

Planar Extensional Motion of an Inertially-Driven Liquid Sheet

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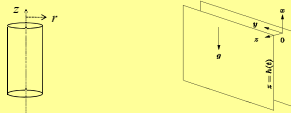
Abstract:

We derive a time-dependent exact solution of the free surface problem for the Navier-Stokes equations that describes the planar extensional motion of a viscous sheet driven by inertia. The linear stability of the exact solution to one- (1D) and two-dimensional (2D) perturbations is examined in the inviscid and viscous limits within the framework of the long-wave approximation. Both transient growth and long-time asymptotic stability are considered. For 1D perturbations in the axial direction, viscous and inviscid sheets are asymptotically marginally stable, though depending on the Reynolds and Weber numbers transient growth can have an important effect. For 1D perturbations in the transverse direction, inviscid sheets are asymptotically unstable to perturbations of all wavelenghts. For 2D perturbations, inviscid sheets are unstable to perturbations of all wavelenghts with the transient dynamics controlled by axial perturbations and the long-time dynamics controlled by transverse perturbations. The asymptotic stability of viscous sheets to 1D transverse perturbations and to 2D perturbations depends on the capillary number (Ca); in both cases, the sheet is unstable to longwave transverse perturbations for any finite Ca.

Extensional Free Surface Flows

Basic Geometries

Axisymmetric Filament Planar Sheet



Properties of an incompressible extensional flow

- Flow contracts uniformly in one direction $u = (-\dot{\epsilon}r/2, 0, \dot{\epsilon}z)$ (cylinder radius or sheet thickness) $u = (\dot{\epsilon}x, 0, -\dot{\epsilon}z)$
- Flow expands in the axial direction $\dot{\epsilon} = \text{stretch rate}$

Examples of extensional flows:

Droplet formation, filament stretching rheometry, fiber drawing, fiber spinning, mixing processes, film blowing, spray atomization, and shaped charges.

Planar Sheet Flow – Free Boundary Problem

Consider symmetric motions of a spatially uniform infinite sheet.

Governing Equations

$$\rho[\partial_t u + (u \cdot \nabla)u] = -\nabla p + \mu \nabla^2 u - \rho g$$

$$\nabla \cdot u = 0,$$

Boundary Conditions

$$\text{at } z = \pm h(t) \quad \text{at } z = 0$$

$$\partial_z h + u \partial_x h + v \partial_y h = w, \quad w = 0,$$

$$n \cdot (T - pI) \cdot n = -\gamma \nabla \cdot n, \quad \partial_z p = 0,$$

$$n \cdot (T - pI) \cdot t_1 = 0, \quad \partial_z u = 0,$$

$$n \cdot (T - pI) \cdot t_2 = 0.$$

$\rho = \text{density}$,
 $\mu = \text{dynamic viscosity}$,
 $\gamma = \text{surface tension}$.

$$u = (u, v, w), \quad T = (\nabla u + \nabla u^T)$$

$$n = \pm(-\partial_x h, -\partial_y h, \partial_z h) / \sqrt{(\partial_x h)^2 + (\partial_y h)^2 + 1},$$

$$t_1 = (1, 0, \partial_x h), \quad t_2 = (0, 1, \partial_y h).$$

An Exact Solution for the Unperturbed Flow

$$\dot{\epsilon}_0(t) = \frac{1}{\tau} \quad (\text{stretch rate})$$

$$h_0(t) = \frac{K}{\tau} \quad (\text{sheet half - thickness})$$

$$u_0 = \left(\frac{x-x_0}{\tau}, \frac{y-y_0}{\tau}, 0, -\frac{z}{\tau} \right) \quad (\text{velocity field})$$

$$p_0(z, t) = \rho \left(\frac{h^2 - z^2}{\tau^2} \right) - \frac{2\mu}{\tau} + p_{\text{amb}} \quad (\text{pressure field})$$

$$\tau = t + t_0, \quad K > 0, t_0 > 0, x_0, y_0, p_{\text{amb}} \in \mathbb{R}$$

Observations

- The stretch rate and the free surface depend only on the inertia of the flow.
- The velocity field represents a balance of inertial and gravitational forces.
- The pressure field is independent of surface tension and uniform in the axial direction.

Stability Analysis

The exact solution also satisfies the long-wave model (in which the flow is averaged over the sheet thickness). We consider the stability of the sheet within the framework of these simpler equations.

Nondimensional Long-Wave Model

$$\partial_t h + \nabla \cdot (hu) = 0$$

$$\partial_t u + (u \cdot \nabla)u = \frac{1}{\text{Re}} \nabla \cdot (h[\nabla u + \nabla u^T + 2(\nabla \cdot u)I]) + \frac{1}{\text{We}} \nabla(\nabla^2 h) - \frac{1}{\text{Fr}^2} \hat{e}_z,$$

$$u = u(x, y, t), \quad v(x, y, t)$$

The dimensionless parameters are the Reynolds, Weber and Froude numbers :

$$\text{Re} = \frac{\rho UL}{\mu}, \quad \text{We} = \frac{\rho U^2 L^2}{H\gamma}, \quad \text{Fr} = \frac{U}{\sqrt{gL}}$$

where L and U are characteristic length and velocity scales. We perturb about the base solution

$$h(x, y, t) = h_0(t) + \delta h_1(x, y, t),$$

$$u(x, y, t) = u_0(t) + \delta u_1(x, y, t),$$

$$v(x, y, t) = 0 + \delta v_1(x, y, t).$$

with $\delta \ll 1$, and change variables to characteristic coordinates

$$\chi = \frac{x-x_0}{\tau} + \frac{\tau}{\text{Fr}^2}, \quad \tau = t + 1.$$

In terms of the characteristic coordinates, the perturbed variables have the form

$$h_1(\chi, y, \tau) = \int_{-\infty}^{\infty} H_1(k_x, k_y, \tau) e^{i(k_x \chi + k_y y)} dk_x dk_y,$$

$$u_1(\chi, y, \tau) = \int_{-\infty}^{\infty} U_1(k_x, k_y, \tau) e^{i(k_x \chi + k_y y)} dk_x dk_y,$$

$$v_1(\chi, y, \tau) = \int_{-\infty}^{\infty} V_1(k_x, k_y, \tau) e^{i(k_x \chi + k_y y)} dk_x dk_y.$$

Substituting the expansions for the free surface and the velocity field into the long-wave model, and keeping only linear terms results in a coupled system of ODEs for the Fourier coefficients, namely

$$\frac{dH_1}{dt} = -\frac{H_1}{\tau} - \frac{ik_x U_1}{\tau^2} - \frac{ik_x V_1}{\tau}, \quad (1a)$$

$$\frac{dU_1}{dt} = -\frac{U_1}{\tau} - \frac{1}{\text{Re}} \left(-\frac{4ik_x H_1}{\tau} + \frac{4k_x^2 U_1}{\tau^2} + \frac{3k_x k_y V_1}{\tau} + k_y^2 U_1 \right) - \frac{i}{\text{We}} \left(\frac{k_x^3}{\tau^2} + \frac{k_x k_y^2}{\tau} \right) H_1, \quad (1b)$$

$$\frac{dV_1}{dt} = -\frac{1}{\text{Re}} \left(-2ik_x H_1 + \frac{k_x^2 V_1}{\tau} + \frac{3k_x k_y U_1}{\tau} + 4k_y^2 V_1 \right) - \frac{i}{\text{We}} \left(\frac{k_x^2 k_y}{\tau^2} + k_y^3 \right) H_1. \quad (1c)$$

The stability of the solution depends on the temporal growth of the perturbed variables relative to the growth of the base solution. In particular, the solution is stable if:

$$h_0 \gg h_1, \quad u_0 \gg u_1, \quad |v_1| < \infty \quad \forall \tau$$

1D Axial Perturbations (Set $k_y = 0$ in (1))

For short and long waves and inviscid and viscous fluids we find that

$$H_1(\tau) \sim \frac{\alpha}{\tau} + o(\tau^{-1}) \quad \text{as } \tau \rightarrow \infty.$$

Not all axial perturbations are marginally asymptotically stable, however, since

$$\alpha \equiv \lim_{\tau \rightarrow \infty} \left[\frac{H_1(\tau)/h_0(\tau)}{H_1(1)/h_0(1)} \right],$$

can become large as a result of transient growth of the perturbations.

Rescaling Eqs. (1) introduces an initial time scale:

$$H_1(\tau) = H(T), \quad \tau = \left(\frac{k_x^2}{\text{We}} \right)^{1/3} T, \quad H(T_0) = 1 \quad \text{at } T_0 = \left(\frac{\text{We}}{k_x^2} \right)^{1/3}.$$

The stability of $H(T)$ depends on the size of T_0 relative to other relevant timescales.

Inviscid Case

For large T_0 (corresponding to long waves), perturbations begin in the long-time regime with

$$H_1(\tau) \leq \frac{1}{\tau} \quad \text{as } \tau \rightarrow \infty,$$

hence, the sheet is marginally stable.

For small T_0 (corresponding to short waves) and for small times

$$H(T) \sim \frac{(T/T_0)^{-3/4}}{\cos \theta_0} \cos \left(\frac{2}{3} [T - T_0]^{3/2} + \theta_0 \right),$$

hence

$$H/h_0 = O(\tau^{1/4}) \quad \text{for } T_0 \leq T \leq T_1,$$

corresponding to transient growth.

For small T_0 , in the long-time limit

$$|H_1(\tau)| \leq \frac{\alpha}{\tau},$$

where

$$\alpha(k_x) = O(k_x^{1/3} \text{We}^{-1/12}).$$

The minimum wave number for this regime corresponds to

$$k_x > k_{x,1} \approx 1.42 \text{We}^{1/4}.$$

Summarizing: If

- $k_x < k_{x,1}$, then perturbations do not grow from their initial amplitude and the sheet is marginally stable.
- $k_x \gg k_{x,1}$, then perturbations can yield large deviations to the long-time sheet thickness.

Connection to the Weber number:

Increasing We delays the instability to higher wave numbers.

In the limit, $\text{We} \rightarrow \infty$, transient growth is absent for all finite wave numbers and the sheet is marginally stable.

For $\text{We} < \infty$, inviscid sheets are weakly unstable to transient perturbations.

Viscous Case

For large T_0 (long waves),

$$H_1(\tau) \leq \frac{1}{\tau} \quad \text{as } \tau \rightarrow \infty,$$

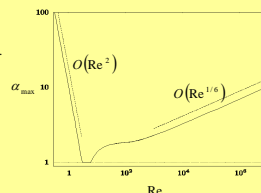
hence the sheet is marginally stable.

For small T_0 (short waves),

$$H_1(\tau) \sim \frac{\alpha_{\text{max}}}{\tau} + o(\tau^{-1}) \quad \text{as } \tau \rightarrow \infty.$$

For $0 < \text{Re} < \infty$, the sheet is marginally stable since $\alpha_{\text{max}} < \infty$.

In the case of Stokes and inviscid flows, the sheet is weakly unstable since $\alpha_{\text{max}} \rightarrow \infty$.



ID Transverse Perturbations

(Set $k_x = 0$ in (1))

Inviscid Case

Here

$$H(T) \sim \frac{(T/T_0)^{-3/4}}{\cos \theta_0} \cos [2\sqrt{T - \sqrt{T_0}} + \theta_0] \quad \text{as } T \rightarrow \infty \quad \text{and } \forall k_y, \quad z = h_0(t)$$

so that $H/h_0 = O(\tau^{1/4})$

Therefore, perturbations of all wavelenghts grow relative to the base state and inviscid sheet are unstable to transverse perturbations.

Viscous Case

We define the long-time asymptotic growth of the perturbations by

$$|H_1(\tau)/h_0(\tau)| = O(\tau^\beta),$$

where

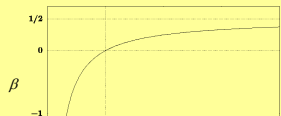
$$\beta \equiv \lim_{\tau \rightarrow \infty} \left[\frac{H_1(\tau)/h_0(\tau)}{\ln \tau} \right] = \frac{2\sigma - 1}{4\sigma} = \frac{1}{2} - \frac{k_y^2 \text{Re}}{4\text{We}}$$

and

$$\sigma = \frac{\text{We}}{k_y^2 \text{Re}} = \frac{\text{Ca}}{k_y^2} > 0.$$

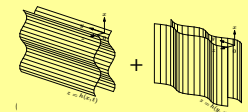
The sheet is unstable if the following equivalent conditions hold

$$\beta > 0 \Leftrightarrow \sigma > 1/2 \Leftrightarrow k_y < \sqrt{2\text{Ca}}$$



Hence, for a fixed capillary number, viscous sheets are unstable to long wave transverse perturbations.

2D Oblique Perturbations



Inviscid Case

In the transient limit,

$$H_1(\tau) \approx O\left(\tau^{-3/4} \cos \left[\frac{2}{3} k_y^2 \text{We}^{-1/2} \tau^{-3/2} \right]\right),$$

which means the sheet is unstable to axial perturbations.

And in the long-time limit,

$$H_1(\tau) \approx O\left(\tau^{-3/4} \cos \left[2k_y^2 \text{We}^{-1/2} \tau^{1/2} \right]\right),$$

which means the sheet is unstable to transverse perturbations.

Since $H_1/h_0 = O(\tau^{1/4}) \forall \tau$,

an inviscid sheet is unstable to 2D perturbations for all time.

Viscous Case

In the transient limit, the behavior is controlled by axial perturbations

$$H_1(\tau) \approx O\left(\tau^{-3/4} e^{2k_y^2/(t \text{Re})} \cos \left[k_y^2 \text{We}^{-1/2} \left(\frac{2}{3} \tau^{-3/2} - 4\text{We} \text{Re}^{-2} \tau^{-1/2} \right) \right]\right),$$

with the perturbations decaying exponentially in time.

In the long-time limit, the behavior is controlled by transverse perturbations

$$H_1(\tau) \approx O\left(\tau^{-\frac{1}{2} + \frac{1}{2} \text{Re} [4\text{We}]}\right),$$

such that for a fixed capillary number,

$$k_y < \sqrt{2\text{Ca}}$$

the sheet is unstable to long wave transverse perturbations. Therefore, viscous sheets are unstable to 2D long wave perturbations.

Conclusions

- We have derived an exact solution for the time-dependent extensional motion of a planar sheet that is driven by inertia.
- The linear stability of the sheet is analyzed within the framework of the long-wave model.
- We find that inviscid and viscous sheets are most unstable to transverse perturbations.
- In experiments, we expect that lateral-edge effects will dominate and destabilize the sheet which is consistent with our stability results.

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