ALTERNATE POWERS IN SERRIN'S SWIRLING VORTEX SOLUTIONS 2

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Introduction

Recent radar studies indicate the fractal nature of the tornado-related vorticity field with respect to the grid size, more specifically, natural log of the vorticity and natural log of the grid spacing have a linear relationship, with a constant ratio. As the slope decreased to -1.6 ([2]) the likely hood of strong tornados increased. We conjecture that stretching of the vortex lines could lead to a fractalization of the vortex and, possibly, to tornado genesis. The possibility of fractalization in the process of vortex stretching was pointed out by Chorin in [3]. In 1972, J. Serrin discovered ([4]) a special class of swirling vortex solutions to the Navier-Stokes Equations. Depending on kinematic viscosity and the value of a "pressure" parameter, he described three cases:

- Down-draft core with radial outflow
- Downdraft core with a compensating radial inflow
- Updraft core with radial inflow (single-cell vortex)

The Main Question

Are there fractal Serrin type solutions to the Navier-Stokes Equations with velocity field proportional to $\mathbf{1/r^{\beta}}$ where $\boldsymbol{\beta}$ is a non-integer?

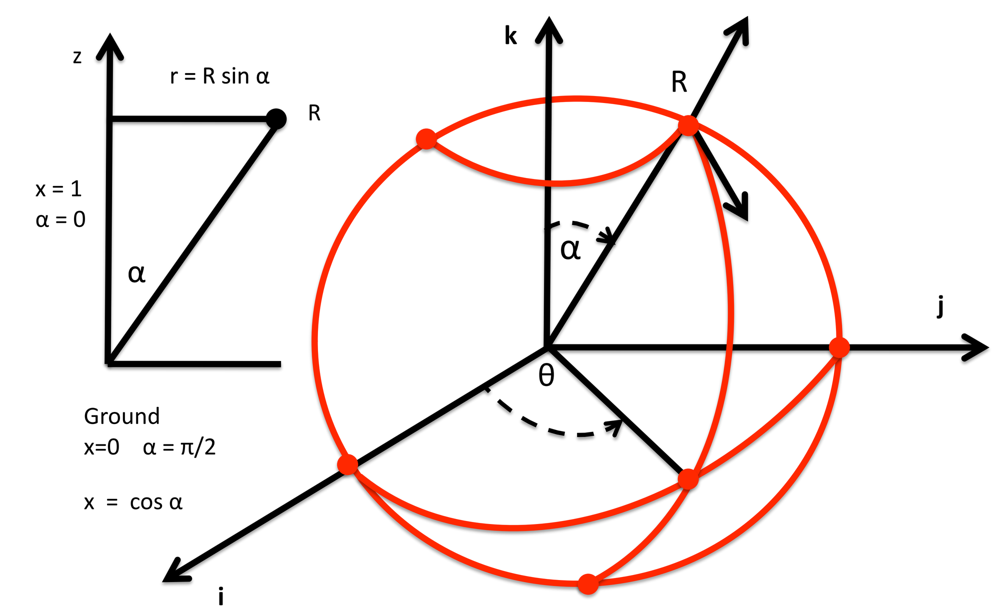
Outline of the Study

- Serrin solutions to Navier-Stokes with $\beta \neq 1$
- Shooting method & Numerical Results
- Comparison with flow in *Vortex Simulator*
- Implications for Tornadogenesis

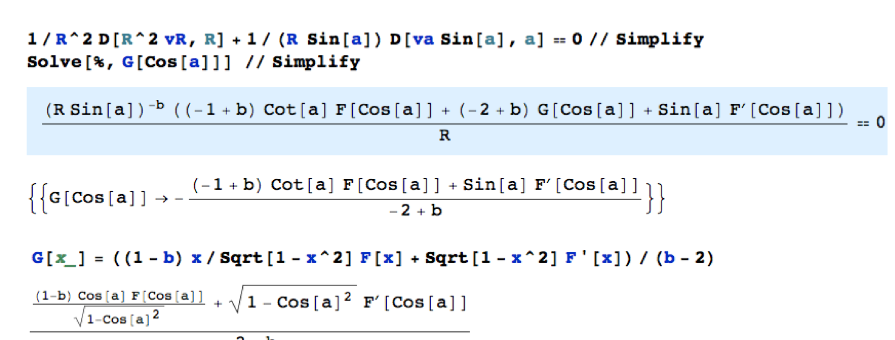
Kinematic Quantities

Position vector $\mathbf{r}(R,\alpha,\theta) = R\cos\theta\sin\alpha\,\mathbf{i} + R\sin\theta\sin\alpha\,\mathbf{i} + R\cos\alpha\,\mathbf{k}$. Orthonormal basis $\mathbf{e}_R(\alpha,\theta) = \cos\theta\sin\alpha\,\mathbf{i} + \sin\theta\sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{k}$, $\mathbf{e}_\alpha(\alpha,\theta) = \cos\theta\cos\alpha\,\mathbf{i} + \sin\theta\cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{k}$, $\mathbf{e}_\theta(\alpha,\theta) = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + 0\,\mathbf{k}$. Velocity vector $\mathbf{v}(R,\alpha,\theta,t) = v_R(R,\alpha,\theta,t)\,\mathbf{e}_R(\alpha,\theta) + v_\alpha(R,\alpha,\theta,t)\,\mathbf{e}_\alpha(\alpha,\theta) + v_\theta(R,\alpha,\theta,t)\mathbf{e}_\theta(\alpha,\theta)$. Modified Serrin variables $r = R\sin\alpha$, $x = \cos\alpha$, $v_R(R,\alpha,\theta) = G(x)/r^{\beta}$, $v_\alpha(R,\alpha,\theta) = F(x)/r^{\beta}$, $v_\theta(R,\alpha,\theta) = \Omega(x)/r^{\beta}$.

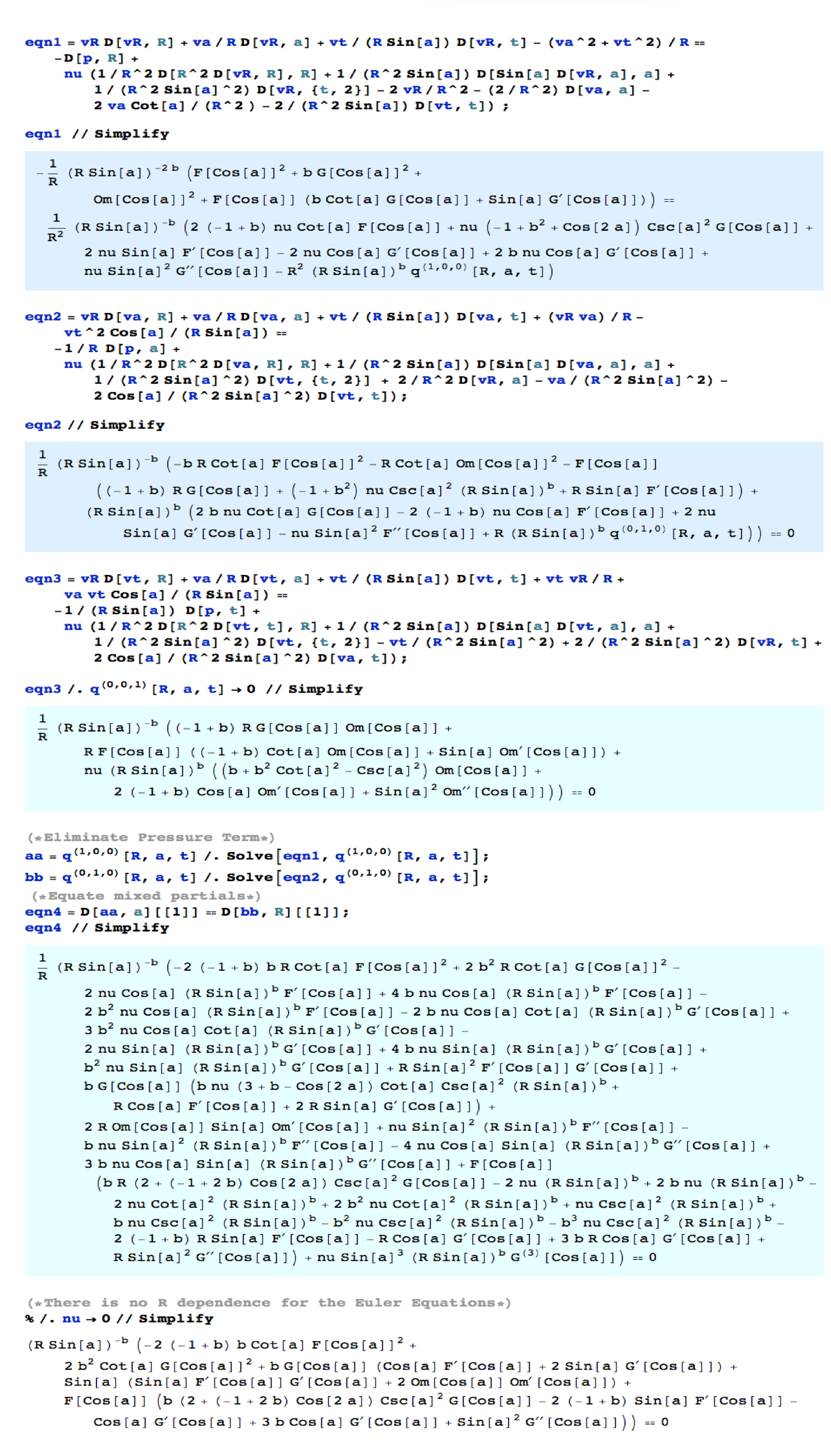
Serrin Variables



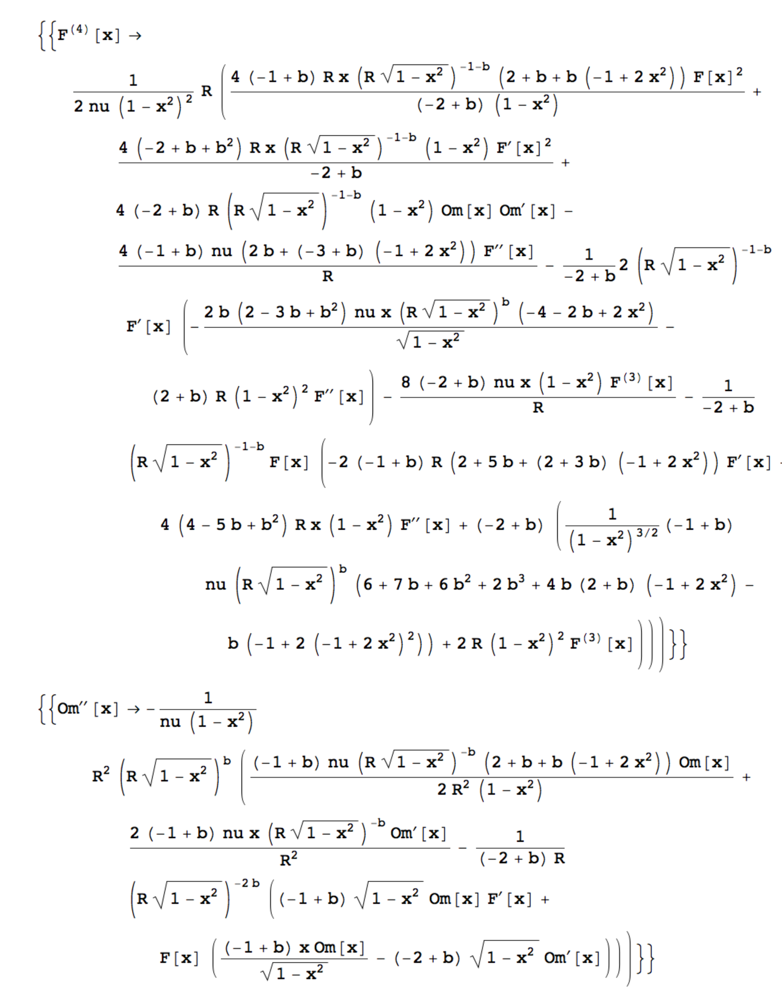
Spherical Continuity Equation



Reduction of the Spherical Navier-Stokes Equations

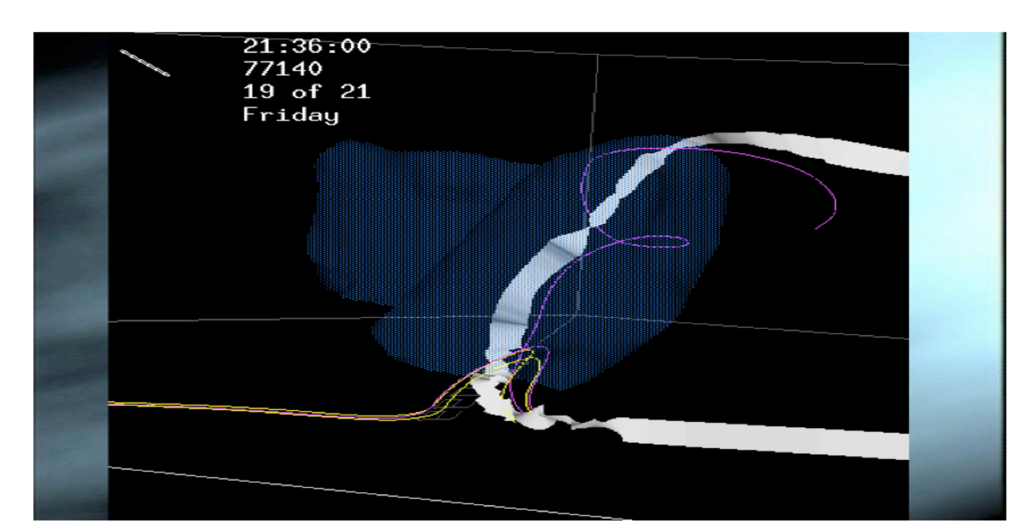


ODE System & Dependence on R



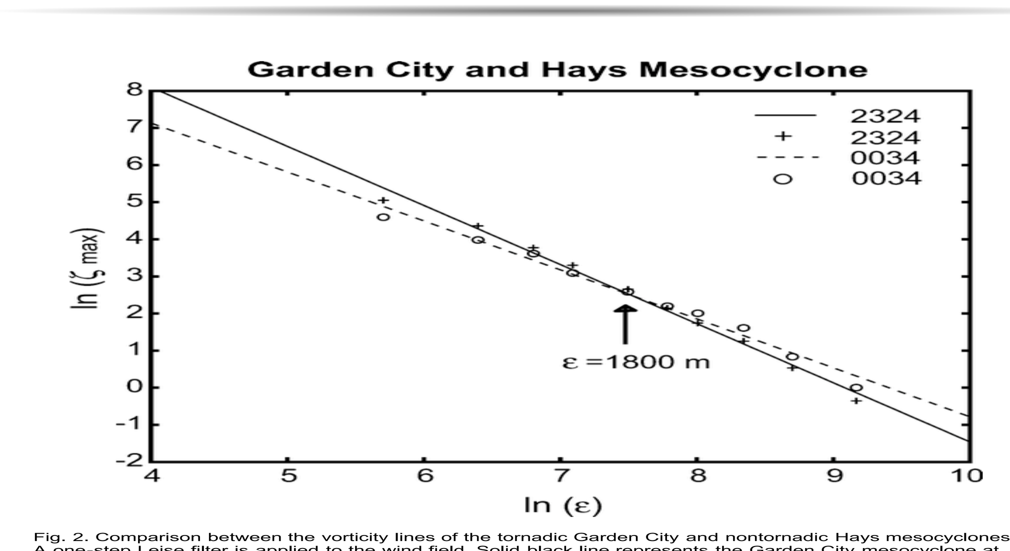
Dependence on R in the right-hand side contradicts the original assumption which shows that Navier-Stokes in this case does not have solutions of the form $F(x)/r^{\beta}$ unless $\beta = 1$ or $\beta = 2$. However, the solutions of this form exist for Euler Equations for all $\beta > 0$.

Tornadogenesis



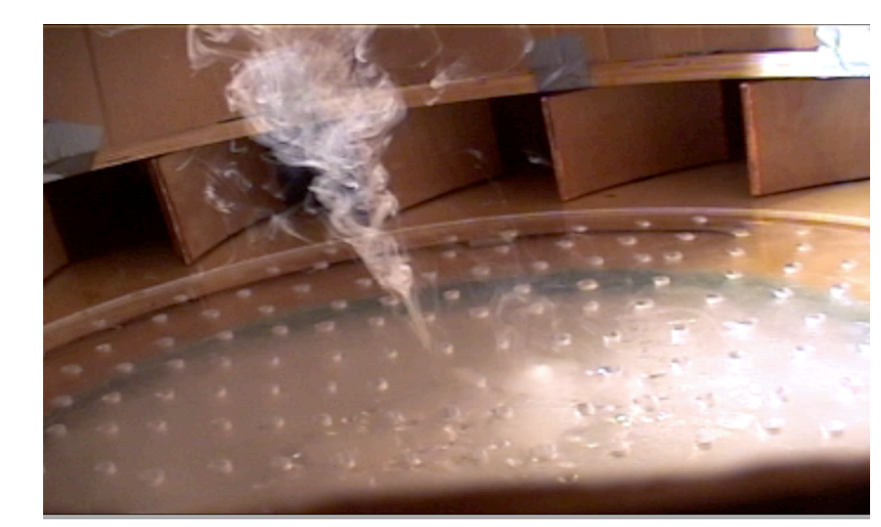
The color lines are the arching vortex lines, white ribbon is a streamline. The blue isosurface is 20 m/s updraft in a supercell thunderstorm (see [7] and [6]).

Cai [2] Results



A possible argument in favor of Cai's result is the following. From Kelvin's Circulation Theorem vorticity times the area of vortex tube is constant for Eulerian barotropic flows. Recent numerical and radar studies of tornadic storms suggest that vortex lines produced on the rear flank gust front of a super cell thunderstorm that are captured by the updraft form arches to produce counter-rotating vortices. A result of a numerical study of Kelvin-Helmholtz instability by Baker and Shelley([1]), identifies a relationship between the thickness of the vortex sheet, h, and the cross sectional area of the vortices, A. They found A scales like $O(h^{1.55})$ as h goes to 0. Cai's paper suggests that the tornados are fractal. If the arching vortex lines combine to form the self-similar (fractal) vortex, the self-similarity of the vortex suggests the largest scales are similar to the smallest scales. Therefore, $\omega = C * h^{-1.55}$, where C is a constant.

Vortex Fractalization?



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[4] J. Serrin, Phil. Trans. Roy. Soc. London, Series A, Math & Phys. Sci. **271** (1972), 325–360

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[6] D. P. Dokken, K. Scholz, P. Shanahan, L. Edholm, CSUMS Report (2009)

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