\ a_2 = 1$

$$\begin{array}{ll} \max & f(w^\top \chi^I) \\ \text{s.t.} & I \in S \end{array} \quad \text{reduces to} \quad \begin{array}{ll} \max & w^\top \chi^I \\ \text{s.t.} & I \in S \end{array}$$

Theorem 1 (Lee, Onn, W. '07)

For every fixed m and p, there is an algorithm that, given $a_1, \ldots, a_p \in \mathbf{Z}$, $W \in \{a_1, \ldots, a_p\}^{m \times n}$, and a function $f : \mathbf{R}^n \to \mathbf{R}$, finds a matroid base B minimizing $f(W\chi^B)$ in time polynomial in n and $\langle a_1, \ldots, a_p \rangle$.

Iterated matroid intersection, Edmonds '79

Theorem 2 (Lee, Onn, W. '07)

Let $a_1 < \ldots < a_p \in \mathbf{Z}_+$ be a sequence of divisible integers. Letting $q = \max\{\frac{a_{i+1}}{a_i} \mid i = 1, \ldots, p-1\}$, the algorithm solves the univariate optimization problem over an independence system by $O(n^{pq})$ calls of a linear optimization oracle.

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Some results along these lines

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A combinatorial example: Integer convex minimization

Convex N-fold integer minimization

$$\min \left\{ \sum_{i=1}^{N} f^{(i)}\left(x^{(i)}\right) : \sum_{i=1}^{N} x^{(i)} = b^{(0)}, Ax^{(i)} = b^{(i)}, 0 \le x^{(i)} \le u^{(i)}, x^{(i)} \in \mathbf{Z}^n, i = 1, \ldots, N \right\},$$

where $f^{(i)}(x^{(i)}) := \sum_{j=1}^{s} f_j^{(i)}\left(c_j^\intercal x^{(i)}\right)$ with convex functions $f_j^{(i)} : \mathbf{R} \to \mathbf{R}$

Ingredient 1: Superadditivity of objective function

- Let $g : \mathbf{R} \to \mathbf{R}$ be a convex function and let $z, y_1, \ldots, y_r$ be real numbers.
- If $y_1, \ldots, y_r$ have same signs:

$$g(z) - g\left(z + \sum_{j=1}^{r} y_j\right) \le \sum_{j=1}^{r} \left[g(z) - g\left(z + y_j\right)\right].$$

(Murota, Saito, W. '04)

$$\begin{pmatrix} I_n & I_n & \cdots & I_n \\ A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots & A \end{pmatrix}$$

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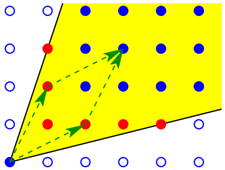
Ingredient 2: Graver bases (Graver, 1975)

Hilbert bases and Graver bases

- **Hilbert basis** HB(C) of a rational polyhedral cone $C \subseteq \mathbf{R}^n$ is a set of integer vectors of C that generate all integer vectors of C by non-negative integer linear combinations
- **Graver basis** $\mathcal{G}(A)$ of a matrix $A \in \mathbf{Z}^{d \times n}$ is the union $\bigcup_j \mathrm{HB}(\ker(A) \cap \mathcal{O}_j)$ over all 2^n orthants of $\mathbf{R}^n$.



Integer linear programs

Graver basis $\mathcal{G}(A)$ provides an optimality certificate for linear integer programs (Graver '75)

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbf{Z}^n_+ \end{aligned}$$

Convex integer minimization problems

Graver basis $\mathcal{G}\begin{pmatrix} A & I_n & 0 \\ C & 0 & -I_s \end{pmatrix}$ provides an optimality certificate for convex integer minimization problems

$$\min\left\{ \sum_{i=1}^{s} g_i(c_i^\top z) : Az \le b,\ z \in \mathbf{Z}^n_+ \right\}$$

with convex functions $g_i : \mathbf{R} \to \mathbf{R}$ (Murota, Saito, W. '04).

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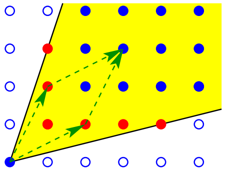

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Graver basis $\mathcal{G}\begin{pmatrix} A & I_n & 0 \\ C & 0 & -I_s \end{pmatrix}$ provides an optimality certificate for convex integer minimization problems

$$\min\left\{ \sum_{i=1}^{s} g_i(c_i^\intercal z) : Az \leq b,\ z \in \mathbf{Z}^n_+ \right\}$$

with convex functions $g_i : \mathbf{R} \to \mathbf{R}$ (Murota, Saito, W. '04).

Ingredient 3: Efficient augmentation
(cf. Ahuja, Magnanti, Orlin '93; Schulz, W. '99)

Theorem (Santos, Sturmfels '03)

The Graver basis $\mathcal{G}$ of

$$A^{(N)} := \begin{pmatrix} I_n & I_n & \cdots & I_n \\ A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots & A \end{pmatrix}$$

increases only polynomially in N.

Integer line search algorithm

To solve $\min\{f(z_0 + \alpha t) : \alpha \in \mathbf{Z}_+\}$ for convex f, use binary search with a comparison oracle for f $\rightarrow$ polynomially many bisection steps

Greedy augmentation algorithm

1. Find feasible solution z_0.
2. While z_0 is not optimal, find $t \in \mathcal{G}$ and $\alpha \in \mathbf{Z}_+$ such that
   - $z_0 + \alpha t$ is a feasible and
   - $f(z_0 + \alpha t)$ is minimal.
3. Return z_0 as optimal solution.

Theorem (Hemmecke, Onn, W. '07)

For fixed A, C, convex N-fold integer minimization is polynomial-time solvable.

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General polynomial optimization

Problem type

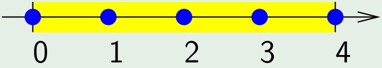
$$\begin{array}{rl} \max & f(x_1, \ldots, x_n) \\ \text{subject to} & (x_1, \ldots, x_n) \in P \cap \mathbf{Z}^n, \end{array}$$

where

- P is a polytope,
- f is a polynomial function non-negative over $P \cap \mathbf{Z}^n$,
- the dimension n is fixed.

Ingredient 1: Generating functions

$$g_P(z_1, \ldots, z_n) = \sum_{\alpha \in P \cap \mathbf{Z}^n} z^\alpha$$



$$g_P(z) = z^0 + z^1 + z^2 + z^3 + z^4$$

Theorem (Barvinok, 1994)

Let the dimension n be fixed. There is a polynomial-time algorithm for computing $g_P(z)$ of $P \cap \mathbf{Z}^n$ in the form of a rational function. In particular, $N = |P \cap \mathbf{Z}^n| = g_P(1)$ can be computed in polynomial time.

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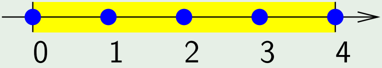

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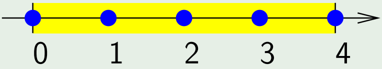

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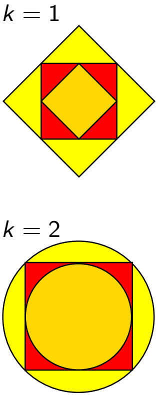
Approximation properties of ℓ_p norms

Let

$$P \cap \mathbf{Z}^n = \{x^1, \ldots, x^N\} \qquad \text{and} \qquad f = \begin{pmatrix} f(x^1) \\ \vdots \\ f(x^N) \end{pmatrix} \in \mathbf{R}^N.$$

Then:

$$L_k := N^{-1/k} \|f\|_k \le \|f\|_\infty \le \|f\|_k =: U_k$$



Convergence

$$U_k - L_k \le \left(\sqrt[k]{N} - 1\right) f(x^{\max}) \le \epsilon f(x^{\max}) \text{ for } k \ge (1 + 1/\epsilon)\log N.$$

Complexity

encoding length and degree of f^k are bounded polynomially in the input size and $\frac{1}{\epsilon}$

How to evaluate?

- $\|f\|_k = \left(\sum_{x \in P \cap \mathbf{Z}^n} f^k(x)\right)^{1/k}$

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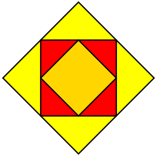

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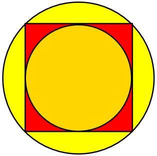

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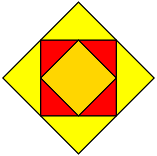

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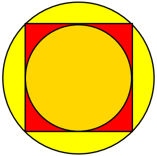

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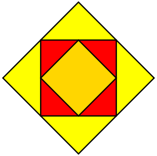

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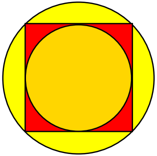

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Ingredient 3: Differential operators on generating functions

Euler differential operator

$$g_P(z) = z^0 + z^1 + z^2 + z^3 + z^4$$

Apply differential operator:

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) g_P(z) = 1z^1 + 2z^2 + 3z^3 + 4z^4$$

Apply differential operator again:

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) g_P(z) = 1z^1 + 4z^2 + 9z^3 + 16z^4$$

Theorem (De Loera, Hemmecke, Köppe, W., 2004 / Barvinok, 2004)

Let $g_P(z)$ be the Barvinok representation of the generating function of the lattice points of P. Let f be a polynomial in $\mathbf{Z}[x_1, \ldots, x_n]$ of maximum total degree D.

We can compute, in time polynomial in D and the size of the input data, a Barvinok rational function representation $g_{P,f}(z)$ for the function $\sum_{\alpha \in P \cap \mathbf{Z}^n} f(\alpha) z^\alpha$.

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$$g_P(z) = z^0 + z^1 + z^2 + z^3 + z^4 = \frac{1}{1-z} - \frac{z^5}{1-z}$$

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$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right) g_P(z) = 1z^1 + 2z^2 + 3z^3 + 4z^4 = \frac{1}{(1-z)^2} - \frac{-4z^5 + 5z^4}{(1-z)^2}$$

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Taking all together

Theorem (De Loera, Hemmecke, Köppe, W., 2006)

Let the dimension n be fixed. There exists a fully polynomial-time approximation scheme for the problem

$$\begin{aligned} \max \quad & f(x_1, \ldots, x_n) \\ \text{subject to} \quad & (x_1, \ldots, x_n) \in P \cap \mathbf{Z}^n, \end{aligned}$$

where

- P is a polytope,
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Fully Polynomial-Time Approximation Scheme (FPTAS)

For every $\epsilon > 0$, there exists an algorithm $\mathcal{A}_\epsilon$ with running time polynomial in the input size and $1/\epsilon$, which computes an approximation x_ϵ with

$$|f(x_\epsilon) - f(x^{\max})| \leq \epsilon f(x^{\max}),$$

where $x^{\max}$ denotes an optimal solution of the optimization problem.

A view towards the mixed integer world: convex mixed integer programs.

An operation

For a lattice point free polyhedron $L = \{x \in \mathbf{R}^n \mid \Pi x \leq \pi\} \subseteq \mathbf{R}^n$ and a closed convex set C, define

$$R(L) := \operatorname{conv}\left(C \setminus \operatorname{rint}(L)\right).$$

Basic properties of such an operation:

- $R(L) \subseteq C$.
- $R(L) \supseteq \operatorname{conv}\left(C \cap (\mathbf{Z}^d \times \mathbf{R}^{n-d})\right)$.
- $R(L) \neq C$ iff there exists an extreme point (x, y) of C such that x is in the interior of L.

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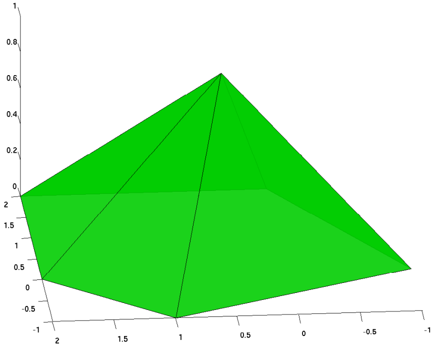

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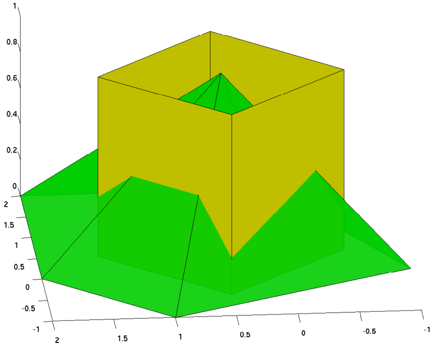

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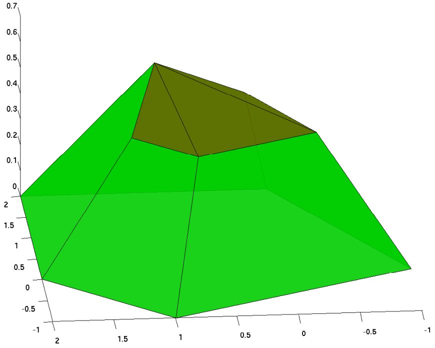

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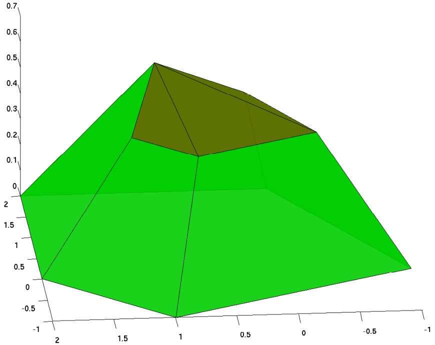

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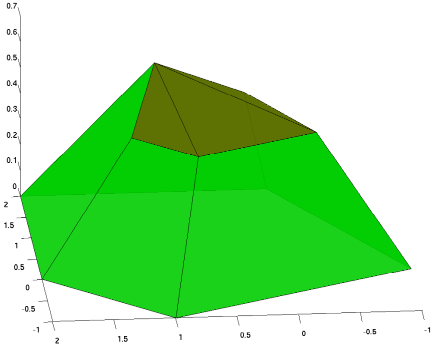

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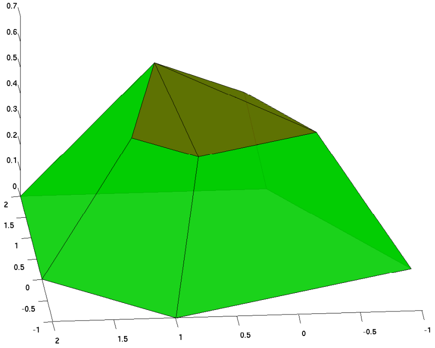

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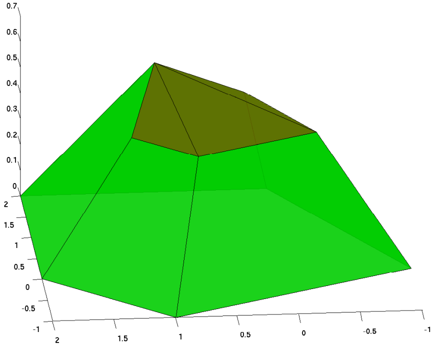

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A closed convex set C is called ...

- LMI-representable if $C = \{x \mid A_0 + A_1 x_1 + \ldots + A_n x_n \succeq 0\}$ with real symmetric matrices A_i.
- SDP-representable if C is the projection of a LMI-representable set.

For a closed convex set C and

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Theorem: When C is xxx, then $R(L) = \mathrm{conv}(C \setminus \mathrm{rint}(L))$ is xxx.

with xxx = polyhedral	with xxx = SDP-representable
[Balas 1979]	[Helton, Nie 2007]

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about when convex sets are closed w.r.t. one such operation

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$$C_k := C \cap \{x \mid \Pi_{k\cdot}^T x \geq \pi_k\},$$ we observe that $$C \setminus \mathrm{rint}(L) = \bigcup_k C_k.$$

Theorem: When C is xxx, then $R(L) = \mathrm{conv}(C \setminus \mathrm{rint}(L))$ is xxx.

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What happens if one applies this operation over and over again?

The closure of split bodies

For a family $\mathcal{F}$ of split bodies, let

$$\mathrm{Cl}(\mathcal{F}, C) := \bigcap_{L \in \mathcal{F}} R(L).$$

Letting $C^0(\mathcal{F}, C) = C$, define for $i \geq 1$,

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We search for meta-theorems of Type A ...

Let $\mathcal{F}$ be any family of split bodies satisfying meaningful property **xxx** and let C be any convex set satisfying a meaningful property **yyy**, so that **yyy** is closed under the $R(L)$-operation. Then $\mathrm{Cl}(\mathcal{F}, C)$ satisfies again property **yyy**.

and also for meta-theorems of Type B ...

Let $\mathcal{F}$ be any family of split bodies satisfying **xxx**. Let C be any convex set satisfying **yyy** that it is closed under the $\mathrm{Cl}(\mathcal{F}, C)$ -operation. There exists finite t such that $\mathrm{conv}(C \cap (\mathbf{Z}^d \times \mathbf{R}^{n-d}) = C^t(\mathcal{F}, C)$.

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Meaningful properties: C being polyhedral sets and $\mathcal{F}$?

The max facet width

- For a split body $L = \{x \in \mathbf{R}^n \mid \Pi x \leq \pi\}$, let

$$w(k) = \pi_k - \min\{\Pi_{k\cdot}^T x : x \in L\}.$$

- The number $\max_k\{w(k)\}$ is a measure of complexity of L.

Theorem [Andersen, Louveaux, W 08].
Let $\mathcal{F}$ be the family of all split bodies of max facet width at most ω^*. For every polyhedron P, $\mathrm{Cl}(\mathcal{F}, P)$ is a polyhedron.
The special case of $\omega^* = 1$ was shown in [Cook, Kannan, Schrijver 90].

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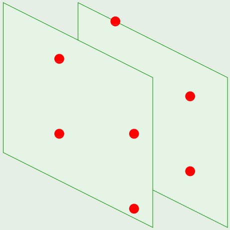

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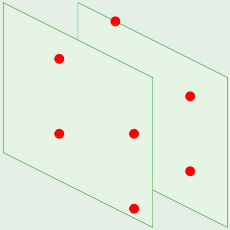

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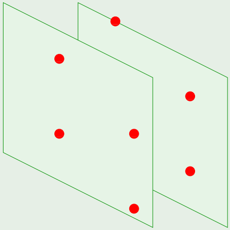

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