Mathematical problems in ad hoc wireless networks

Wing Shing Wong
Department of Information Engineering
Chinese University of Hong Kong
Agenda

- Networking Fairness (with F. J. Li)

- UWB/Wireless Sensor network based estimation (with Q. Li)
Agenda on Fairness Index

- Fairness index for peer-to-peer network
- Basic Properties
- Fairness-Price duality
- Von Neumann equilibrium
- Distributed algorithms for controlling fairness index
- Some preliminary results on routing (Fengjun Li)
Prior work on network fairness

- Focus on fairness in resource sharing
- Example, fairness in multiple access, link scheduling
- Asymmetric: Different roles between resource consumers and providers
Designing an efficient, fair, distributed, convergent algorithm (Chiu and Jain 89)
Efficient: Total allocation
Fair
Distributed
  - A version based on an underload/overload condition
Convergent
Increase decrease algorithm

- Observation (Chiu & Jain)
  \[ y(t) = \begin{cases} 
  1 & \text{if } X(t - 1) = \sum x_i(t - 1) > X_{goal}, \\
  0 & \text{if } X(t - 1) \leq X_{goal}. 
\end{cases} \]

- Affine type feedback control: \( \forall i \)
  \[ x_i(t) = \begin{cases} 
  a_I + b_I x_i(t - 1) & \text{if } y(t) = 0, \\
  a_D + b_D x_i(t - 1) & \text{if } y(t) = 1. 
\end{cases} \]

- For convergence to optimal fairness
  \[ a_I > 0, \quad b_I \leq 1, \quad a_D = 0, \quad 0 \leq b_D < 1. \]

- Additive increase, multiplicative decrease
Fairness for ad hoc networks

- Peer-to-peer structure implies symmetric role
- Nodes are critical resources and accounting authorities
- Power saving a key consideration
- Distributed controllers that require low input data rate
Lessons learned from power control

- Classical model for circuit-switched connections

\[
\max_{(P_1, \ldots, P_n)} \min_i \Gamma_i = \frac{p_i}{\sum_{i \neq j} Z_{ij} P_j + \eta_i}
\]

subject to \( P_i \geq 0 \)

- Solution is optimal when all the \( \Gamma_i \) are equal.

- If thermal noise is 0, then the optimal signal to noise that can be achieved is:

\[
\gamma^* = \frac{1}{\lambda_Z - 1}
\]

\( \lambda_Z \) is the Perron-Frobenius eigenvalue of \( Z \)
Distributed Algorithms

- How to determine the power vector?

- Zander’s Algorithm:

  \[ P^{(0)} = P_0 \]

  \[ P_i^{(n+1)} = \beta^{(n)} P_i^{(n)} \left[ 1 + \frac{1}{\Gamma_i^{(n)}} \right], \quad \beta^{(n)} > 0 \]

  \[ \lim_{n \to \infty} P^{(n)} = P^* \]

  \[ \lim_{n \to \infty} \Gamma_i^{(n)} = \gamma^* \]
QoS Tracking Model

- Given QoS requirements, $\gamma_i$, requires
  \[ \Gamma_i \geq \gamma_i \]
- A feasibility problem
- Matrix formulation:
  \[
  (I - B)P \geq \eta \\
  B_{ij} = \begin{cases}
    0 & i = j \\
    \gamma_i Z_{ij} & i \neq j
  \end{cases}
  \]
A nonnegative solution $P$ exists for any nonzero, nonnegative $\eta$, if and only if $\lambda_B < 1$. In this case, there is only one solution $P^*$, which is strictly positive and given by

$$P^* = (I - B)^{-1} \eta$$

The solution $P^*$ is Pareto optimal in the sense that for other feasible solution:

$$P \geq P^*$$
Distributed Algorithms

- Foschini-Miljanic Algorithm:

\[ P_i^{(n+1)} = \frac{\gamma_i}{\Gamma_i^{(n)}} P_i^{(n)} \]

- \( P^{(n)} \) converges to the optimal solution if a feasible solution exists
Distributed Algorithms

- Discrete Algorithm (Sung-Wong):

\[ P_i^{(n+1)} = \begin{cases} 
\delta P_i^{(n)} & \Gamma_i^{(n)} < \delta^{-1} \gamma_i \\
\delta^{-1} P_i^{(n)} & \Gamma_i^{(n)} > \delta \gamma_i \\
P_i^{(n)} & \text{otherwise}
\end{cases} \]

- Theorem: If a feasible solution exists, the discrete algorithm converges to a solution with the property:

\[ \delta^{-1} \gamma_i \leq \Gamma_i(\hat{P}) \leq \delta \gamma_i \]

- Relation with low data rate control problem
How to measure fairness in a peer-to-peer network?

- Incoming traffic
- Transient traffic
- Bilateral Peering Agreements
- Outgoing traffic
- Multilateral Peering Agreements MPLA
- $k$ nodes
Definition of Fairness Indices

\[ r_i \quad \text{total input traffic from node } i \]
\[ M_{ij} \quad \text{traffic distribution from node } j \text{ to node } i \]
\[ M_{ij}r_j \quad \text{traffic flow from node } j \text{ destined to node } i \]
\[ L_{ij} \quad \text{traffic flow from node } j \text{ passing through node } i \]
\[ M_{ii} = L_{ii} = 0 \]
\[ r_i + \sum_{j=1}^{K} (L_{ij} + M_{ij})r_j \quad \text{total traffic handled by node in the peer-to-peer network} \]

"Useful" traffic? Depends
Definition of Fairness Indices

Source-fairness index for user $i$

$$l_i = \frac{r_i}{r_i + \sum_{j=1}^{K} (M_{ij} + L_{ij})r_j}$$

Destination-fairness index for user $i$

$$o_i = \frac{\sum_{j=1}^{K} M_{ij}r_j}{r_i + \sum_{j=1}^{K} (M_{ij} + L_{ij})r_j}$$

Fairness index for user $i$

$$\chi_i = \frac{r_i + \sum_{j=1}^{K} M_{ij}r_j}{r_i + \sum_{j=1}^{K} (M_{ij} + L_{ij})r_j}$$

Locally computable by each node
Maxmin Fairness

What is the optimal maxmin solution

\[ M = \max_r \min_{i,j} \left( \frac{\chi_i}{\chi_j}, \frac{\chi_j}{\chi_i} \right) \]

\[ \chi_i = \chi_j \quad \forall i, j \]

This leads to a perfectly fair solution
A Perfectly Fair Solution

A perfectly fair solution exists if there exists a non-zero, non-negative rate vector so that all fairness indices are equal

\[ \gamma_S (I + L + M)r = r, \]
\[ \gamma_D (I + L + M)r = Mr, \]
\[ \gamma_C (I + L + M)r = (I + M)r \]

I identity matrix
**Proposition:** If \( L + M \) is irreducible, then a perfectly source-fair solution exists. The index, \( \gamma_s \), is uniquely defined and the corresponding rate vector is positive and unique up to a scalar constant.

\[
\gamma_s (I + L + M)r = r,
\]

For the other 2 types of indices, existence issue is extremely complicated. (Paper by Mangasarian)

For any \( K \)-by-\( K \) matrix, \( R \), define the cone, \( C(R) \), to be the set of the form

\[
\{ \sum_i c_i R_i : c_i \geq 0 \}.
\]
Example:

\[ R = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}, \quad a, b > 0, \quad c(R) = \mathbb{R}_+^2 \]

\[ R_a = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} \]

\[ R_{1/2} = R_2 \]

\[ R_0 = R_\infty = \mathbb{R}_+^2 \]
Some Basic Properties (cont)

**Proposition:**

1. Suppose \( c(I+L) \subseteq c(M) \), then a perfectly destination-fair solution exists with \( 0 < r_o < 1 \).

2. In addition, if \( c(I+L) \) is contained in the interior of \( c(M) \) and \( M \) is of full rank, then \( r_o \) is uniquely defined and \( r \) is positive and unique up to a scalar constant.
Proposition:

1. Suppose $c(I+M) \subseteq c(L)$, then a perfectly fair solution exists with $0 < \gamma_c < 1$.

2. In addition, if $c(I+M)$ is contained in the interior of $c(L)$ and $L$ is of full rank, then $\gamma_c$ is uniquely defined and $r$ is positive and unique up to a scalar constant.
Definition: A single routing node network is a network such that traffic between any source-destination pair is delivered either directly or routed through a unique node.

\[
L = \begin{bmatrix}
0 & L_{12} & \cdots & L_{1K} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & L_{12}^T \\
0 & 0
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
0 & M_{12}^T \\
M_{21} & M_{22}
\end{bmatrix}
\]
Proposition: A single routing node network has a perfectly destination-fair index if

\[(M_{12} - \rho(M_{22})L_{12}^T)z > 0\]

where $M_{22}$ is the spectral radius of $\rho(M_{22})$ and $z$ is a corresponding eigenvector.
Example of a single routing node networks

\[
M = \begin{bmatrix}
0 & 0.19 & 0.36 \\
0.50 & 0 & 0.64 \\
0.50 & 0.81 & 0
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
0 & 0.81x & 0.64x \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Recall this is a destination-fair case
Example of a single routing node networks (cont)
Pricing Duality

Define *shadow prices*, $\mathbf{p} = [p_1, \ldots, p_K]$.

Nodes are allowed to charge each other for handling traffic, including traffic it generates and receives as well as the transitory traffic it delivers for others, under the same pricing scheme.

Define *combined pricing indices*:

$$
\chi^p_j = \frac{p_j + \sum_{i=1}^{K} M_{ij} p_i}{p_j + \sum_{i=1}^{K} (M_{ij} + L_{ij}) p_i}
$$
Pricing Index

\[
\left[ \sum_{i=1}^{K} (M_{ij} + L_{ij}) p_i \right] \cdot r_j
\]

is the total shadow price for handling the traffic from node \( j \)

reflect the economic benefits generated by traffic from node \( j \) under different scenario (assuming such benefits are uniformly proportional to the shadow prices.)

The price ratio reflects a benefit cost efficiency ratio – *payment return ratio*

Can define other price indices similarly
Problem: For non-negative matrices, find non-negative vectors such that:

\[
\begin{cases}
\alpha pA \leq pB \\
\beta Ar \leq Br
\end{cases}
\] (*)

for some positive \(\alpha\) and \(\beta\).
Von Neumann Economic Model (cont)

\[ \beta \sum_{j=1}^{K} a_{ij} r_j \geq \sum_{j=1}^{K} b_{ij} r_j \quad \text{or} \]
\[ \sum_{j=1}^{K} a_{ij} r_j / \sum_{j=1}^{K} b_{ij} r_j \geq 1 / \beta \]

1/\beta \quad \text{is the minimum fairness factor among the users}

\[ \alpha \sum_{i=1}^{K} a_{ij} p_i \leq \sum_{i=1}^{K} b_{ij} p_i \quad \text{or} \]
\[ \sum_{i=1}^{K} a_{ij} p_i / \sum_{i=1}^{K} b_{ij} p_i \leq 1 / \alpha \]

1/\alpha \quad \text{is the maximum payment return ratio among the users}
**Definition**: An equilibrium solution is defined as a set of non-negative, non-zero vectors, \( \mathbf{r} \) and \( \mathbf{p} \), satisfying equation (*) for some positive constants \( \alpha \) and \( \beta \), with the property that for any index \( i \), if

\[
\beta \sum_{j=1}^{K} a_{ij} r_j > \sum_{j=1}^{K} b_{ij} r_j
\]

then \( p_i = 0 \). 

For all index \( j \), if

\[
\alpha \sum_{i=1}^{K} a_{ij} p_i < \sum_{i=1}^{K} b_{ij} p_i
\]

then \( r_j = 0 \). 

---

**Von Neumann Equilibrium**
Proposition: If $A$ and $B$ are non-negative matrices and $A + B > 0$, then there exists a equilibrium solution to equation (*). Moreover,

$$\alpha = \beta = 1/\sigma_c$$

$$\sigma_c = \max_r \min_i \chi_i$$

$$= \min_p \max_i \chi_i^p$$
Example of Von Neumann Equilibrium
Low Complexity, Low Data Rate, Robust Controller

- **Tri-state Control Algorithm:**

  
  \[
  x_i^{(n+1)} = \begin{cases} 
  \delta x_i^{(n)} & y_i^{(n)} < \varepsilon^{-1} \gamma_i \\
  \delta^{-1} x_i^{(n)} & y_i^{(n)} > \varepsilon \gamma_i \\
  x_i^{(n)} & \text{otherwise}
  \end{cases}
  \]

  
  \[
  \gamma_i + \varepsilon \\
  \gamma_i \\
  \gamma_i - \varepsilon
  \]

  
  - **Can it converge?**
Counter-example to convergence

\[ y_1 = \frac{x_1 + \frac{1}{2} x_2}{x_3} \quad y_2 = \frac{\frac{1}{2} x_1 + x_2}{x_4} \quad x_i^{(n+1)} = \begin{cases} \delta x_i^{(n)} & y_i^{(n)} < \varepsilon^{-1} y_i \\ \delta^{-1} x_i^{(n)} & y_i^{(n)} > \varepsilon y_i \\ x_i^{(n)} & \text{otherwise} \end{cases} \]

\[ y_3 = \frac{x_3 + \frac{1}{2} x_4}{x_1} \quad y_4 = \frac{\frac{1}{2} x_3 + x_4}{x_2} \]

- Performance target \( \gamma = 1.5 \), \( \forall i \) is feasible

\[
\begin{align*}
x_1^{(0)} &= 2 \\
x_2^{(0)} &= 0.5 \\
x_3^{(0)} &= 2 \\
x_4^{(0)} &= 0.5 \\
x_5^{(0)} &= 0 \\
\Gamma_1^{(0)} &= 1.125 \\
\Gamma_2^{(0)} &= 3 \\
\Gamma_3^{(0)} &= 1.125 \\
\Gamma_4^{(0)} &= 3 \\
x_1^{(1)} &= 2\delta \\
x_2^{(1)} &= 0.5\delta^{-1} \\
x_3^{(1)} &= 2\delta \\
x_4^{(1)} &= 0.5\delta^{-1} \\
\delta &> 1 \\
\Gamma_1^{(1)} &= \Gamma_3^{(1)} = 1 + .125\delta^{-2} \\
\Gamma_2^{(1)} &= \Gamma_4^{(1)} = 2 + \delta^2 \\
\end{align*}
\]

- Divergent!
Results on convergence

A set of positive performance targets, \((\lambda_1,\ldots,\lambda_K)\), is said to be feasible if there exists a non-zero traffic rate vector, \((\tilde{r}_1,\ldots,\tilde{r}_K)\), so that,

\[
\lambda_i = \sum_{j=1}^{K} M_{ij} \tilde{r}_j
\]

\[
T_{ij} = M_{ij} + L_{ij}
\]

\[
\tilde{r}_i + \sum_{j=1}^{K} T_{ij} \tilde{r}_j.
\]

There exists a rate vector, \(\mathbf{r}^* = (r_1(0)\delta_i^1,\ldots,r_K(0)\delta_i^K)\) such that

\[
\varepsilon^{-1} \lambda_i \leq \gamma_i^* \leq \varepsilon \lambda_i.
\]
**Convergence Result**

**Technical Assumption:** For \( i = 1, \ldots, K \) and all \( j \neq i, \ T_{ij} \neq 0 \)

\[
\lambda_i \leq \frac{1}{\varepsilon} \cdot \frac{M_{ij}}{T_{ij}}.
\]

**Theorem:** Assume technical assumption holds for a set of feasible targets, \((\lambda_1, \ldots, \lambda_K), \ \delta^2 \leq \varepsilon\) and for all \( i, \)

\[
\varepsilon^2 \min_{j, M_{ij} \neq 0} (M_{ij} / T_{ij}) \geq \max_{j, M_{ij} \neq 0} (M_{ij} / T_{ij})
\]

The tri-state algorithm is convergent, such that

\[
\varepsilon^{-1} \lambda_i \leq \gamma_i^* = \frac{\sum_{j=1}^{K} M_{ij} r_j^*}{r_i^* + \sum_{j=1}^{K} T_{ij} r_j^*} \leq \varepsilon \lambda_i.
\]
Idea of proof

Define \( a(i,n) \) by
\[
 r_i^{(n)} = r_i^* \delta^{a(i,n)} 
\]
Note that \( |a(i, n + 1) - a(i, n)| \leq 1 \)

Let \( K(n) = \max_i |a(i, n)| \)

Claim: \( K(n) \) is non-increasing as a function of \( n \)
Define a peak-slide of length $k$ for user $i$, if for $m < n$

\[
\begin{align*}
  r_{i}^{(m-1)} &= r_{i}^{(m)} \delta \\
  r_{i}^{(m)} &= r_{i}^{(n)} \delta^k \quad k \geq 1
\end{align*}
\]
Idea of proof (cont)

Claim: Existence of a peak-slide of length $k$ implies there is another user $j$, and $m-1 < m \leq t < n$ so that

$$\left| r_j^{(m-1)} / r_j^{(t)} \right| \geq (k + 1) \delta$$

If there are cycles, then there will be an increasing sequences of peak-slides. A contradiction.
# Numerical Results 1

<table>
<thead>
<tr>
<th>Simulation No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Nodes</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Fairness Index</td>
<td>0.2462</td>
<td>0.2337</td>
<td>0.2344</td>
<td>0.1940</td>
<td>0.1892</td>
<td>0.1592</td>
<td>0.1583</td>
</tr>
<tr>
<td>Converged Index</td>
<td>0.2448</td>
<td>0.2358</td>
<td>0.2314</td>
<td>0.2326</td>
<td>0.2357</td>
<td>0.2351</td>
<td>0.2323</td>
</tr>
<tr>
<td>No. of Iterations</td>
<td>266</td>
<td>122</td>
<td>324</td>
<td>207</td>
<td>219</td>
<td>243</td>
<td>165</td>
</tr>
</tbody>
</table>
Numerical Results 2

Feasible targets

Infeasible targets
Heuristic Algorithm
Some Preliminary Routing Results (Class 1) (F J Li)

- Link Capacity Limit = 100 unit;
- Input traffic rate = [70 40 60 80 50 90 80 50]’;
Some Preliminary Routing Results (Class 1)

Left figure shows the fairness indices of all 6 nodes under DBR
Right figure shows the fairness indices of all 6 nodes under FIBR
Some Preliminary Routing Results (Class 1)

- The total traffic rate (TTR)
  - summation of the traffic on all links; it reveals the effectiveness of the routing algorithm.
  - TTR using DBR is 1340 units;
  - TTR using FIBR is 1109 units.

- The total link delay cost (TLDC)
  - summation of the link delay cost among all links.
  - TLDC using DBR is 50.3 units;
  - TLDC using FIBR is 53.1 units.
### Some Preliminary Routing Results (Class 1)

<table>
<thead>
<tr>
<th>TTR</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBR</td>
<td>1070</td>
<td>1035</td>
<td>1087</td>
<td>1205</td>
<td>1163</td>
</tr>
<tr>
<td>FIBR</td>
<td>1069</td>
<td>1040</td>
<td>1088</td>
<td>1211</td>
<td>1171</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TLDC</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBR</td>
<td>41.9</td>
<td>35.0</td>
<td>50.5</td>
<td>70.3</td>
<td>54.1</td>
</tr>
<tr>
<td>FIBR</td>
<td>46.1</td>
<td>46.1</td>
<td>65.1</td>
<td>86.7</td>
<td>62.8</td>
</tr>
</tbody>
</table>

5 experiments under the different network topology and different traffic proportion among routes.
Some Preliminary Routing Results (Class 2)

- Link Capacity Limit = 40 unit;
- Input traffic rate = [70 40 60 80 50 90 80 50]’;
Some Preliminary Routing Results (Class 2)

Left figure shows the fairness indices of all 8 nodes under DBR

Right figure shows the fairness indices of all 8 nodes under FIBR
Some Preliminary Routing Results (Class 2)

- The total traffic rate (TTR)
  - summation of the traffic on all links; it reveals the effectiveness of the routing algorithm.
  - TTR using DBR is 1106 units;
  - TTR using FIBR is 792 units.

- The total link delay cost (TLDC)
  - summation of the link delay cost among all links.
  - TLDC using DBR is 104 units;
  - TLDC using FIBR is 3996 units.
UWB Systems

Implementation via: CDMA, OFDM, pulse radio
A simple receiver structure robust against timing jitter

- Oversampling
- Combination via the maximum function
- Decision via a threshold function
- Threshold adjust by Kiefer-Wolfowitz type algorithm
More application example

Estimation with a lattice of sensors
Estimation with unlabelled observers

$I = 1, \ldots, I$ observers defined by:

\[ y_i(n) = h_i(z(n)) + w_i(n) \]

\[ z(n) = z_j \text{ with prob } \rho_j \]

\[ w_i(n) \text{ independent Gaussian noise with variance } \sigma^2 \]

Combining by summation will not work well if

\[ \sum h_i \approx 0 \]

\[ m(z) = \max_i h_i(z) \]
Structure of optimal decision function

- Minimize the probability of error
  \[ P_e = \int_{-\infty}^{\infty} \sum_{j=1}^{N} (1 - \chi_{\theta(\tau)=z_j}) \Pr(m = \tau \mid z_j) \rho_j d\tau \]

- Well known that the Bayes decision function is optimal
  \[ \theta(\tau) = j \quad \text{where} \quad j \quad \text{is the index satisfying} \]
  \[ \max_k \Pr(m = \tau \mid z_k) \rho_k = \Pr(m = \tau \mid z_j) \rho_j. \]
Large noise variance example

\( N_s = 3; \)
\( \alpha_1 = 0.1; \)
\( \alpha_2 = 0.2; \)
\( \alpha_3 = 1; \)
\( \beta_1 = 0.55; \)
\( \beta_2 = 0.55; \)
\( \beta_3 = 0.6; \)
\( \sigma = 0.6; \)
When noise variance drops

\( N_s = 3; \)
\( \alpha_1 = 0.1; \)
\( \alpha_2 = 0.2; \)
\( \alpha_3 = 1; \)
\( \beta_1 = 0.55; \)
\( \beta_2 = 0.55; \)
\( \beta_3 = 0.6; \)
\( \sigma = 0.15 \)
Structure result

**Definition 1:** A system of sensors is said to be of *pure* type if the observation functions are either all strictly increasing or all strictly decreasing.

**Definition 2:** A system of odd, strictly monotonic observers is *positively tempered* if for every strictly decreasing, $h_i$, one can associate with it a uniquely paired strictly increasing, $h_j$, so that the sum of the two function is increasing.

\{-2z, -3z, 2z, 4z, 5z\} \quad \{2z, 3z, -2z, -4z, -5z\} \quad \{-5z, 2z, 3z\}

**Theorem:** For these system the optimal decision function is a threshold function and $P_e$ has a global minimum when the noise variance is small enough.