Max Plus Decomposition of Supermartingale with application to Portfolio Insurance

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The classical Doob-Meyer décomposition of supermartingale states the existence of a previsible increasing process A and of a martingale N such that a supermartingale Z can be decomposed into $N = Z + A$.
The Max Plus Algebra is obtained in exchanging the operations $(+, x)$ into $\max, +)$. Therefore, the Max-Plus decompositon becomes :
• find a martingale N and and adapted increasing process Λ with max-plus density $L \in [-\infty, +\infty]$ such that

$$M = \max(Z, \Lambda), \quad \Lambda_t = \sup_{s \leq t} L_s$$

The existence is proved via convex analysis argument.
The martingale M is also characterized as the optimal solution of the following optimisation martingale problem :
• Find the smallest martingale $M^\star$ with respect to the stochastic order of their terminal value among the martingales majorizing Z and with initial value Z_0.
In mathematical terms, let $\mathcal{M}$ be the set of martingales with $M_0 = Z_0$, and T the horizon of the study. The problem is to find $M^\star \in \mathcal{M}$ such that for any *convex* function g

$$E(g(M_T)) \geq E(g(M_T^\star)), \quad \forall \quad M \geq Z$$

In Finance, the problem of tailoring an self-financing portfolio V submitted to the American constrainted $V \geq F$, where F is the dynamic floor (the guarantee), optimal for some with expected utility criterium on the terminal value may be solved by the previous result, given the optimal solution of the non-constrained problem, and a change of probability measure.