A One-Dimensional Van der Waals Charged System & Lattice Path Mappings

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• **Statistical Mechanics**
  - A Branch of Mathematical Physics
  - Derives Macro Laws From Microscopic Ones
  - Classical Mechanics $\Rightarrow$ Thermodynamics
  - Potential Energy $\Rightarrow$ Free Energy
  - $H(\vec{x}, \vec{p}) = K + U \Rightarrow Z = \int e^{-\beta H} \Rightarrow F(T, S, N)$
  - Thermodynamic Limit: $\lim_{N \to \infty} \frac{\ln Z}{-\beta N} = F$

• **Charged Systems (Plasmas)**
  - Complications: Long-Range Potentials
  - Interest: Basic Component of Matter
Initial Goals & Results

• Goals: Derive the **Exact** Thermodynamics of a Model & Show (Non-)Existence of Phase Transitions

• Result #1: Modified 1-D Two Component Plasma Solved Exactly

• Result #2: It Exhibits No Phase Transitions

• **Result #3:** Suitable Modifications Lead To a Van Der Waals Equation of State.
  
  • Original Van Der Waals: Modified Ideal Gas Law
    
    \[ p = \frac{kT}{v} \Rightarrow p = kT \left( \frac{1}{v-b} - \frac{\alpha}{v^2} \right) \]
  
  • Equation with a “wiggle” flattened by the Maxwell “equal area” construction
The Main Result

- (1999) D. Chelst: Derived Van Der Waals (Liquid-Vapor) Phase Transition For Charged System (1-D Modified TCP’s)
  - Basic TCP’s thermodynamics solved by Pringsheim (1961) and Lenard (1961)
  - Earlier Van der Waals Derivations: Kac, Uhlenbeck and Hemmer (1963-65) ⇒ Lebowitz and Penrose (1966)
  - Modified L. and P.’s Arguments For Charged Systems
  - Added New Combinatorial Theorem Invoking Charge “Screening”
One-Dimensional Two-Component Plasma

- 2N Ions, Charges $\sigma_i = \pm \sigma$, Neutral $\sum \sigma_i = 0$, Positions $x_i \in [0, L]$
- One Dimension: Order Positions $x_i \leq x_{i+1}$
- Charge Ordering $\vec{\sigma} \rightarrow$ Electric Field $\vec{E}$
  - $E_0 = 0$, $E_i = E_{i-1} + \sigma_i$, $E_{2N} = 0$
- System’s Energy: $U(\vec{E}, \vec{x}) = \sum_{i=1}^{2N-1} E_i^2 (x_{i+1} - x_i)$
- Additional Modifications Include: Hard Core Exclusions, Nearest-Neighbor Potentials, Additional Dipoles
System Illustrations

Basic Charges & Electric Field

Ignoring Positions
Focusing On Smaller Parts

- $\vec{E} \rightarrow (\vec{E}^{(1)}, \vec{E}^{(2)})$
- Ions: $N^{(1)} = N_+^{(1)} + N_-^{(1)}$, $N_+^{(1)} - N_-^{(1)} = a$.
- If $a = 0$, $\vec{E}^{(1)}$ and $\vec{E}^{(2)}$ Function Independently.
Smaller Parts, Cont.

- Physically, Lower Energy Is Better.
- Prove: Fix $N^{(1)}$ (Even); Then $a = 0$ Is Best Choice.
Let’s Be Specific!

Let $C(N, a, b) = \{ \vec{E} : \text{Unit Steps}, \, E_0 = a, \, E_N = b \}$

**Theorem:** A Map Exists $\kappa : C(N, a, b) \rightarrow C(N, a \mod 2, b \mod 2)$

That Is

1. Injective, and

2. Lowers Energy: If $\vec{E}' = \kappa(\vec{E})$, Then $|E'_i| \leq |E_i|$ For All $i$.

**Idea of Proof:** Plot Sample $\vec{E}$ to the points $(i, E_i)$ and Use A Composition of “Reflections”.

![Graph showing the plot of a vector $\vec{E}$ to the points $(i, E_i)$ and the use of a composition of reflections to transform the vector.]
The Lattice Paths Connection

- Rotate The plot $45^\circ$
- Lattice Path From $(0, 0)$ To $(N_-, N_+)$

- Compare Paths of Equal Length With Different Endpoints
- Map To Path Closer To $x = y$ Using “Conjugation”
Lattice Paths Continued

• Question #1: Do Equivalent Lattice Path Theorems Exist? Usually: Counting, Fixed Endpoint, Different Criteria (Domination)

• Question #2: Have Additional Useful Methods Been Developed?
  • Alternative Representations
  • Different Mapping Techniques (Not Just Conjugation)
  • Different Lines, e.g. $x = 2y$
Food For Thought

- $3N$ Unbalanced Ions $\sigma_i \in \{-1, 2\}$, $\sum_i \sigma_i = 0$
- Nothing Proven: Conjugation-Reflection Fails
- Lattice Path $(0, 0)$ to $(2N, N)$
- Stay Close To $x = 2y$ Line.
Conclusions

- There is a connection between one-dimensional plasma and two-dimensional lattice path.
- One uses a lattice path conjugation argument to prove a physically relevant result.
- Lattice path analyses may hold more information for charged systems.
- Charged system analysis may interest lattice path devotees.