A One-Dimensional Van der Waals Charged System & Lattice Path Mappings

Dov Chelst

dchelst@nj.devry.edu

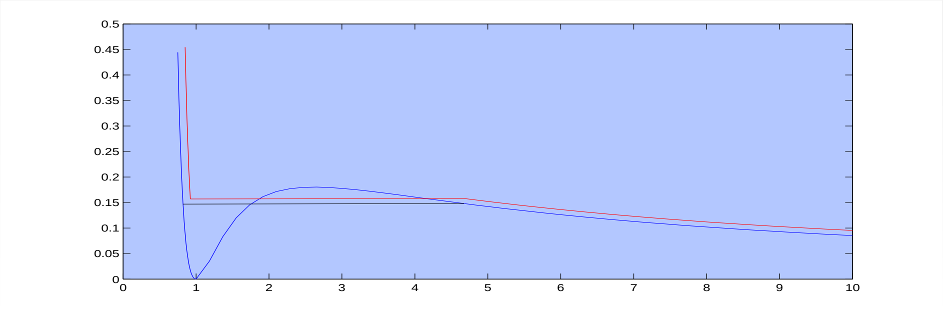
DeVry University

The Basics: Intro to Stat. Mech.

- **Statistical Mechanics**
  - A Branch of Mathematical Physics
  - Derives Macro Laws From Microscopic Ones
  - Classical Mechanics $\Rightarrow$ Thermodynamics
  - Potential Energy $\Rightarrow$ Free Energy
  - $H(\vec{x}, \vec{p}) = K + U \Rightarrow Z = \int e^{-\beta H} \Rightarrow F(T, S, N)$
  - Thermodynamic Limit: $\lim_{N \to \infty} \frac{\ln Z}{-\beta N} = F$
- **Charged Systems (Plasmas)**
  - Complications: Long-Range Potentials
  - Interest: Basic Component of Matter
  - Simplified Models: Ising Model, “Plasmas”, “Electrolytes”

Initial Goals & Results

- Goals: Derive the **Exact** Thermodynamics of a Model & Show (Non-)Existence of Phase Transitions
- Result #1: Modified 1-D Two Component Plasma Solved Exactly
- Result #2: It Exhibits No Phase Transitions
- **Result #3**: Suitable Modifications Lead To a Van Der Waals Equation of State.
  - Original Van Der Waals: Modified Ideal Gas Law $p = \frac{kT}{v} \Rightarrow p = kT\left(\frac{1}{v-b} - \frac{\alpha}{v^2}\right)$
  - Equation with a "wiggle" flattened by the Maxwell "equal area" construction



The Main Result

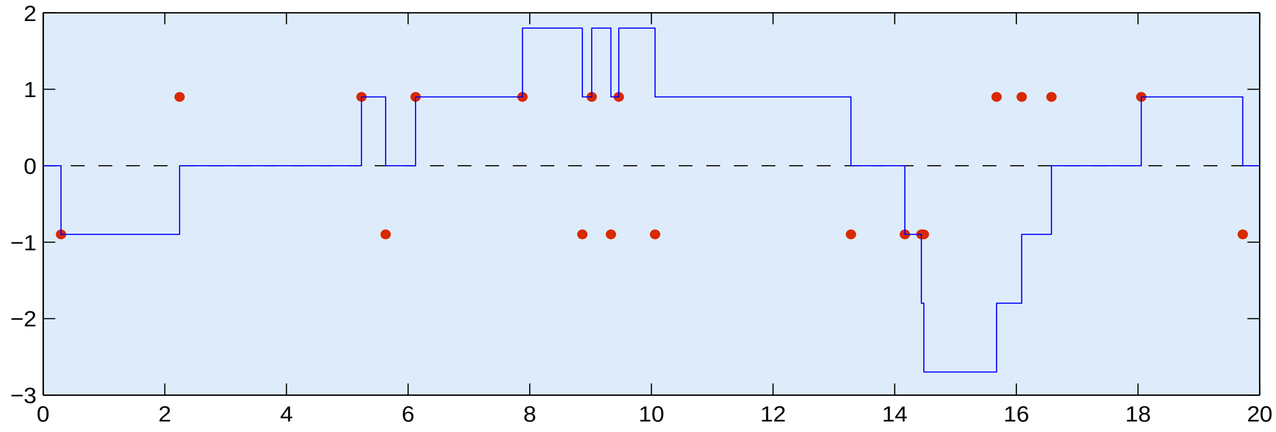
- (1999) D. Chelst: Derived Van Der Waals (Liquid-Vapor) Phase Transition For Charged System (1-D Modified TCP's)
  - Basic TCP's thermodynamics solved by Pringsheim (1961) and Lenard (1961)
  - Earlier Van der Waals Derivations: Kac, Uhlenbeck and Hemmer (1963-65) $\Rightarrow$ Lebowitz and Penrose (1966)
  - Modified L. and P.'s Arguments For Charged Systems
  - Added New **Combinatorial** Theorem Invoking Charge "Screening"

One-Dimensional Two-Component Plasma

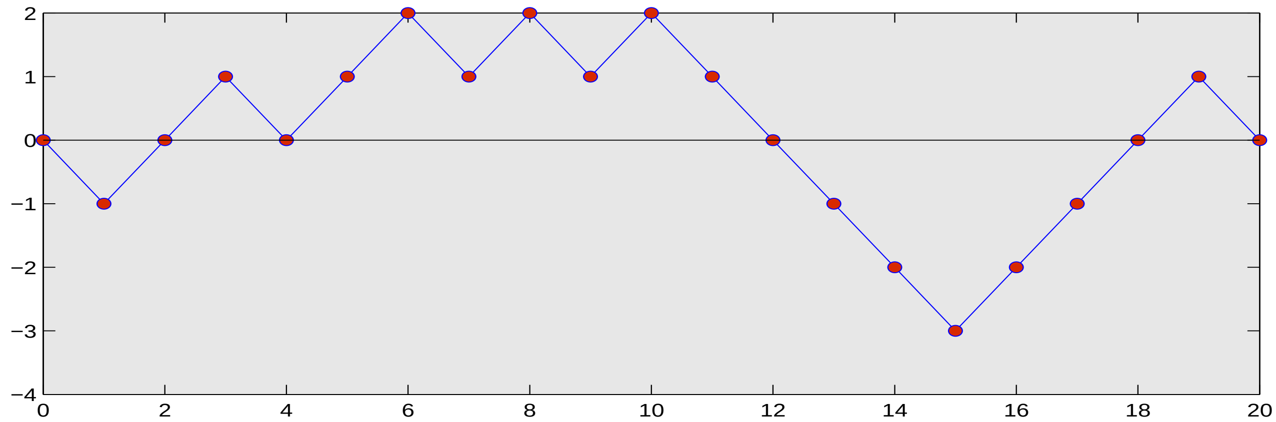
- $2N$ Ions, Charges $\sigma_i = \pm\sigma$, Neutral $\sum \sigma_i = 0$, Positions $x_i \in [0, L]$
- One Dimension: Order Positions $x_i \leq x_{i+1}$
- Charge Ordering $\vec{\sigma} \longrightarrow$ Electric Field $\vec{E}$
  - $E_0 = 0$, $E_i = E_{i-1} + \sigma_i$, $E_{2N} = 0$
- System's Energy: $U(\vec{E}, \vec{x}) = \sum_{i=1}^{2N-1} E_i^2 (x_{i+1} - x_i)$
- Additional Modifications Include: Hard Core Exclusions, Nearest-Neighbor Potentials, Additional Dipoles

System Illustrations

Basic Charges & Electric Field

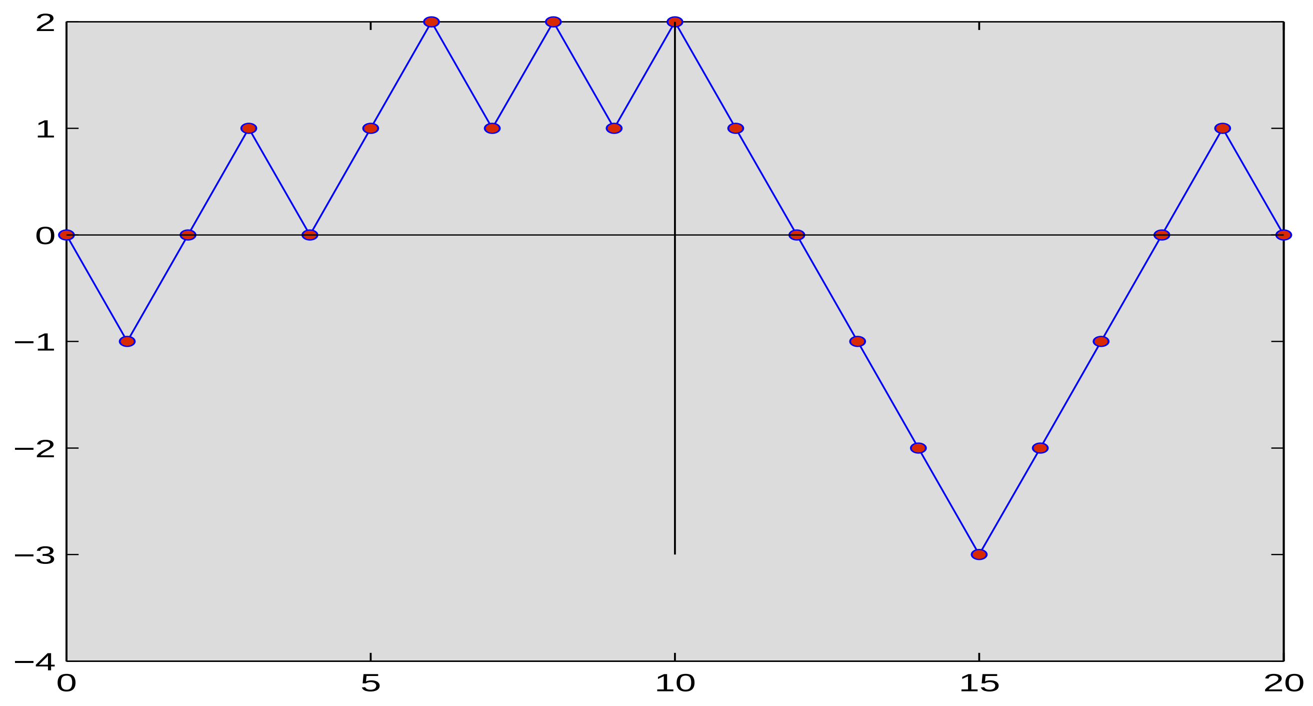


Ignoring Positions



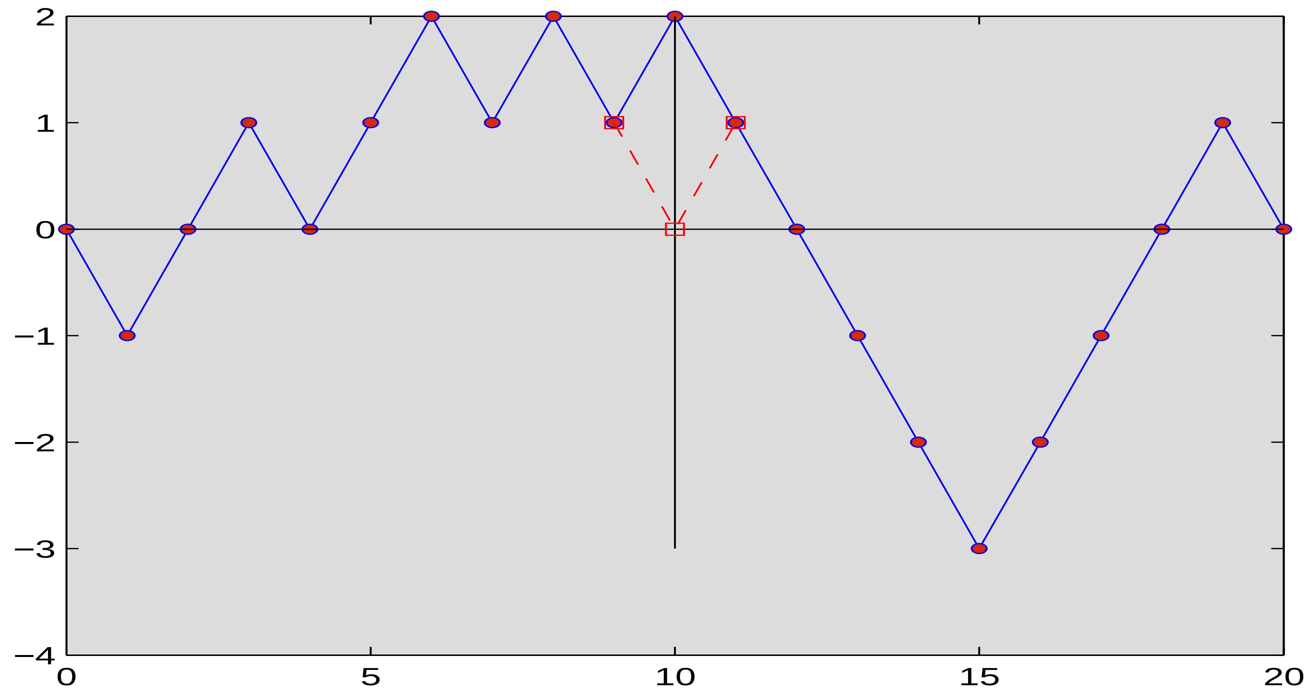
Focusing On Smaller Parts

- $\vec{E} \longrightarrow (\vec{E}^{(1)}, \vec{E}^{(2)})$
  - Ions: $N^{(1)} = N_+^{(1)} + N_-^{(1)}$, $N_+^{(1)} - N_-^{(1)} = a$.
  - If $a = 0$, $\vec{E}^{(1)}$ and $\vec{E}^{(2)}$ Function Independently.



Smaller Parts, Cont.

- Physically, Lower Energy Is Better.
- Prove: Fix $N^{(1)}$ (Even); Then $a = 0$ Is Best Choice.



Let's Be Specific!

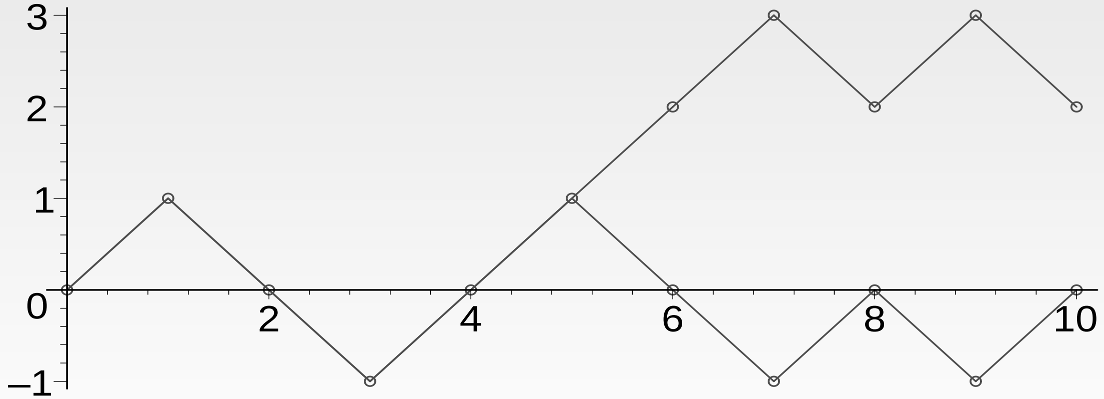
Let $C(N, a, b) = \{\vec{E} : \text{Unit Steps}, E_0 = a, E_N = b\}$

Theorem: A Map Exists $\kappa : C(N, a, b) \rightarrow C(N, a \bmod 2, b \bmod 2)$

That Is

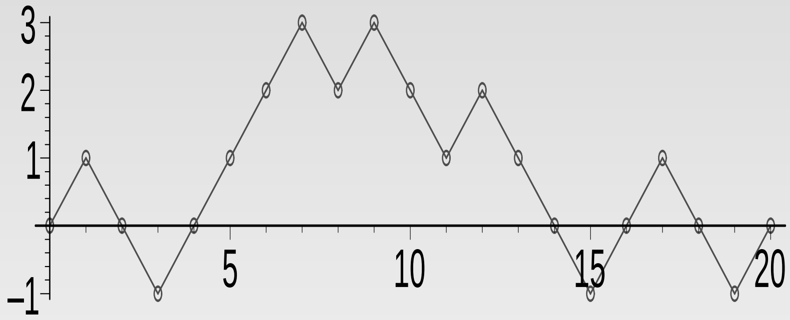
1. Injective, and
2. Lowers Energy: If $\vec{E}' = \kappa(\vec{E})$, Then $|E_i'| \leq |E_i|$ For All i.

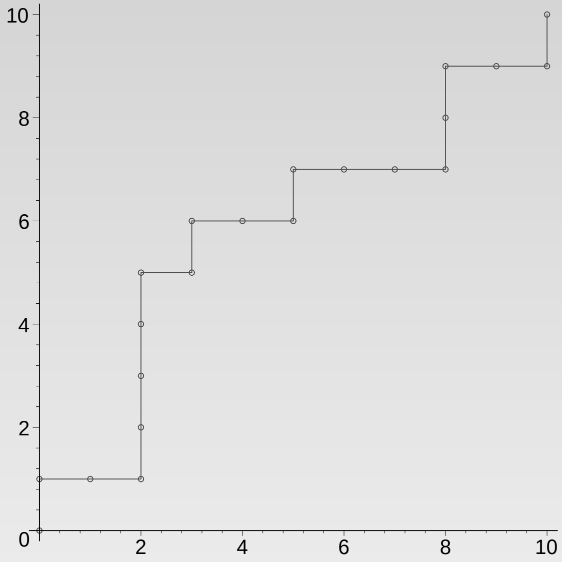
Idea of Proof: Plot Sample $\vec{E}$ to the points (i, E_i) and Use A Composition of "Reflections".



The Lattice Paths Connection

- Rotate The plot 45°
- Lattice Path From $(0,0)$ To (N_-, N_+)





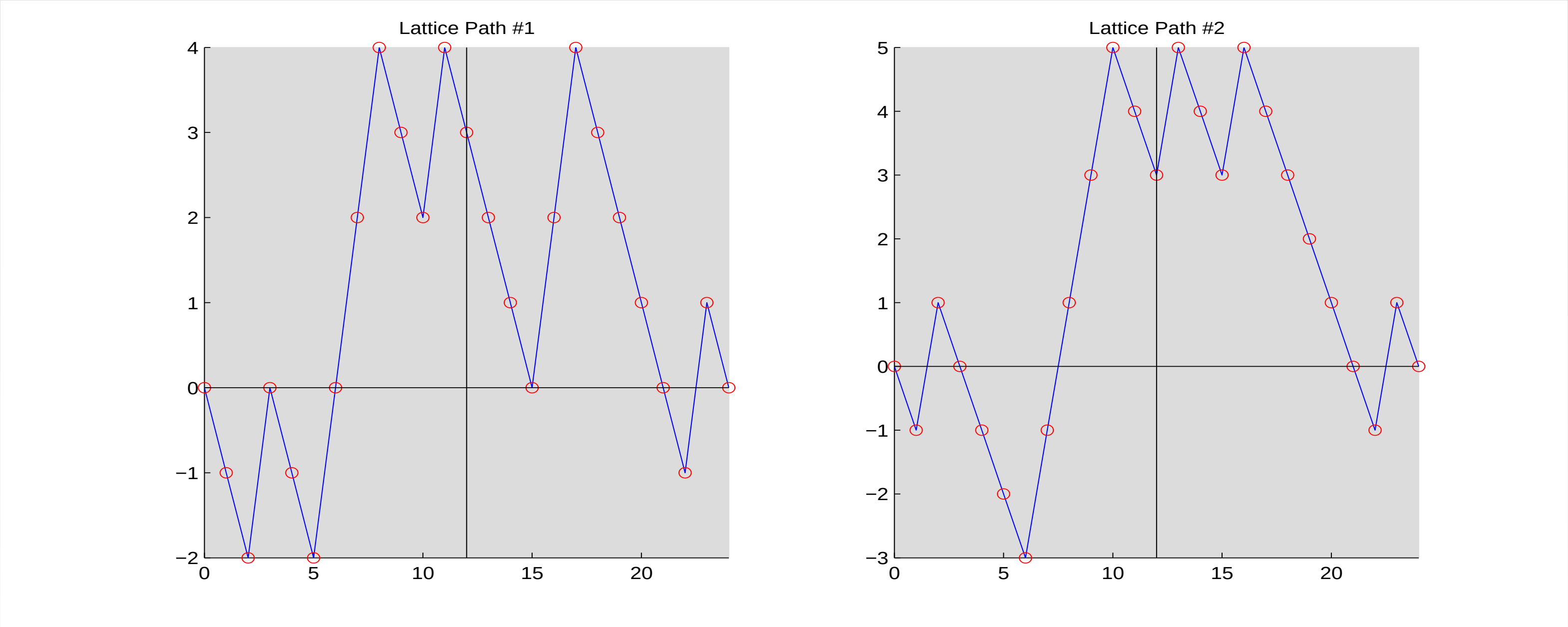
- **Compare** Paths of Equal Length With **Different Endpoints**
- Map To Path **Closer To** $x = y$ Using “Conjugation”

Lattice Paths Continued

- Question #1: Do Equivalent Lattice Path Theorems Exist? Usually: Counting, Fixed Endpoint, Different Criteria (Domination)
- Question #2: Have Additional Useful Methods Been Developed?
  - Alternative Representations
  - Different Mapping Techniques (Not Just Conjugation)
  - Different Lines, e.g. $x = 2y$

Food For Thought

- $3N$ Unbalanced Ions $\sigma_i \in \{-1, 2\}$, $\sum_i \sigma_i = 0$
- Nothing Proven: Conjugation-Reflection Fails
- Lattice Path $(0, 0)$ to $(2N, N)$
- Stay Close To $x = 2y$ Line.



Conclusions

- There Is A Connection Between One-Dimensional Plasma and Two-Dimensional Lattice Path
- One Uses A Lattice Path Conjugation Argument To Prove A Physically Relevant Result.
- Lattice Path Analyses May Hold More Information For Charged Systems.
- Charged System Analysis May Interest Lattice Path Devotees.