Where does Combinatorial Analysis fit in?

David Bressoud
Macalester College

(1) What’s in chapter CM
(2) My favorite bits of Combinatorial Analysis
Topics for a Chapter on Combinatorial Analysis

(1) counting techniques
   (inclusion-exclusion, generating functions etc.)
(2) lattice paths
(3) permutations
(4) set partitions
(5) integer partitions
(6) algebraic combinatorics
(7) Polya theory
(8) graph theory
(9) partially ordered sets
(10) combinatorial geometry
(11) combinatorial designs

Maybe this is too ambitious.
Chapter 24 of HMF:

(1) binomial and multinomial coefficients,
    Stirling numbers
(2) partitions (unrestricted and into distinct parts)
(3) arithmetic functions: Möbius, Euler, divisor;
    primitive roots; factorization and primes

Chapter NT will pick up unrestricted partitions
and arithmetic functions

factorization & primality testing:
Is there any point to a printed table of factors or primes?

What about tables of partition numbers, Stirling numbers?
(HMF has partitions to $n = 500$.
    Stirling numbers to $n = 25$.)
Chapter CM of DLMF:

(1) lattice paths:
   binomial coefficients, multinomial coefficients,
   Catalan numbers, others
(2) set partitions: Bell and Stirling numbers
(3) integer partitions: various restrictions, compositions
(4) plane partitions
(5) permutations:
   statistics on words, Eulerian numbers, rook polynomials,
   multiset permutations
(6) counting techniques:
   12-fold way, inclusion/exclusion, Polya theory
(7) applications
(8) primes and factoring
What are some of the most exciting applications of combinatorial analysis emerging today?

EXPLOITING SYMMETRY

- multidimensional extensions of classical special functions
- Selberg integral and its generalizations
- Macdonald identities
- mock theta functions
- ties to representation theory
- ties to quantum mechanics
Vandermonde formula:
(Cauchy, *J. de l’École Polytechnique* 1815)

\[
\prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \prod_{i=1}^{n} x_i^{n-\sigma(i)}.
\]

**Proof:** RHS is an alternating fcn. of \(x_1, \ldots, x_n\), so divisible by LHS, both sides of degree \(n(n - 1)/2\). Coefficient of \(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}\) is 1 on each side.
Proof of Jacobi’s triple product identity:

\[ [x; q]_\infty = (x; q)_\infty (q / x; q)_\infty = \prod_{i=1}^{\infty} (1 - x q^{i-1}) (1 - x^{-1} q^i) \]

\[ [xq; q]_\infty = [x; q]_\infty \frac{1 - x^{-1}}{1 - x} = -x^{-1} [x; q]_\infty \]

Let \( [x; q]_\infty = \sum_{n=-\infty}^{\infty} a_n(q) x^n \).

\[ a_n(q) q^n = -a_{n+1}(q) \quad \Rightarrow \quad a_n(q) = (-1)^n q^{n(n-1)/2} a_0(q) \]

\[ [x; q]_\infty = a_0(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n \]
Macdonald
(Inventiones Mathematicae 1972)

$R$ a root system spanning $\mathbb{R}^n$, $L$ a lattice in $\mathbb{Z}^n$

$$(q; q)_\infty^n \prod_{\alpha \in R^+} [\alpha \cdot q]_\infty$$

$$= \sum_{\mu \in L} \left( \sum_{\omega \in W(R)} \text{sgn}(\omega) e^{\mu \cdot \rho - \mu \cdot (\rho - \mu)} \right) e^{\mu \cdot \rho} q^{\mu \cdot \rho / 2} - \rho \cdot \rho / \rho$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

E.g.

$A_n = \{ \epsilon_i - \epsilon_j | 0 \leq i < j \leq n \}$,

$L = \{ (n + 1) \sum_{j=0}^n k_j \epsilon_j | \sum k_j = 0 \}$,

$\rho = \frac{n}{2} \epsilon_0 + \left( \frac{n}{2} - 1 \right) \epsilon_1 + \left( \frac{n}{2} - 2 \right) \epsilon_2 + \cdots - \frac{n}{2} \epsilon_n$,

$g = n + 1$
Dyson’s conjecture

The constant term (w.r.t. \(x_1, \ldots, x_n\)) in

\[
\prod_{i \neq j} (1 - x_i/x_j)^{a_i}
\]

is

\[
\binom{a_1 + \cdots + a_n}{a_1, \ldots, a_n}.
\]

**Proof:**

\[
1 = \sum_j \prod_{i \neq j} \frac{x - x_j}{x_i - x_j} \implies 1 = \sum_j \prod_{i \neq j} \frac{1}{1 - x_i/x_j}.
\]

\[
\prod_{i < j} (1 - x_i/x_j)^{a_i} \quad \text{and} \quad \binom{a_1 + \cdots + a_n}{a_1, \ldots, a_n}
\]
satisfy the same recursion and the boundary values of the constant term are (by induction) the appropriate multinomial coefficients.
Andrews’s $q$-Dyson conjecture:
(Zeilberger & B., *Discrete Math.* 1985)

The constant term (w.r.t. $x_1, \ldots, x_n$) in

$$\prod_{i<j} [x_i/x_j; q]_{(a_i, a_j)}$$

is

$$\frac{(q; q)_{a_1 + \ldots + a_n}}{(q; q)_{a_1} \cdots (q; q)_{a_n}}.$$

\[ [z; q]_{(a,b)} = (z; q)_a (q/z; q)_b \]

\[ (z; q)_a = \prod_{j=1}^a (1 - z q^{j-1}) \]

Due for a new proof.
Variation on Gessel’s proof of Vdm:

\[
\prod_{i<j} (x_i - x_j) = \sum_{T \in \mathcal{T}} (-1)^{U(T)} \prod_{i} x_{i}^{\omega(i)}
\]

where \( \mathcal{T} \) is the set of tournaments (complete directed graphs),
\( U(T) \) is the number of upsets (directed edges from \( j \) to \( i \) with \( j > i \)) and
\( \omega(i) \) is the out-degree of vertex \( i \).

**Proof:**

Put an order on pairs of vertices and find the first pair with the same out-degree.
Switch labels.
This pairs tournaments with weights of opposite sign.
The remaining tournaments correspond to permutations.

\[
\prod_{i<j} (x_i - x_j) = \sum_{\sigma \in \mathcal{S}} (-1)^{Z(\sigma)} \prod_{i} x_{i}^{n-\sigma(i)}
\]
Jacobi’s adjoint matrix theorem:

$$\det(M) = \frac{\det(M_n^1) \det(M_n^n) - \det(M_1^n) \det(M_1^1)}{\det(M_{1,n}^{1,n})}.$$ 

Robbins-Rumsey $\lambda$-determinant:

$$\det_\lambda(M) = \frac{\det_\lambda(M_1^1) \det_\lambda(M_n^n) + \lambda \det_\lambda(M_1^n) \det_\lambda(M_1^1)}{\det_\lambda(M_{1,n}^{1,n})}.$$

$$\det_\lambda(x_i^{n-j}) = \prod_{1 \leq i < j \leq n} (x_i + \lambda x_j).$$
\[
\text{det}_\lambda \begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{pmatrix}
= aei + \lambda(bdi + afh) + \lambda^2(bfg + cdh) + \lambda^3ceg + \lambda(1 + \lambda)kde^{-1}fh
\]

\[
\text{det}_\lambda \begin{pmatrix}
  a & b & c & d \\
  e & f & g & h \\
  i & j & k & l \\
  m & n & o & p \\
\end{pmatrix}
= \cdots + \lambda^3(1 + \lambda)bcf^{-1}hk + \lambda^2(1 + \lambda)cf^{-1}hi + \lambda^{1}(1 + \lambda)hij^{-1}kn + \cdots
\]
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
\det_\lambda (a_{ij}) = \sum_{B \in A_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{N(B)} \prod_{i,j=1}^n a_{ij}^{B_{ij}}.
\]

\(\mathcal{I}(B)\) is the inversion number of \(B\)
\(N(B)\) is the number of \(-1\)s in \(B\)

\[
\prod_{1 \leq i < j \leq n} (x_i + \lambda x_j) = \sum_{B \in A_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{N(B)} \prod_{i,j=1}^n x_i^{(n-j)B_{ij}}.
\]

Okada (J. Alg. Comb., 1993) has found similar formulas for other root systems.
Mills-Robbins-Rumsey Conjecture:
(Zeilberger Electronic J. of Combinatorics 1996)

\[ |A_n| = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} \]

Also the number of Descending Plane Partitions with parts \( \leq n \)
Also the number of Totally Symmetric Self-Complementary Plane Partitions
in a \( 2n \times 2n \times 2n \) box.
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
\begin{array}{cccccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
\uparrow & \downarrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
\downarrow & \uparrow & \uparrow & \downarrow & \uparrow & \uparrow & \uparrow \\
\rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
\downarrow & \downarrow & \uparrow & \uparrow & \downarrow & \uparrow & \uparrow \\
\rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
\end{array}
\]

\# horiz = n + N, \quad \# vert = N
\# NE = \#SW = I - N
\# NW = \#SE = \left( \binom{n}{2} \right) - I
Cauchy:  
(J. de l’École Polytechnique 1815)

\[
det \left( \frac{1}{(x_i + y_j)} \right) \prod_{i,j=1}^{n} (x_i + y_j) \prod_{1 \leq i < j \leq n} (x_i - x_j)^{-1}(y_i - y_j)^{-1} \]

\[= 1.\]
Izergin and Korepin:
(Quantum Inverse Scattering Method and Correlation Functions 1993)

\[
\det \left( \frac{1}{(x_i + y_j)(ax_i + y_j)} \right) \prod_{i,j=1}^{n} (x_i + y_j)(ax_i + y_j) \prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)
\]

\[= \sum_{A \in A_n} (-1)^{N(A)} (1 - a)^{2N(A)} a^{\chi(A)} b^{\gamma(A)}
\times \prod_{\text{vert}} x_i y_j \prod_{\text{NE, SW}} (ax_i + y_j) \prod_{\text{NW, SE}} (x_i + y_j),
\]

**Proof:** Left side is symmetric polynomial in \(x_i\) and in \(y_i\), of degree \(n - 1\) in \(x_1\), so uniquely determined by limiting value as \(x_1 \to -y_1\). By Baxter’s triangle-to-triangle relation, same is true of right side. Limiting values are equal by induction.
Set $a = 1$:

$$\det \left( \frac{1}{(x_i + y_j)^2} \right) = \prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j) \sum_{\sigma \in S_n} \prod_{i=1}^{n} (x_i + y_{\sigma(i)})^{-1}. $$

Set $a = e^{2\pi i/3}$, $x_j = -\omega$, $y_j = 1$:

$$(-3\omega)^{n^2-n} \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n + j)!} = (-3\omega)^{n^2-n} |A_n|. $$