

Dynamics of Shallow Water Layer Models

Stability, Wave Breaking and Mixing.

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IMA, April 12, 2010

Collaborators

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- ▶ Tivon Jacobsen, Anakewit Boonkasame

Overview

Two-layer flows: Stability

- Motivation and formulation

- Stability results

- Non-Boussinesq

Two-layer flows: Mixing

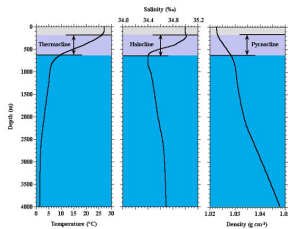
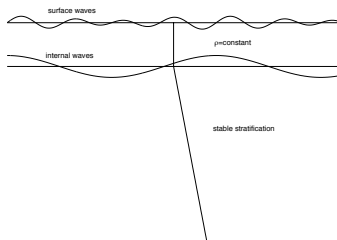
- Formulation

- Mixing closure

- Results

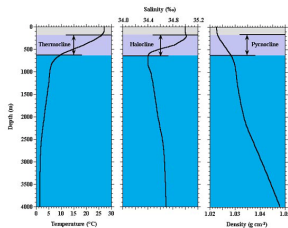
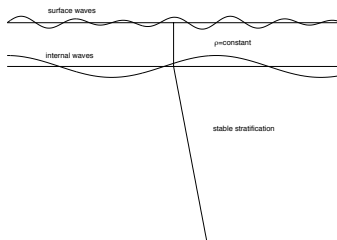
Conclusion

Motivation



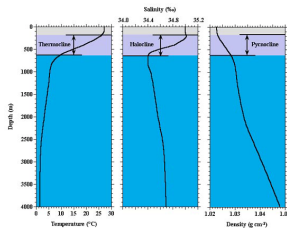
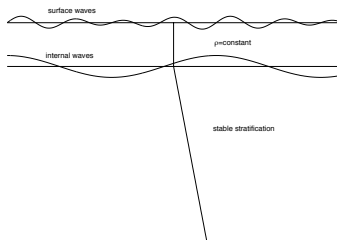
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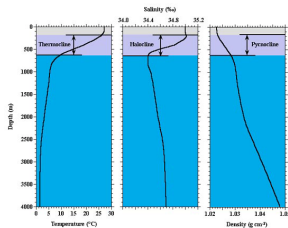
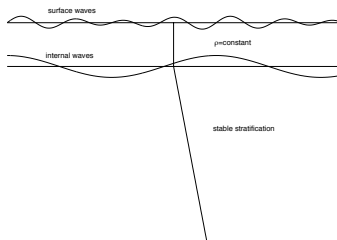
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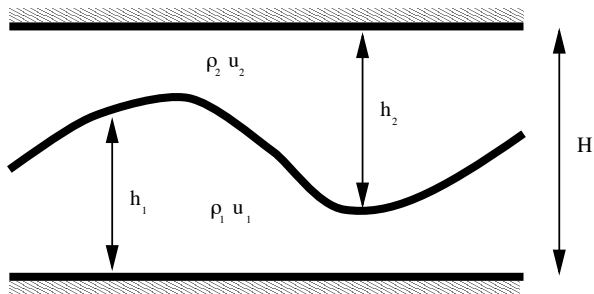
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Motivation



- ▶ Stratification is basic state atmosphere/ocean flows.
- ▶ Picture inverted in atmosphere: upper ocean = troposphere.
- ▶ Breaking waves and turbulent mixing changes the stratification.
- ▶ Waves are slow: $c = \sqrt{\frac{\Delta\rho}{\rho}gh}$. For the ocean $\Delta\rho/\rho \approx 1/200$, mixed layer depth ≈ 200 m. So $c \approx 3$ m/s ≈ 10 km/hour. Decoupling.

Assumptions



- ▶ Shallow Water approximation $u(x, t), h(x, t)$. (Hydrostatic)
- ▶ Boussinesq. (Density affects buoyancy only).

Governing Equations (Long)

Using h as the scaled lower layer depth, and u as the scaled lower layer velocity (they are proportional to h_1 and u_1):

$$\begin{aligned}
 h_t + (hu)_x &= 0 \\
 u_t + \frac{1-3h}{1-h}uu_x + \left((1-h) - \frac{1}{(1-h)^2}u^2 \right) h_x &= 0.
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System is of mixed type, with eigenvalues, or characteristic speeds:

$$\lambda = \frac{1-2h}{1-h} u \pm \sqrt{\frac{h}{1-h} ((1-h)^2 - u^2)}$$

Hyperbolic when:

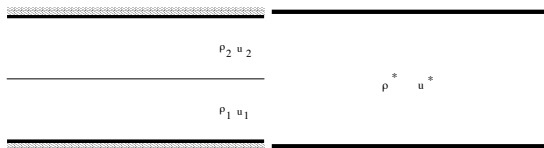
$$\frac{(1-h)^2}{u^2} \equiv Ri > 1$$

Initial data not satisfying this condition everywhere is ill-posed.

Interpretation of Richardson number: Stability and Mixing

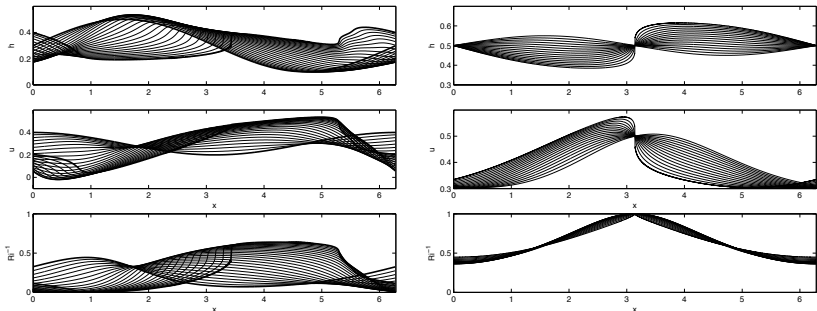
Dimensionally, equations are hyperbolic when:

$$\frac{g}{H} \frac{\Delta\rho/\rho}{\left(\frac{u_1}{h_2} - \frac{u_2}{h_1}\right)^2} \equiv Ri > 1, \quad \text{hyperbolic : } Ri > 1.$$



- ▶ In the scenario above, there is sufficient shear energy to fully mix the fluid when $Ri < 1$
- ▶ Ill-posedness (ellipticity, instability) coincides with mixing energetics.
- ▶ Continuous stratification: the Howard stability criterion $Ri < 1/4$ is *more restrictive* than the mixing criterion there $Ri < 1/2$.

Numerical Solutions



The short wavelength KH instability is always present in stratified shear flows. Here, the shallow water approximation has filtered it since it governs long wave-phenomena. One can imagine a separation into two behaviors: a thin KH mixing layer superposed on the long wave behavior.

Is 2-layer flow well-posed? “nonlinearly stable?”

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Theorem: (i) A necessary condition for a system of mixed type $u_t + A(u)u_x = 0$ to be nonlinearly stable is that, in phase space, the plane tangent to the sonic surface $S(u)$ (i.e., the surface where the system changes type) must be invariant under the evolution. (ii) A necessary condition for this invariance is that the tangent surface must, at every point in S include the right eigenvector of A corresponding to the eigenvalue that degenerates on the surface.

Riemann Invariants for 2-layer flows

Introducing $r = \frac{u}{1-h}$, the Riemann invariants are

$$R^{\pm} = 2\sqrt{h(1-h)(1-r^2)} \mp r(2h-1),$$

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$$R_t^- + c^-(R^+, R^-) R_x^- = 0.$$

$$c^+ = \frac{3}{4}R^+ - \frac{1}{4}R^-, \quad c^- = -\frac{3}{4}R^- + \frac{1}{4}R^+.$$

- ▶ Since the speeds are smooth functions of the Riemann invariants, the system is well posed. *Nonlinear KH stability.*
- ▶ The system is now identical to (one layer) shallow water in Riemann invariant form! Thus we constructed an *explicit* 2-layer to 1-layer (St. Venant equations) map.

More Layers

Consider similarly N -layer flows. The main mathematical obstruction is the fact that we lose Riemann invariants. The principal results that we can show are:

- ▶ 2-periodic N -layer flows are also mapped onto the one-layer shallow-water model. For geophysical flows we provide a map between the nonlinear first baroclinic (internal structure modes) and the barotropic modes.

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Consider similarly N-layer flows. The main mathematical obstruction is the fact that we lose Riemann invariants. The principal results that we can show are:

- ▶ 2-periodic N-layer flows are also mapped onto the one-layer shallow-water model. For geophysical flows we provide a map between the nonlinear first baroclinic (internal structure modes) and the barotropic modes.
- ▶ 3-layer flows of all kinds *fail* the necessary condition for well-posedness (eigenvector condition on the sonic surface). That is, “catastrophic” shear instabilities can arise after finite-time smooth nonlinear wave motion. Numerical calculations confirm this. [current work: sharp stability result]

Non-Boussinesq

Non-Boussinesq two-layer flows have the parameter $r = (\rho_2 - \rho_1)/2$ and rescaled dimensionless variables

$$\begin{aligned}t &= \tau\sqrt{2r}, \\h &= h_2 - h_1, \\m &= (\rho_2 u_2 - \rho_1 u_1)/\sqrt{2r}.\end{aligned}$$

Yielding the governing equations

$$h_t + \left(\frac{(1 - h^2)m}{2(1 - rh)} \right)_x = 0, \quad (1)$$

$$m_t + \left(\frac{[r(1 + h^2) - 2h]m^2}{4(1 - rh)^2} + \frac{h}{2} \right)_x = 0, \quad (2)$$

in the frame of reference where $h_1 u_1 + h_2 u_2 = 0$.

Boussinesq limit and instability

- ▶ *In this scaling* the Boussinesq approximation is clear and obtained in limit $r \rightarrow 0$.

$$h_t + \left(\frac{(1-h^2)m}{2} \right)_x = 0, \quad (3)$$

$$m_t + \left(\frac{hm^2}{2} + \frac{h}{2} \right)_x = 0, \quad (4)$$

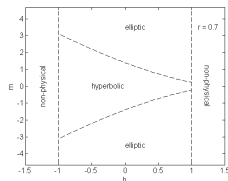
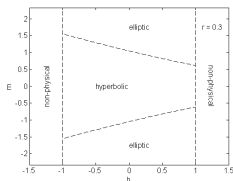
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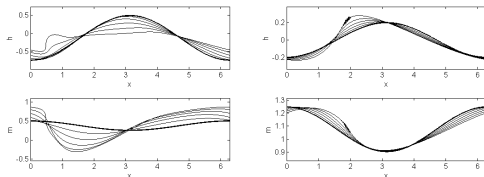
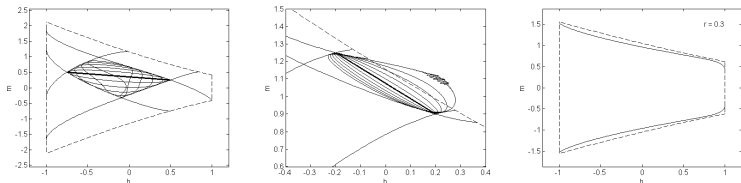
$$m_t + \left(\frac{hm^2}{2} + \frac{h}{2} \right)_x = 0, \quad (4)$$

- ▶ The system now *fails* the necessary condition for stability at the hyperbolic-elliptic boundary. [implying nonsmooth characteristic speeds]

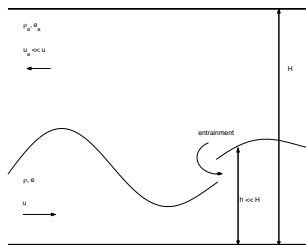


Simple waves and stability

Using nonlinear *simple waves* we can sharply construct the maximal stability region for non-Boussinesq waves.

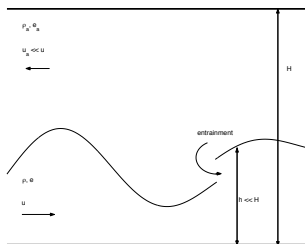


Wave Breaking: Mixing Problem



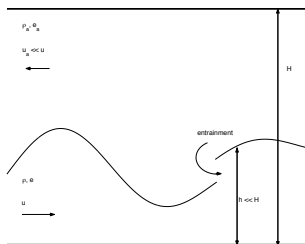
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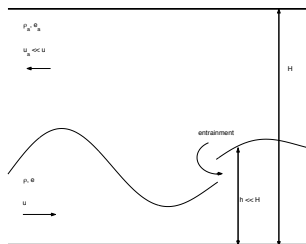
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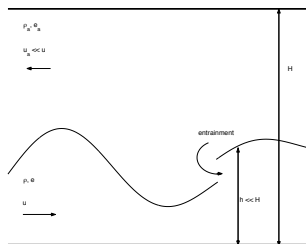
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- ▶ Boussinesq approximation.
- ▶ Physical scenario: downslope gravity flows (ex: Mediterranean outflows), diapycnal mixing in upper ocean, tropopause mixing.

Governing Equations - Conservation Laws

- ▶ Mass: $(bh)_t + (bhu)_x = 0.$
- ▶ Momentum: $(hu)_t + \left(hu^2 + \frac{bh^2}{2}\right)_x = 0.$
- ▶ Energy: $\left(\frac{hu^2}{2} + \frac{bh^2}{2} + he\right)_t + \left(\frac{hu^3}{2} + bh^2u + heu\right)_x = 0.$

- ▶ $b = g \frac{\rho - \rho_a}{\rho_a}$ is the buoyancy, e is the "internal energy" in small scales.

The derivation is obtained by considering both layers and taking $H \rightarrow \infty$.

These are three conservation laws for four unknowns. Constant b gives standard shallow water. The "mass conservation" acts on total mass and does not conserve mass in each layer.

Constant density

For constant b , mass and momentum give a closed problem

$$(h)_t + (hu)_x = 0$$

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and the energy equation gives the evolution of internal energy he :

$$\left(\frac{hu^2}{2} + \frac{h^2}{2} + he \right)_t + \left(\frac{hu^3}{2} + h^2u + heu \right)_x = 0.$$

In smooth parts: $(he)_t + (hue)_x = 0$. By contrast, at shocks,

$$-c[h] + [hu] = 0, \quad -c[hu] + [hu^2 + h^2/2] = 0$$

imply $-c[he] + [uhe] > 0$. The production of internal energy and dissipation of mechanical energy is known without resolving the small scale details of the flow.

Variable density

For nonconstant b , we can *assume*, in smooth parts, in addition to the three conservation laws that, as in constant density flows:

$$e_t + ue_x = 0.$$

Which is equivalent to assuming no mixing in smooth parts:

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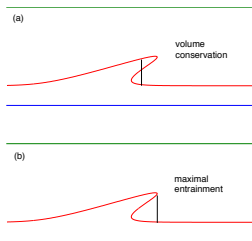
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One can pose the question as: given a certain long scale energy dissipated at a shock, how is it partitioned between increasing turbulence (internal energy) and entraining and mixing (potential energy)? Can we do this *without* looking at the details?

Mixing at shocks: heuristic maximization idea

Let us consider the maximization of mixing (or entrainment). For a simple “Burgers” model $u_t + uu_x = 0$.



Burgers: Choose $c = u^-$, that is, the maximum speed satisfying the Lax condition.

Shallow water: Choose c as the downstream characteristic speed of the shocking characteristic: $c = u_- - \sqrt{b_- h_-}$.

Maximizing mixing at shocks: optimization

Find an *upper bound* to the dynamically allowed mixing by solving a constrained *optimization* problem.

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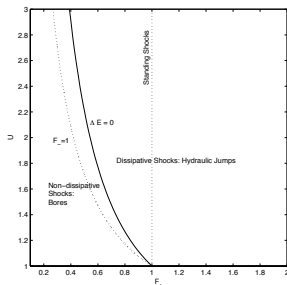
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$$\lambda_- \geq c \geq \lambda_+. \quad \text{Lax condition}$$

λ_{\pm} stand for the characteristic speeds corresponding to the shocking characteristic on the left and right of the shock.

Equivalence of the Two Ideas

Theorem: For sufficiently large upstream Froude numbers, the maximization of entrainment is achieved at the downstream characteristic speed. For less strongly supercritical flows, the energy constraint becomes active.



$$U = \frac{u_-}{u_+}$$

$$F_- = \frac{u_-}{\sqrt{b_- h_-}}$$

$$F_+ = \frac{u_+}{\sqrt{b_+ h_+}}$$

— is upstream

$1 - F_+$ is the speed of the shock

This may provide a mathematical motivation for the physical distinction between mixing bores (small dissipation) and hydraulic jumps (large dissipation) in internal flows. This distinction does not exist in free surface shallow water where the equations are Galilean invariant.

Optimization and Dynamics

Describe the dynamics in terms of the conserved quantities $q = bh$, and $m = hu$. Maximize an entropy of mixing subject to the dynamic constraints

$$\begin{aligned}
 q_t + \left(\frac{qm}{h}\right)_x &= 0 && \text{[global mass]} \\
 m_t + \left(\frac{m^2}{h} + \frac{1}{2}qh\right)_x &= 0 && \text{[momentum]} \\
 h_t + m_x &\geq 0 && \text{[entrainment]} \\
 \left(\frac{m^2}{h} + qh\right)_t + \left(\frac{m^3}{h^2} + 2qm\right)_x &\leq 0 && \text{[dissipation]}.
 \end{aligned}$$

In smooth parts the equations can be combined to yield

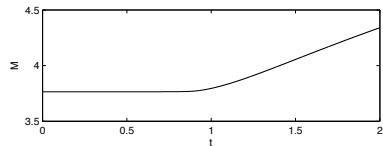
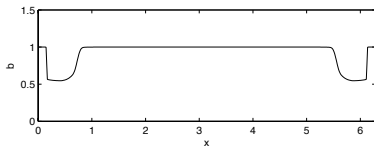
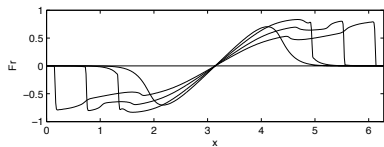
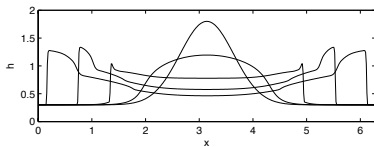
$$(1 - F^2)(h_t + m_x) \leq 0.$$

In this model, for everywhere subcritical flows $|F| < 1$ there is no mixing up to breaking and then mixing occurs in the form of energy-preserving bores. In supercritical flows, mixing is allowed in smooth parts of the flow.

Numerical example

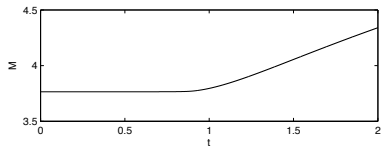
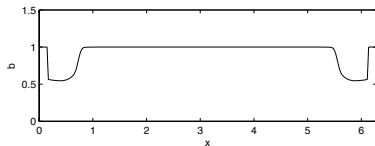
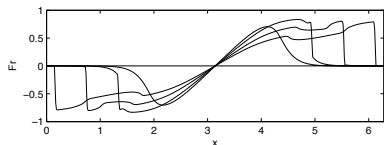
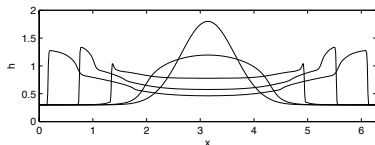
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Example of entraining internal bore:



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The case for $|F| > 1$ is more delicate...in physical problems (such as the Mediterranean outflow) there is a balance between K-H mixing and the formation of hydraulic jumps over topographical features.

Future Work

- ▶ Stability without Riemann Invariants (simple waves).
- ▶ Topographic effects in 2-layer flows.
- ▶ Geophysical application - breaking and mixing at the tropopause due to diurnal forcing. Wang)
- ▶ Non-mixing shock conditions.