Introduction to the 3+1 Einstein equations*

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1 Space-time decomposition

The 3-metric g_{ij} and extrinsic curvature (second fundamental form) K_{ij} are the fundamental variables describing the geometry in any space+time decomposition of the Einstein equations. (g, K) describe the local geometry of a single space-like hypersurface M, and it is then natural to describe the evolution of the space-time geometry by a 1-parameter family $(g(t), K(t))$, describing the local geometry of the (space-like) hypersurfaces M_t. In order to piece these hypersurfaces together, however, we must also specify the lapse N and shift vector X^i, which describe the relation between the time evolution vector ∂_t[1] and the space-time vector n normal to the hypersurfaces:

$$\partial_t = Nn + X^i \partial_i, \qquad n = N^{-1}(\partial_t - X^i \partial_i). \tag{1}$$

The spacetime metric is then fully determined, by

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt), \tag{2}$$

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[1] We assume always that the hypersurfaces M_t form a foliation of the spacetime, so we are justified in treating the evolution vector as a coordinate vector.

and the inverse metric in terms of $[g^{ij}] = [g_{ij}]^{-1}$ is

$$\bar{g}^{ab}\partial_a \otimes \partial_b = -N^{-2}(\partial_t - X^i\partial_i)^2 + g^{ij}\partial_i \otimes \partial_j, \tag{3}$$

where to reduce the risk of confusion we use $\bar{g}_{ab}$ to denote the spacetime metric components. Conversely we have

$$\begin{aligned} N^2 &= -\bar{g}_{00} + \bar{g}_{0i}\bar{g}_{0j}g^{ij} \quad (4) \\ &= -1/\bar{g}^{00}, \quad (5) \\ X_i &= g_{ij}X^j = \bar{g}_{0i}, \quad (6) \\ X^i &= -\bar{g}^{0i}/\bar{g}^{00}, \quad (7) \\ g^{ij} &= \bar{g}^{ij} - \bar{g}^{0i}\bar{g}^{0j}/\bar{g}^{00}. \quad (8) \end{aligned}$$

Any numerical formulation brings with it coordinates (t, x^i) — one important challenge then is to develop good choices of the lapse and shift, so that the space-like hypersurfaces (level sets of t) and the spatial coordinates x^i remain as smooth and regular as possible. In geometric terms this amounts to constructing "good" coordinates, where "good" can mean many things. For example, a popular choice for the time coordinate requires that the hypersurfaces are *maximal*, ie. $trK = 0$. Because this amounts to an elliptic equation (analogous to the minimal surface equation satisfied by soap films), the t coordinate is as smooth as the spacetime allows, so coordinate breakdown signals serious geometrical problems, rather than spurious coordinate effects. On the other hand, the resulting elliptic equation on the lapse

$$\Delta_g N = N|K|^2 = NK_{ij}K^{ij} \tag{9}$$

is expensive to solve numerically, and is "non-local".

The relation between (g, K) and (N, X^i) is

$$K_{ij} = \tfrac{1}{2}N^{-1}(\partial_t g_{ij} - \nabla_i X_j - \nabla_j X_i), \tag{10}$$

where $\nabla_i X_j$ denotes the spatial covariant derivative,

$$\nabla_i X_j = \partial_i(X_j) - \Gamma_{ijk}X^k. \tag{11}$$

This shows that if (g, K) is given, and (N, X^i) chosen appropriately, then $\partial_t g_{ij}$ is determined. By analogy with the usual wave equation, it is perhaps not surprising that (g, K) form the geometric initial data for the Einstein equations. This follows from the details of the proof of local existence, as described in Alan's lecture. However, one point at which the analogy with the wave equation $\Box u = 0$ breaks down, is the matter of constraints.

Whereas the initial data for the wave equation consists of *arbitrary* functions (u_0, u_1) with $u(x, 0) = u_0(x)$, $\partial_t u(x, 0) = u_1(x)$, for the Einstein equations the data (g, K) are not freely specifiable, but must satisfy the constraint equations

$$G_{00} = R(g) + |K|^2 - (\mathrm{tr}_g K)^2 \tag{12}$$

$$G_{0i} = 2(\nabla^j K_{ij} - \nabla_i \mathrm{tr}_g K). \tag{13}$$

where G_{0a}, $a = 0, \ldots, 3$[2] are components of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$. The Einstein equations

$$G_{ab} = 8\pi\kappa T_{ab} \tag{14}$$

connect the spacetime curvature with the stress-energy tensor T_{ab}, which reflects the matter content of any additional fields (perfect fluid, Maxwell, Yang-Mills, dilaton etc) present in the simulation. For the vacuum Einstein equations, the stress energy tensor T_{ab} vanishes, so we can consider $G_{0a} = 0$ for simplicity. Note that T_{00} is interpreted physically as describing the local energy density as measured by an observer with worldline in the direction e_0, and likewise T_{0i}, $i = 1, \ldots, 3$ describes the local momentum density vector.

2 Solving the constraint equations

The standard technique for constructing solutions of the constraint equations is based on conformal deformation of the spatial metric (see [9] for a recent

[2]Here the subscript $_0$ refers to the $e_0 = n$ timelike unit normal component.

survey). Thus, if $\tilde{g}_{ij}$ is an arbitrarily chosen 3-metric and $\phi \in C^\infty(M)$ is a positive function, the scalar curvature of the metric $g_{ij} = \phi^4 \tilde{g}_{ij}$ satisfies

$$\Delta_{\tilde{g}}\phi - \tfrac{1}{8}R(\tilde{g})\phi = -\tfrac{1}{8}R(g)\phi^5. \tag{15}$$

This identity is the key to solving the Hamiltonian constraint (12), which amounts to solving an elliptic equation for the conformal factor ϕ with respect to the unphysical metric $\tilde{g}$.

Solving the momentum constraint (13) is more complicated. The additional data required is a symmetric traceless tensor λ_{ij}; we introduce W_i and the conformal Killing operator

$$LW_{ij} = \tilde{\nabla}_j W_i + \tilde{\nabla}_i W_j - \tfrac{2}{3}\tilde{\nabla}^k W_k \tilde{g}_{ij}. \tag{16}$$

Denoting the (prescribed) mean curvature function by $\tau = \mathrm{tr}_g K$, the extrinsic curvature is

$$K_{ij} = \phi^{-2}(\lambda_{ij} - LW_{ij}) + \tfrac{1}{3}\tau g_{ij} \tag{17}$$

and the constraint equations reduce to an elliptic system for ϕ and W_i,

$$\begin{aligned} 8\Delta_{\tilde{g}}\phi &= R(\tilde{g})\phi - \phi^{-7}\tilde{g}^{ik}\tilde{g}^{jl}(\lambda_{ij} - LW_{ij})(\lambda_{kl} - LW_{kl}) + \tfrac{2}{3}\phi^5\tau^2 && (18) \\ \tilde{\nabla}^j LW_{ij} &= \tilde{\nabla}^j\lambda_{ij} + \tfrac{2}{3}\phi^6\tilde{\nabla}_i\tau. && (19) \end{aligned}$$

For well-chosen data $(\tilde{g}_{ij}, \lambda_{ij})$ this system is solvable for (ϕ, W_i). Now the constraint equations are propagated by the 3+1 evolution equations

$$\begin{aligned} \partial_t g_{ij} &= NK_{ij} + \nabla_i X_j + \nabla_j X_i && (20) \\ \partial_t K_{ij} &= \nabla^2_{ij}N + (\bar{R}_{ij} - R_{ij} + 2K_i^k K_{jk} - \mathrm{tr}_g K g_{ij})N + \mathcal{L}_X K_{ij}, && (21) \end{aligned}$$

where $\bar{R}_{ij}$ is the spacetime Ricci curvature, determined by the stress-energy tensor, R_{ij} is the spatial Ricci curvature determined by the 3-metric g_{ij}, and

$$\mathcal{L}_X K_{ij} = X^k \nabla_k K_{ij} + \nabla_i X^k K_{jk} + \nabla_j X^k K_{ik}. \tag{22}$$

Thus in theory at least it is necessary only to solve the constraint equations on the initial hypersurface $t = 0$. Although in theory there is no difference between theory and practice, in practice there is, and ensuring the constraints remain satisfied is one of the challenges for numerical relativity.

Although the conformal method is widely applicable, there are other techniques for solving the constraint equations which may be of use in some situations. The quasi-spherical ansatz for the 3-metric

$$ds^2 = u^2 dr^2 + r^2((d\vartheta + \beta^1 dr)^2 + (\sin\vartheta d\varphi + \beta^2 dr^2)^2), \tag{23}$$

and the Hamiltonian constraint leads to a radial parabolic equation for u with β^1, β^2 freely specifiable functions [2]. A possible application of this construction is to solving the constraints in the asymptotically flat region exterior to a strong field solution. An extension of the quasi-spherical construction in [11] offers greater flexibility. The quasi-spherical idea, applied to foliate outgoing null cones, was used in [4] to numerically construct vacuum spacetimes with strong radiation.

Finally, there are two further constructions for the constraint equation which are interesting from a theoretical viewpoint, but which can only be applied locally and hence would have only limited use in numerical relativity. These are the thin sandwich formulation [3, 10] and the Corvino-Schoen equation [6].

3 The linearised equations

The spacetime metric in the neighbourhood of the solar system is very close to the flat metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ — if we write

$$g_{ab} = \eta_{ab} + h_{ab} \tag{24}$$

then even in the center of the sun, $|h_{ab}| \leq 10^{-4}$. So it is reasonable as a first approximation to consider the Einstein equations in terms of h_{ab} and ignore all quadratic and higher powers. Now the vacuum Einstein equations in local coordinates take the form

$$0 = R_{ab} \quad = \quad g^{cd} \left(\partial_a \Gamma_{cdb} - \partial_c \Gamma_{adb} \right) + Q$$

$$
\begin{aligned}
= & -\tfrac{1}{2} g^{cd}\partial^2_{cd} g_{ab} + Q && (25)\\
& + \tfrac{1}{2} g^{cd}\left(\partial_a(\partial_c g_{bd} - \tfrac{1}{2}\partial_b g_{cd}) + \partial_b(\partial_c g_{ad} - \tfrac{1}{2}\partial_a g_{cd})\right), && (26)
\end{aligned}
$$

where ${\Gamma_{ab}}^c = \frac{1}{2} g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$ is the Christoffel symbol of g, and Q represents some ignorable expression, quadratic in the first derivatives ∂g. In terms of h_{ab} this becomes

$$
\eta^{cd}\partial^2_{cd} h_{ab} = \partial_a \gamma_b + \partial_b \gamma_a, \tag{27}
$$

where $\gamma_a = \eta^{cd}(\partial_c h_{ad} - \frac{1}{2}\partial_a h_{cd})$. Clearly the linearized equations become the hyperbolic wave equation if we can set $\gamma_a = 0$. This can be achieved by a suitable small coordinate change $x^a \to x^a + \xi^a$, which induces changes

$$
\begin{aligned}
h_{ab} \to h'_{ab} &= h_{ab} - \partial_a \xi_b - \partial_b \xi_a, && (28)\\
\gamma_a \to \gamma'_a &= \gamma_a - \Box \xi_a, && (29)
\end{aligned}
$$

where $\Box = \eta^{ab}\partial_a\partial_b = -\partial_t^2 + \partial_i^2$ is the wave operator. Thus choosing ξ_a to satisfy $\Box\xi_a = \gamma_a$ enforces the required gauge condition $\gamma_a = 0$. This is usually called the de Donder gauge and is directly analogous to the Lorentz gauge $\eta^{ab}\partial_b A_a = 0$ in electromagnetism. Because the coordinates x^a in this gauge satisfy $\Box x^a = 0$, the de Donder gauge is sometimes also called the "harmonic" gauge, by analogy with its Riemannian counterpart, but this terminology is somewhat misleading.

Note that the de Donder gauge is preserved by infinitesimal coordinate changes satisfying $\Box\xi_a = 0$.

The linearized equations come out a little neater in terms of

$$
\bar{h}_{ab} = h_{ab} - \tfrac{1}{2}\eta^{cd} h_{cd}\eta_{ab} \tag{30}
$$

which satisfies the de Donder condition

$$
\eta^{bc}\partial_c \bar{h}_{ab} = 0 \tag{31}
$$

and the linearized Einstein equations become

$$
\Box \bar{h}_{ab} = -16\pi T_{ab}. \tag{32}
$$

These equations clearly admit radiating solutions. To be explicit, assume that h_{ab} takes the form of a plane wave travelling in the x-direction, so $h_{ab} = h_{ab}(t - x)$. Now, the de Donder gauge is preserved by infinitesimal coordinate changes satisfying $\Box\xi_a = 0$ and this remaining freedom can be used to bring h_{ab} into the form

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{bmatrix} \tag{33}$$

See [7] for details. h_{22} and h_{23} represent the two polarization states of the radiation, and have the physical effect of changing the separation distance between points in the (y, z) plane after the plane wave has moved through in the x-direction.

Note that these polarization states will be difficult to detect directly from numerical data, unless the coordinate freedom (in the lapse and shift (N, X^i) for example) is chosen carefully to keep h_{ab} in the above form. This is generally not practicable, even if the radiation direction is known beforehand.

Other measures of radiation have been used in numerical relativity, most notably based on spherical harmonic decompositions around large spheres, based on black hole perturbation theory [5].

4 Energy and radiation

Defining energy in general relativity is a complex issue. The best known and most reliable definition is that of Arnowit, Deser and Misner [1], which applies for an asymptotically flat spacelike hypersurface M. We say (M, g, K) is *asymptotically flat* if there are rectangular coordinates x^i near infinity in M with respect to which the metric and extrinsic curvature satisfy

$$\begin{aligned} g_{ij} - \delta_{ij} &= O(r^{-1}), && (34) \\ \partial_k g_{ij} &= O(r^{-2}), \quad \partial^2_{kl} g_{ij} = O(r^{-3}), && (35) \end{aligned}$$

$$K_{ij} = O(r^{-2}), \quad \partial_k K_{ij} = O(r^{-3}) \tag{36}$$

and has integrable scalar curvature, $R(g) \in L^1(M)$ and $\nabla^j K_{ij} - \nabla_i \mathrm{tr}_g K \in L^1(M)$. These conditions can be weakened, but this is unlikely to be significant for numerical work. The ADM energy is then

$$16\pi E = \lim_{r\to\infty} \oint_{S^2(r)} (\partial_j g_{ij} - \partial_i g_{jj})\, dS^i \tag{37}$$

and the ADM momentum is

$$8\pi p_i = \lim_{r\to\infty} \oint_{S^2(r)} (K_{ij} - \mathrm{tr}_g K g_{ij})\, dS^j, \tag{38}$$

where $dS^i = n^i dS$, n^i is the radial outward spacelike unit normal to the large spheres $S^2(r)$ and dS is the area measure. (E, p_i) together form the ADM energy-momentum 4-vector, and a remarkable theorem states that (E, p_i) is future timelike under physically reasonable matter conditions.

Although the ADM definition is both physically meaningful and robust (I suspect this is also true numerically), it measures gravitational radiation only indirectly. In order to see radiation flux, which is the most convincing evidence that gravitational waves actually carry energy, it is necessary to look not at spacelike infinity as in the ADM definition, but at future null infinity. This may be viewed as the endpoint of outgoing light rays, and there is a separate definition of mass due to Bondi, which hass the property that it decreases in time. It should be possible to use this definition (with adaptions) to compute the Bondi mass and the mass flux numerically, using the expression due to Hawking [8]

$$m_{\mathrm{Bondi}} = \lim_{r\to\infty} \frac{|S^2(r)|}{32\pi} \left(1 - \frac{1}{4\pi} \oint_{S^2(r)} \mu\rho\, dS\right) \tag{39}$$

where μ, ρ are the ingoing and outgoing expansions, determined by null vectors n, ℓ orthogonal to $S^2(r)$.

References

[1] R. Arnowitt, S. Deser, and C. Misner. In L. Witten, editor, *Gravitation*, pages 227–265. Wiley, 1962.

[2] R. Bartnik. Quasi-spherical metrics and prescribed scalar curvature. *J. Diff. Geom.*, 37:31–71, 1993.

[3] R. Bartnik and G. Fodor. On the restricted validity of the thin sandwich conjecture. *Phys. Rev. D*, 48:3596–3599, 1993. gr-qc/9304004.

[4] R. Bartnik and A. Norton. Numerical methods for the Einstein equations in null quasi-spherical coordinates. *SIAM J. Sci. Comp.*, 22:917–950, 2000. gr-qc/9904045, ISE R 99/110.

[5] S. Chandrasekhar. *The Mathematical Theory of Black Holes.* Oxford 1984.

[6] J. Corvino. Scalar curvature deformation and a gluing construction for the Einstein constraint equations. *Comm. Math. Phys.*, 214:137–189, 2000.

[7] R. D'Inverno. *Introducing Einstein's Relativity.* Oxford 1992.

[8] S. W. Hawking. Gravitational radiation in an expanding universe. *J. Math. Phys.*, 9:598–604, 1968.

[9] J. Isenberg. Constructing solutions of the einstein constraint equations. In *GR16 Proceedings*, 2002. gr-qc/0203044.

[10] J. W. York Jr. Conformal "thin sandwich" data for the initial-value problem of general relativity. *grqc/9810051*, 1998.

[11] Brian Smith and Gilbert Weinstein. On the connectedness of the space of initial data for the Einstein equations. *Electron. Res. Announc. Amer. Math. Soc.*, 6:52–63, 2000.