Super-convergence for the $3D$ Navier-Stokes

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Introduction

This work is devoted to the super-convergence in space approximation of a fully discrete scheme for the incompressible time-dependent Navier-Stokes equations in three-dimensional domains. We discrete by stable Finite Element in space and by a semi-implicit backward Euler scheme in time. Using an extension of the duality argument in negative-norm for elliptic linear problems (see for instance [1]) to the mixed problem for velocity-pressure formulation (Theorem 3), we prove some super-convergence results for the velocity with respect to the spatial approximation which are corresponded with the numerical accuracy given by several computational experiments. On the other hand, we obtain optimal error estimates for the pressure without imposing restrictions on the discrete time and spatial parameters.

Navier-Stokes Model

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain in a time interval $(0,T)$:

$$\begin{cases} u_t + (u\cdot\nabla)u - \Delta u + \nabla p = f, \quad \nabla\cdot u = 0, \text{ in } \Omega\times(0,T), \\ u(t)|_{\partial\Omega} = 0 \quad t\in(0,T), \quad u|_{t=0} = u_0 \text{ in } \Omega. \end{cases} \tag{1}$$

where the unknowns are $u:\Omega\times(0,T)\to\mathbb{R}^3$, $p:\Omega\times(0,T)\to\mathbb{R}$, $f:\Omega\times(0,T)\to\mathbb{R}^3$ and $u_0:\Omega\to\mathbb{R}^3$.
Variational formulation of (1):
Find $(u(t),p(t))\in H_0^1(\Omega)^3\times L_0^2(\Omega)$ such that $u(0)=u_0$, and

$$(u_t(t),v) + a((u(t),p(t)),(v,q)) + c(u(t),u(t),v) = (f(t),v) \tag{2}$$

holds a.e. $t\in(0,T)$ and for any $(v,q)\in H_0^1(\Omega)^3\times L_0^2(\Omega)$ $(=\{q\in L^2(\Omega);\ \int_\Omega q=0\})$, where

$a((u,p),(v,q)) = (\nabla u,\nabla v) - (p,\nabla\cdot v) - (q,\nabla\cdot u)$ and $c(u,w,v) = ((u\cdot\nabla)w,v) + \frac{1}{2}(\nabla\cdot u, w\cdot v)$

verifying the skew-symmetric property:

$$c(u,v,v) = 0 \quad \forall\, u,v\in H_0^1(\Omega).$$

Theorem 1. *[6] The following stability* Inf-Sup *condition holds: there exists $\beta>0$ such that*

$$\inf_{q\in L_0^2(\Omega)}\sup_{v\in H_0^1(\Omega)^3}\frac{\int_\Omega q\nabla\cdot v}{||v||_{H^1(\Omega)}||q||_{L^2(\Omega)}} \geq \beta.$$

Problem (2) has a weak solution

$$u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \quad p\in H^{-1}(0,T;L^2(\Omega))$$

and the following Energy Inequality holds $\forall\, t\in[0,T]$:

$$||u(t)||_{L^2}^2 + \int_0^t ||\nabla u(s)||_{L^2}^2 \leq \|u_0\|_{L^2}^2 + \int_0^t ||f(s)||_{H^{-1}}^2 \tag{3}$$

Euler Semi-Implicit finite element scheme

Spatial discretization: (V_h,W_h) Finite Element Spaces in $(H_0^1(\Omega)^3, L_0^2(\Omega))$ with $\mathcal{T}_h$ a regular partition of Ω, $P_m\subset V_h$, $P_{m-1}\subset W_h$ (for some $m\geq 1$) and (V_h,W_h) satisfying the **Discrete Inf-Sup Condition,**

$$\inf_{q_h\in W_h}\sup_{v_h\in V_h}\frac{\int_\Omega q_h\nabla\cdot v_h}{||v_h||_{H^1}||q_h||_{L^2}} \geq \overline{\beta}.$$

Time discretization: $(t_n=nk)_{n=0}^N$, $k=\frac{T}{N}$ and $(f^{n+1})_{n=1}^N \simeq (f(t_{n+1}))_{n=1}^N$.
Initialization: $u_h^0\in V_h$ approximation of u_0. $(=P_{L^2,V_h}u_0)$
Step $n+1$: Let $(u_h^n,p_h^n)\in V_h\times W_h$. Compute $(u_h^{n+1},p_h^{n+1})\in V_h\times W_h$ solving:

$$\left(\delta_t u_h^{n+1}, v_h\right) + a((u_h^{n+1},p_h^{n+1}),(v_h,q_h)) + c(u_h^n,u_h^{n+1},v_h) = (f^{n+1},v_h), \tag{4}$$

for any $(v_h,q_h)\in V_h\times W_h$.

Theorem 2. *[5][Unconditionally Stability] Scheme (4) satisfies the following a priori estimates for velocity (which is a discrete version of (3)): for any $r=1,\dots,N$,*

$$\|u_h^r\|_{L^2}^2 + k\sum_{n=0}^{r-1}\|\nabla u_h^{n+1}\|_{L^2}^2 + \sum_{n=0}^{r-1}\|u_h^{n+1}-u_h^n\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + \|f\|_{L^2(0,t_r;H^{-1})}^2 \tag{5}$$

and a priori estimate for pressure in a weighted norm:

$$\left(k\sum_{n=1}^N \|p_h^{n+1}\|_{L^2}^{4/3}\right)^{3/4} \leq \frac{C}{k^{1/2}}$$

Error Estimates in Velocity

We denote by e^n, z^n the total errors, by e_h^n, z_h^n the discrete errors and by e_i^n, z_i^n the interpolation errors,

$$e^n = u(t_n) - u_h^n = \underbrace{u(t_n) - I_h u(t_n)}_{e_i^n} + \underbrace{I_h u(t_n) - u_h^n}_{e_h^n}$$
$$z^n = p(t_n) - p_h^n = \underbrace{p(t_n) - J_h p(t_n)}_{z_i^n} + \underbrace{J_h p(t_n) - p_h^n}_{z_h^n}$$

where $I_h: H_0^1(\Omega)^3\to V_h$, $J_h: L_0^2(\Omega)\to W_h$ are two interpolation operators.
Comparing the exact problem and the scheme, one arrives at the following error equations:

$$\begin{cases} (\delta_t e_h^{n+1}, v_h) + a(e_h^{n+1}, z_h^{n+1}) = \langle \mathcal{E}^{n+1}, v_h\rangle - (\delta_t e_i^{n+1}, v_h) - (\nabla e_i^{n+1}, \nabla v_h) \\ \qquad\qquad + (z_i^{n+1}, \nabla\cdot v_h) - c(u_h^n, e^{n+1}, v_h) - c(e^n, u(t_{n+1}), v_h), \end{cases} \tag{6}$$

where $\mathcal{E}^{n+1}$ denotes the consistency error in time.

$\mathbb{P}_1-bubble/\mathbb{P}_1$ approximation and I_h an interpolation operator satisfying the discrete divergence-free condition

$$\text{i.e., } (\nabla\cdot(u-I_h u), q_h) = 0 \quad \forall\, q_h\in W_h, \quad \forall\, u\in H^1(\Omega)^3. \tag{7}$$

I_h can be defined verifying the discrete divergence-free condition and the following aproximation and stability properties ([1]):

- $\frac{1}{h}\|I_h u - u\|_{L^2} + \|I_h u - u\|_{H^1} \leq C\, h\, \|u\|_{H^2}$
- $\|I_h u - u\|_{H^{-1}} \leq C\, h \|I_h u - u\|_{L^2} \leq C\, h^2\, \|u\|_{H^1}$
- $\|I_h u\|_{L^\infty\cap W^{1,3}} \leq C\, \|u\|_{H^2}$

Under the regularity assumptions:

$$u\in L^\infty H^{m+1}, \quad p\in L^\infty H^m, \quad u_t\in L^2H^m, \quad u_{tt}\in L^2H^{-1},$$

we obtain

$$||e_h^n||_{l^\infty L^2\cap l^2H^1} \leq C(k+h^m), \text{ and } ||e^n||_{l^\infty L^2\cap l^2H^1} \leq C(k+h^m)$$

By using a duality argument (taking $v_h = A_h^{-1}e_h^{n+1}$, where A_h^{-1} is the discrete Stokes Operator), the l^2L^2-norm can be improved:

$$||e_h^n||_{l^2L^2} \leq C\,(k+h^{m+1}) \quad \text{and} \quad ||e^n||_{l^2L^2} \sim O(k+h^{m+1}).$$

$\mathbb{P}_m/\mathbb{P}_{m-1}$ $(m\geq 2)$ approximation and $(I_h,J_h)(u,p)$ the Stokes Projector of (u,p) onto $V_h\times W_h$,

i.e., $(I_h u, J_h p)\in V_h\times W_h$ s. t. $a((u-I_h u, p-J_h p),(v_h,q_h)) = 0 \quad \forall\,(v_h,q_h)\in V_h\times W_h$.
The following approximation property holds ([4]):

$$\|I_h u - u\|_{H^1} + \|J_h p - p\|_{L^2} \leq C\, h^m\, (\|u\|_{H^{m+1}} + \|p\|_{H^m})$$

Theorem 3. *Given $(f,g)\in H^{-1}(\Omega)^3\times L_0^2(\Omega)$, let $(u,p)\in H_0^1(\Omega)^3\times L_0^2(\Omega)$ such that*

$$a((u,p),(v,q)) = \langle (f,g),(v,q)\rangle \quad \forall\,(v,q)\in H_0^1(\Omega)^3\times L_0^2(\Omega).$$

Let $s\geq 1$ be an integer. Assume the assumptions:
1. $||u||_{H^{s+1}(\Omega)} + ||p||_{H^s(\Omega)} \leq C(||f||_{H^{s-1}(\Omega)} + ||g||_{H^s(\Omega)})$,
2. $\inf_{v_h\in V_h}||u-v_h||_{H^1(\Omega)} + \inf_{q_h\in W_h}||p-q_h||_{L^2(\Omega)} \leq C\, h^s(||u||_{H^{s+1}(\Omega)} + ||p||_{H^s(\Omega)})$.
Then:

$$||u-I_h u||_{H^{-(s-1)}} + ||p-J_h p||_{H^{-s}} \leq C\, h^s(||u-I_h u||_{H^1(\Omega)} + ||p-J_h p||_{L^2(\Omega)})$$

where $H^{-(s-1)}$ is the dual space of $H^{s+1}(\Omega)\cap H_0^1(\Omega)$ and H^{-s} the dual of $H^s(\Omega)\cap L_0^2(\Omega)$.

Lemma 1. *Let $(u_h,p_h)\in V_h\times W_h$ s.t. $a((u_h,p_h),(v_h,q_h)) = (f,v_h)\ \forall\,(v_h,q_h)\in V_h\times W_h$, then*

$$\|u_h\|_{W^{1,6}} + \|p_h\|_{H^1} \leq C\,\|f\|_{L^2}.$$

Corollary 1. $\|I_h u\|_{W^{1,6}} + \|J_h p\|_{H^1} \leq C\,(\|u\|_{H^2} + \|p\|_{H^1})$ *holds for any* $(u,p)\in H^2\times H^1$.
Imposing regularity $u_t\in L^2H^{m+1}$ and $p_t\in L^2H^m$, we obtain

$$||e_h^n||_{l^\infty L^2\cap l^2H^1} \sim O(k+h^{m+1}) \text{ and } ||e^n||_{l^\infty L^2} \sim O(k+h^{m+1}).$$

And combining Theorem 3 ($s=2$ and assuming the $H^3\times H^2$-regularity of the Stokes Problem) with duality results,

$$\|e_h^n\|_{l^2(L^2)} \leq C\,(k+h^{m+2}).$$

Error Estimates for Pressure

With the Stokes Projector the error equations (6) can be written as follows:

$$(\delta_t e_h^n, v_h) + a((e_h^n, z_h^n),(v_h,q_h)) = (g, v_h)$$

where $g = \mathcal{E}^n + \delta_t e_i^n + C(u_h^{n-1}, e^n) + C(e^{n-1}, u(t_n))$. Using Lemma 1 we can deduce

$$\|\delta_t e_h^n\|_{l^2L^2}^2 + \|e_h^n\|_{l^\infty H^1}^2 + \|e_h^n\|_{l^2W^{1,6}}^2 + \|z_h^n\|_{l^2H^1}^2 \leq C\, e^{(k\sum_n K(n))}\left(k^2+h^{2m}\right).$$

where

$$K(n) = \|e_h^n\|_{H^1}^4 + \|e_h^{n-1}\|_{H^1}^2\|e_h^n\|_{H^1}^2$$

Estimates with $h << f(k)$: Since $\|e_h^{n-1}\|_{H^1}^2, \|e_h^n\|_{H^1}^2 \leq C(k^2+h^{2(m+1)})/k$, then

$$k\sum_n K(n) \leq C\left(k + \frac{h^{2(m+1)}}{k}\right)(k^2+h^{2(m+1)}) \leq C \quad \text{If (H1): } \frac{h^{4(m+1)}}{k} \leq C.$$

Estimates with $k << g(h)$: By using the Inverse Inequality $\|e_h\|_{H^1} \leq C\, h^{-1}\|e_h\|_{L^2}$, then

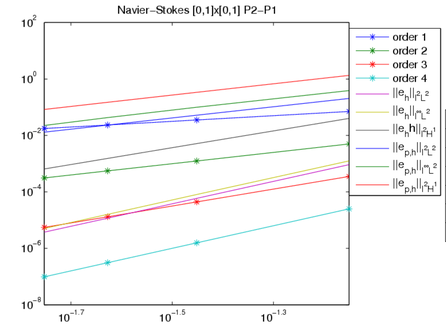
$$k\sum_n K(n) \leq C\,\frac{1}{h^2}(k^2+h^{2(m+1)})^2 \leq C \quad \text{If (H2): } \frac{k^4}{h^2} \leq C.$$

Combining both arguments, we deduce the following estimates without restrictions on the discrete parameters h,k:

$$\|\delta_t e_h^n\|_{l^2L^2}^2 + \|e_h^n\|_{l^\infty H^1}^2 + \|e_h^n\|_{l^2W^{1,6}}^2 + \|z_h^n\|_{l^2H^1}^2 \leq C\,(k^2+h^{2m}).$$

Numerical Simulations

- $\Omega=(0,1)\times(0,1)$, $T=(0,1)$, $k=\frac{1}{10000}$, $\mathbb{P}_2/\mathbb{P}_1$ approximation.
- A uniform triangulation of Ω with $h=\frac{1}{10},\frac{1}{20},\frac{1}{30},\frac{1}{40}$ and structured meshes.
- Crank-Nicolson scheme for the linear term and a 'linearization' of the convective term, considering the term $c(u_h^k, u_h^{k+1}, v_h)$ in the $k+1$ step.
-
$$u_1(x,y,t) = e^{-t}(-cos(2\pi x)sin(2\pi y) + sin(2\pi y))$$
$$u_2(x,y,t) = -e^{-t}(-cos(2\pi x)sin(2\pi y) + sin(2\pi y))$$
$$p(x,y,t) = 2\pi(cos(2\pi y) - cos(2\pi x))$$



CONVERGENCE RATES			
$\|\|I_h u - u_h\|\|_{l^2L^2}$	3.9537	3.9971	3.9931
$\|\|I_h u - u_h\|\|_{l^2H^1}$	2.9598	2.9828	2.9892
$\|\|I_h u - u_h\|\|_{l^\infty L^2}$	3.9668	4.0000	3.9490
$\|\|J_h p - p_h\|\|_{l^2L^2}$	1.9636	2.0927	2.2133
$\|\|J_h p - p_h\|\|_{l^\infty L^2}$	1.9659	1.9973	2.0102
$\|\|J_h p - p_h\|\|_{l^2H^1}$	1.9957	2.0102	2.0182

References

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