

## Exercises for the course *Variational Modelling* by Mark Peletier

**Exercise 1.** In the lectures we made the claim that using primal and dual spaces makes the equations automatically coordinate-independent, as a physical theory should be.

Verify this for a gradient flow

$$\mathcal{Z} = \mathbb{R}^2, \quad \mathcal{F} : \mathcal{Z} \rightarrow \mathbb{R}, \quad \mathcal{G}(z) \in \mathbb{R}^{2 \times 2},$$

for instance by applying a linear transformation of the type

$$y := Az \quad \text{with} \quad A \in \mathbb{R}^{2 \times 2} \quad \text{invertible.}$$

How do the expressions for  $\mathcal{F}$  and  $\mathcal{G}$  in terms of  $y$  relate to those in terms of  $z$ ? And  $\mathcal{K}$ ?

**Exercise 2.** For given  $\mathcal{G}$  and  $\mathcal{K}$ , define the norms and bilinear forms on  $\mathcal{Z}$  and  $\mathcal{Z}'$  respectively:

$$\begin{aligned} \|s\|_{\mathcal{G},z}^2 &:= \langle \mathcal{G}(z)s, s \rangle & \|\xi\|_{\mathcal{K},z}^2 &:= \langle \mathcal{K}(z)\xi, \xi \rangle \\ (s_1, s_2)_{\mathcal{G},z} &:= \langle \mathcal{G}(z)s_1, s_2 \rangle & (\xi_1, \xi_2)_{\mathcal{K},z}^2 &:= \langle \mathcal{K}(z)\xi_1, \xi_2 \rangle. \end{aligned}$$

Prove that if any of the six objects  $\mathcal{G}$ ,  $\mathcal{K}$ ,  $\|\cdot\|_{\mathcal{G},z}^2$ ,  $\|\cdot\|_{\mathcal{K},z}^2$ ,  $(\cdot, \cdot)_{\mathcal{G},z}$ , or  $(\cdot, \cdot)_{\mathcal{K},z}$  is chosen, then the other five follow automatically. (For this exercise a form of non-degeneracy can be assumed).

**Exercise 3.** The norms  $\|\cdot\|_{\mathcal{G},z}$  and  $\|\cdot\|_{\mathcal{K},z}$  defined above satisfy a duality inequality:

$$\forall s \in \mathcal{Z}, \xi \in \mathcal{Z}' : \quad \langle s, \xi \rangle \leq \|s\|_{\mathcal{G},z} \|\xi\|_{\mathcal{K},z}.$$

Equality happens if and only if  $s = \lambda \mathcal{K}(z)\xi \iff \xi = \lambda^{-1} \mathcal{G}(z)s$  with  $\lambda > 0$ . Prove these statements.

**Exercise 4.** Formulate a model for the following system. Particle  $A$  has mass  $m$  but zero physical size; particle  $B$  is a ball of radius  $r$  with no mass. The two are connected by a linear spring with spring constant  $k$ . Both particles are embedded in a viscous fluid with mass density  $\rho$  and dynamic viscosity  $\nu$  and subject to gravity.

**Exercise 5.** An important entropic force is *osmotic pressure*; in this exercise we derive the Van 't Hoff formula for this pressure. (Jacobus Henricus van 't Hoff, Jr., 1852–1911, was a Dutch physical and organic chemist and the first winner of the Nobel Prize in Chemistry. There aren't many Dutch Nobel laureates, so it's good to emphasize one when we find one ...)

Consider a container with solvent and particles, separated into two parts by a movable membrane. The membrane is semi-permeable: it is invisible to the water molecules, but the particles can not pass through it. We place the particles on one side of the membrane.

Choosing a one-dimensional setup, assume the water occupies the region  $[0, L]$ , the membrane is at  $x = \varphi L$ ,  $\varphi \in [0, 1]$ , and the particles are confined to the region  $[0, \varphi L]$ . The total number of particles is  $N$ .

1. Calculate the entropy of the particles as a function of the parameter  $\varphi$ .
2. Calculate the corresponding force, i.e. minus the derivative of the entropy with respect to the position  $\varphi L$ . This force is the so-called osmotic pressure.
3. Repeat the calculation for the case where the particles are present on both sides (but maybe with different concentrations). If you express the pressure in terms of the particle concentrations on both sides, then the result is known as the Van 't Hoff formula.