Factors of i.i.d. processes on graphs and groups

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I.i.d. processes on groups and graphs

Let X be a set with a (discrete) group G acting on it. We will be concerned in particular with the cases when

- $X = G$, so G acts on itself by multiplication, and
- X is an infinite graph and G is a group of automorphisms of X.

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- X is an infinite graph and G is a group of automorphisms of X.

We consider i.i.d. processes on X.

- $\xi_m = (\xi_m(x))_{x \in X}$ is the *full* m*-shift* on (X, G) if it is i.i.d. and each $\xi_m(x)$ takes values uniformly in $\{0, \ldots, m-1\}$.
- $\xi_{[0,1]} = (\xi_{[0,1]}(x))_{x \in X}$ is the *full* $[0,1]$*-shift* on (X, G) if it is i.i.d. and $\xi_{[0,1]}(x)$ takes values uniformly in $[0,1]$.

Groups act on processes

Let ξ be an i.i.d. process on (X, G). Then G acts on configurations of ξ:

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Examples:

- $X = G = \mathbb{Z}$. Then ξ_2 is the 2-sided i.i.d. Bernoulli-(1/2) process. $\mathbb{Z}$ acts on this process by shifting sequences.

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow 1 \longrightarrow 0 \longrightarrow$$

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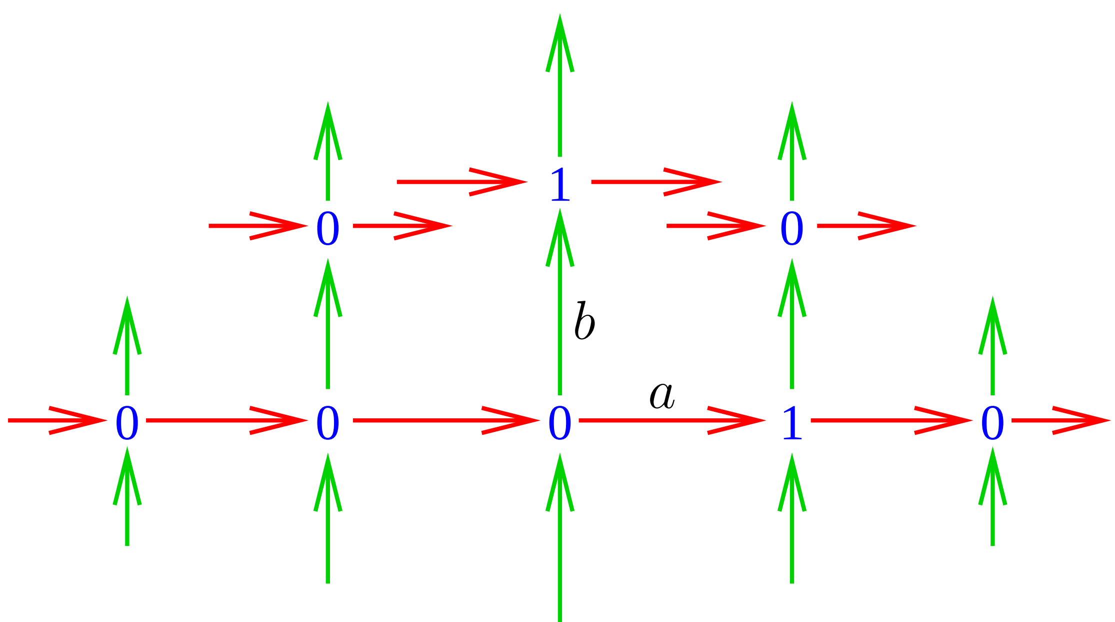
- $X = G = \mathbb{F}_2 =$ the free group on 2 generators:



Question

Under what conditions on (X, G) do there exist $m < n$ such that there is a G-factor from ξ_m to ξ_n on (X, G)?

Definition of a G-factor:

- Let ξ, ζ be processes on a set X with G acting.
- A map $F : \xi \to \zeta$ is a G-*factor* if F commutes with the G-action on ξ and $F(\xi) = \zeta$:

$$g \cdot F(\xi) = F(g \cdot \xi).$$

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$$(\xi(x))_{x \in X} \xrightarrow{g \in G} (\xi(g \cdot x))_{x \in X}$$

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$$\begin{array}{ccc}
(\xi(x))_{x\in X} & \xrightarrow{g\in G} & (\xi(g\cdot x))_{x\in X} \\
F \big\downarrow & & \big\downarrow F \\
(\zeta(x))_{x\in X} & \xrightarrow[g\in G]{} & (\zeta(g\cdot x))_{x\in X}
\end{array}$$

Motivation

Consider case where $X = G = \mathbb{Z}$ and $\mathbb{Z}$ acts on a process ξ by shifting:

$$n \cdot (\ldots, \xi(-1), \xi(0), \xi(1) \ldots) \mapsto (\ldots, \xi(n-1), \xi(n), \xi(n+1) \ldots)$$

Theorem. There is a $\mathbb{Z}$-factor from ξ_m to ξ_n on $\mathbb{Z}$ $\iff$ $m \geq n$.

- In particular, the full 4-shift is not a $\mathbb{Z}$-factor of the full 2-shift.
- This is a baby version of Sinai's Factor Theorem, which can be considered part of Ornstein's Isomorphism Theory. It is a fundamental result in ergodic theory.

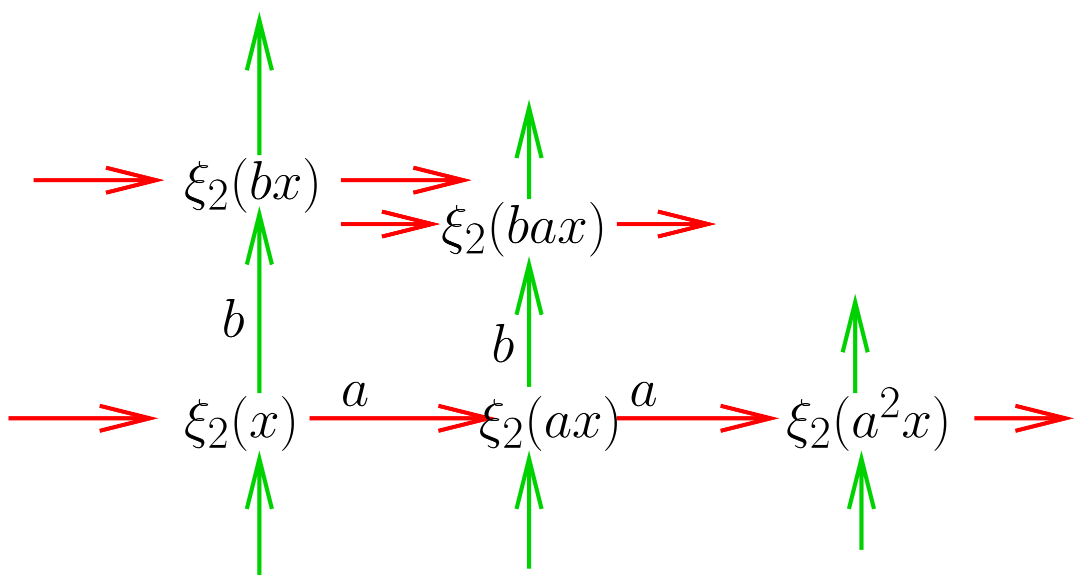
Motivation

Now consider the case where $X = G = \mathbb{F}_2$, the free group on two generators, a and b.

Proposition. (Ornstein-Weiss, 1987) There is an $\mathbb{F}_2$-factor from ξ_2 to ξ_4 on $\mathbb{F}_2$.

Proof. Define the factor map F on the full 2-shift ξ_2 by:

$$F(\xi_2)(x) = (\xi_2(a \cdot x) \oplus \xi_2(x), \xi_2(b \cdot x) \oplus \xi_2(x))$$



Motivation

Observation (Adam Timar):
If there exists a G-factor from ξ_2 to ξ_4 on (X, G), then there exists a G-factor from ξ_2 to $\xi_{[0,1]}$ on (X, G).

Proof. We will code a number in $[0, 1]$ in binary.

Let F be a G-factor taking ξ_2 to ξ_4 on (X, G). We use it to build a G factor F' from ξ_2 to $\xi_{[0,1]}$.

- Apply F to ξ_2 to get ξ_4.
- Code the coordinates of $\xi_4 = F(\xi_2)$ as bits: $00, 01, 10, 11$.
- Let the first bit of $F'(\xi_2)(x)$ be the first bit of $F(\xi_2)(x)$.
- Apply F again to the second bits of $F(\xi_2)$.
- Repeat the previous 2 steps (infinitely many times).

Amenability

Definition. A discrete group G is *amenable* if for every finite set $C \subset G$ and $\epsilon > 0$, there exists a finite $K \subset G$ such that

$$|CK \triangle K| < \epsilon |K|.$$

Definition. A graph G is *amenable* if for every $\epsilon > 0$, there exists a finite set of vertices $K \neq \emptyset$ such that

$$\frac{\text{\# edges in } G \text{ w/ exactly 1 endpt in } K}{\text{\# vertices in } K} < \epsilon.$$

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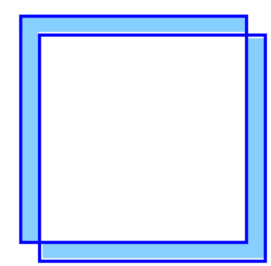

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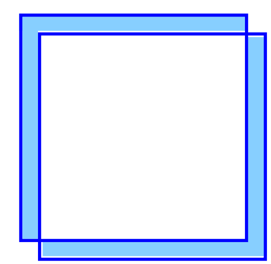


Example. $\mathbb{F}_2$ is not amenable: easy to see balls don't work.

Existence of factors

Take $X = G$.

Theorem. (B.) A finitely-generated group G is nonamenable $\iff \exists\, m > 0$ with a G-factor taking ξ_{2^m} on G to $\xi_{2^{m+1}}$ on G.

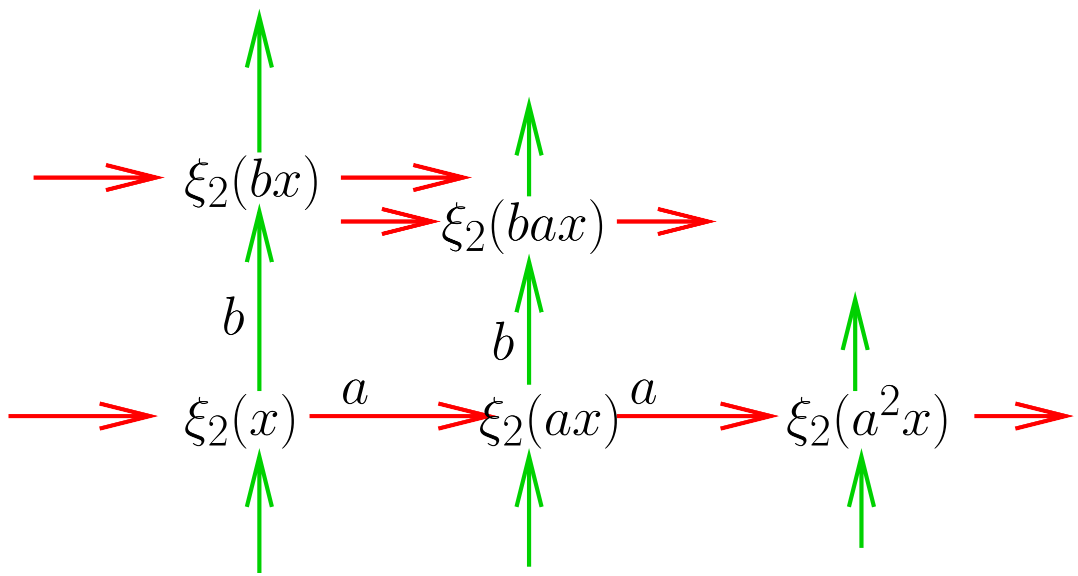
Notes:

- $(\Leftarrow)$ follows easily from entropy considerations for amenable group actions.
- If G is nonamenable and has a subgroup isomorphic to $\mathbb{F}_2$, then $(\Rightarrow)$ can be proved using the Ornstein-Weiss factor.
- There are nonamenable groups which do not have an $\mathbb{F}_2$-subgroup, as proved by Ol'shanskii. These are mysterious objects.

Strategy

Recall the Ornstein-Weiss factor:

$$F(\xi_2)(x) = (\xi_2(a \cdot x) \oplus \xi_2(x), \xi_2(b \cdot x) \oplus \xi_2(x))$$

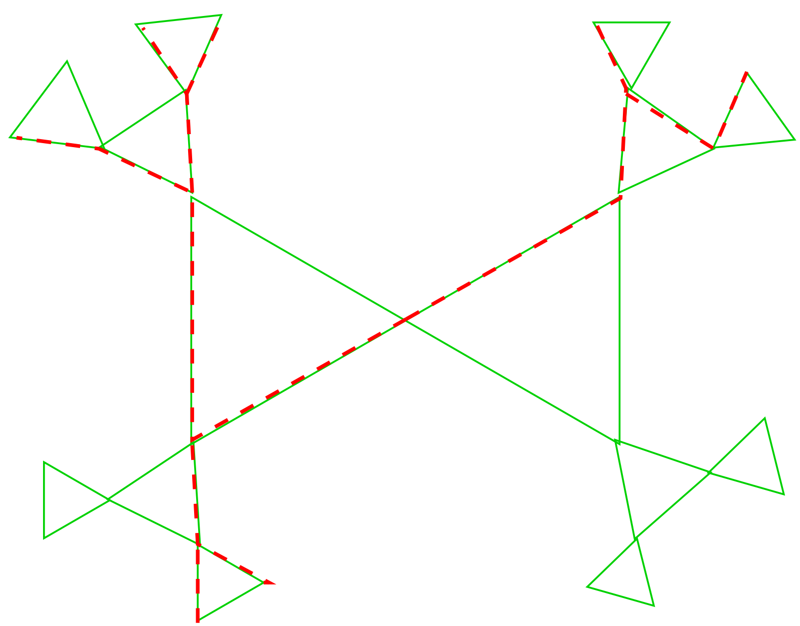


The independence of the random variables depends on the fact that the Cayley graph of $\mathbb{F}_2$ above is a many-ended tree.

Strategy

For a general finitely generated nonamenable graph G, our goal will be to split the information in $\xi_{2^m} = (\xi_{2^{m-1}}, \xi_2)$ and

- Step 1. Let X be a Cayley graph of G and choose at least one tree $T \subseteq X$ as a G-factor of $\xi_{2^{m-1}}$.



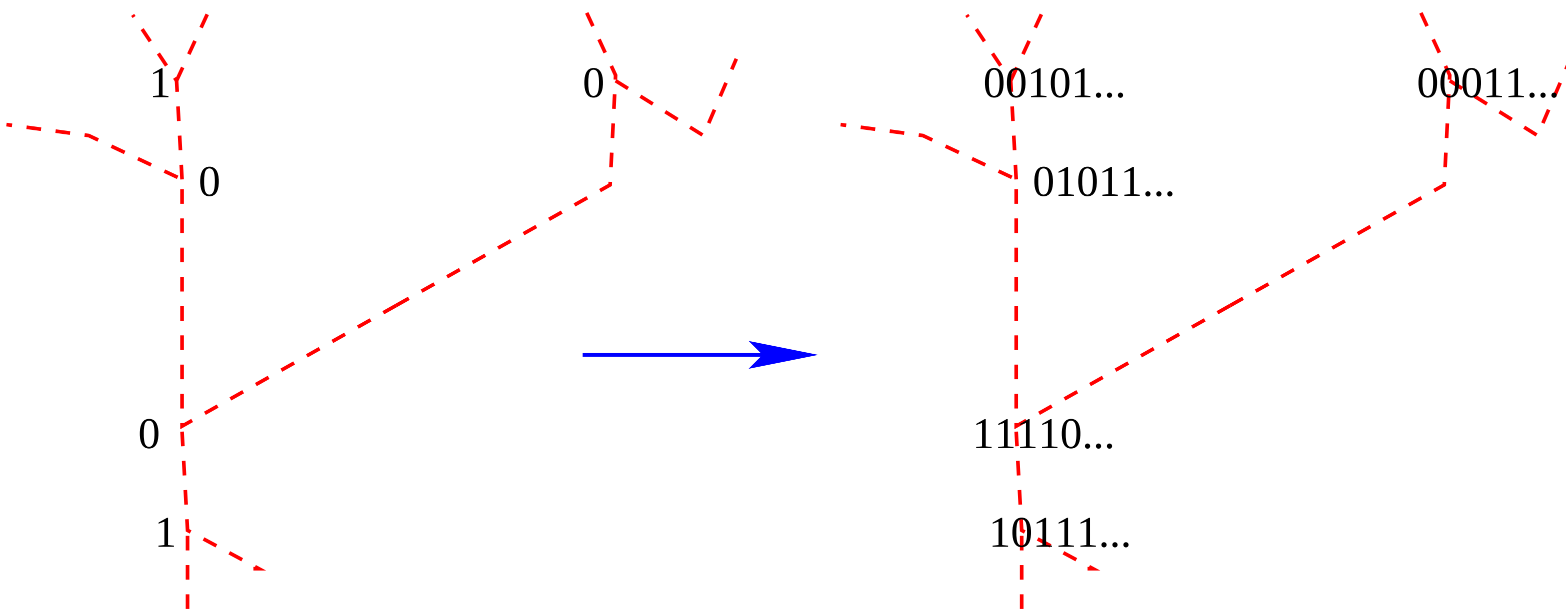
We want to choose trees which

- have no leaves (vertices of degree one) and
- have at least three ends (paths to infinity).

Strategy

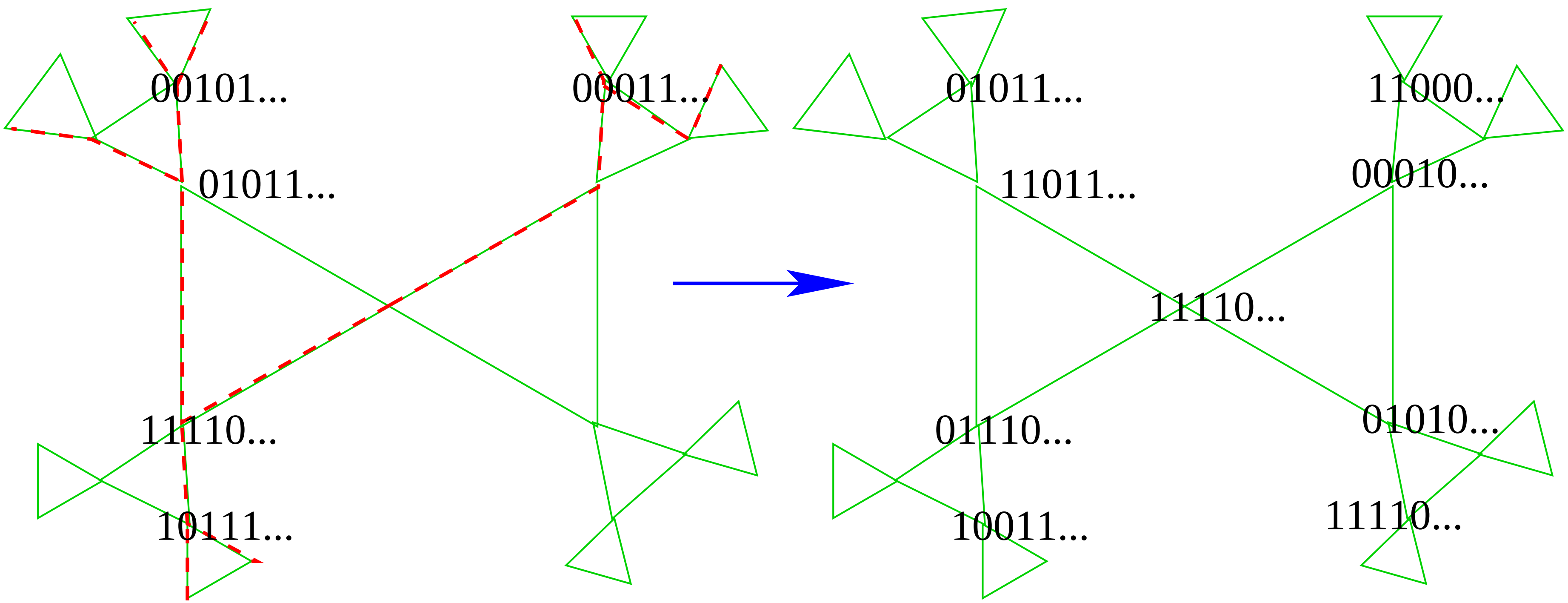
Step 2.

- Modify the O-W factor above to give an Aut(T)-factor $F : \xi_2 \to \xi_4$ on a tree T with bounded degree, no leaves, and ≥ 3 ends.
- Use Adam's observation to get infinitely many indep. bits at each $x \in T$.



Strategy

- Step 3. Distribute the bits to the rest of the $x \in X$.



Step 1. Trees as factors

Goal: To choose a percolation of G containing a tree with at least three ends as a G-factor of $\xi_{2^{m-1}}$.

We appeal to some results from percolation theory.
Let X be a graph. Then define

- $p_c(X) = \inf\{p :$ Bernoulli(p) has an infinite comp.$\}$
- $p_u(X) = \inf\{p :$ Bernoulli(p) has a unique ∞-comp.$\}$

Theorem. (Pak, Smirnova-Nagnibeda 2000) If G is a nonamenable, f.g. group, then there exists a Cayley graph X of G such that $p_c(X) < p_u(X)$.

Theorem. (Benjamini, Schramm 1996) Consider Bernoulli(p) percolation with $p_c(X) < p < p_u(X)$. For every n there is an infinite component with more than n ends.

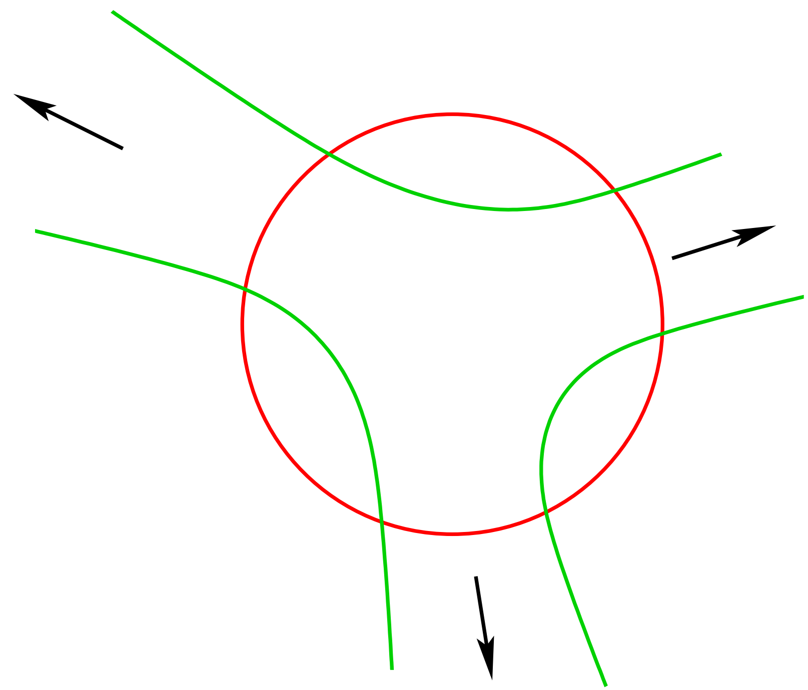
Step 1. Trees as factors

- Let X be a Cayley graph of G with $p_c(X) < p_u(X)$.
- Choose m large enough that there is a value of k such that

$$p_c(X) < k/2^{m-1} < p_u(X).$$

- Use $\xi_{2^{m-1}}$ to generate a Bernoulli$(k/2^{m-1})$ percolation on X which has a component C with at least three ends!

C has $\geq$ 3 ends:



$\exists$ a finite set $K \subset C$ s.t. $C \setminus K$ has at least 3 components.

Step 1. Trees as factors

Have: A component with at least 3 ends.
Want: A *tree* with at least 3 ends.

- Use minimal spanning forests:
  - Let X be a graph with distinct labels $L(e)$ at each edge e.
  - Remove an edge e from $X \iff L(e)$ is the largest label along some cycle of X.

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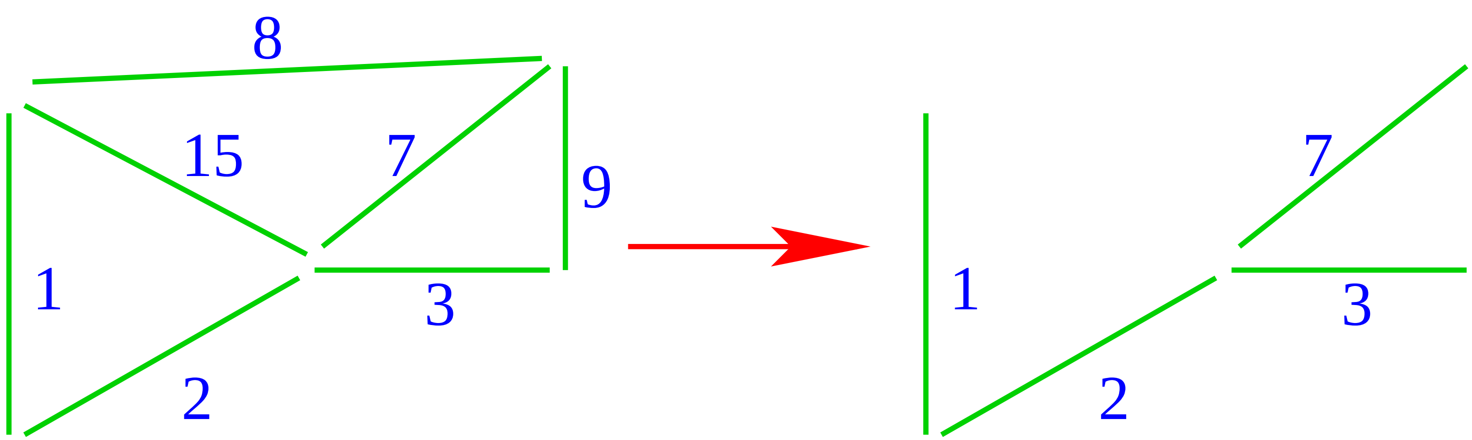
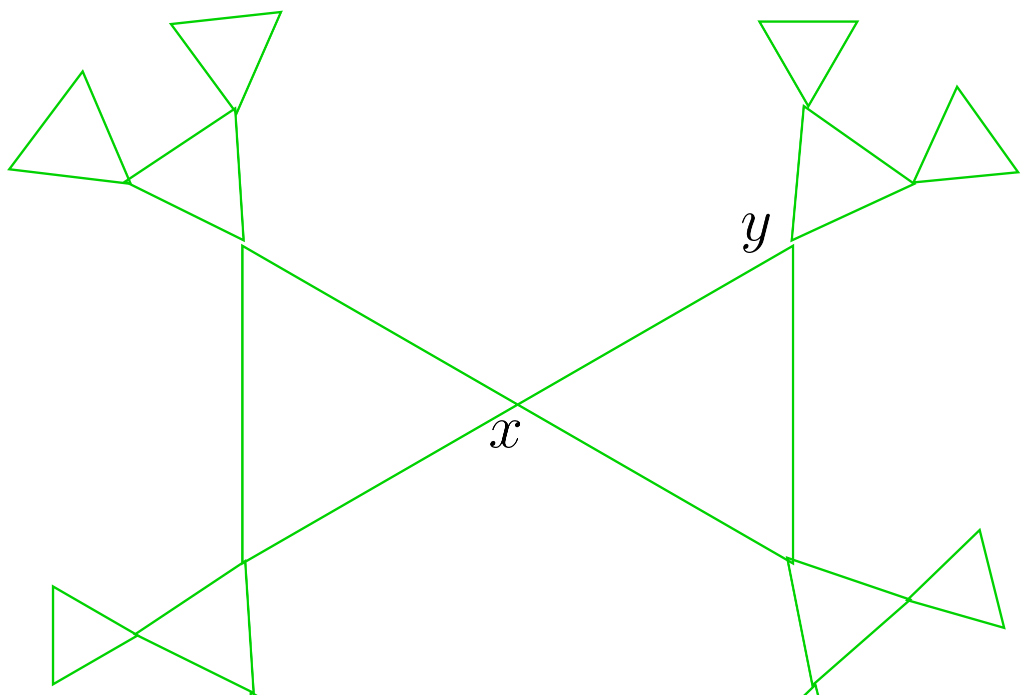


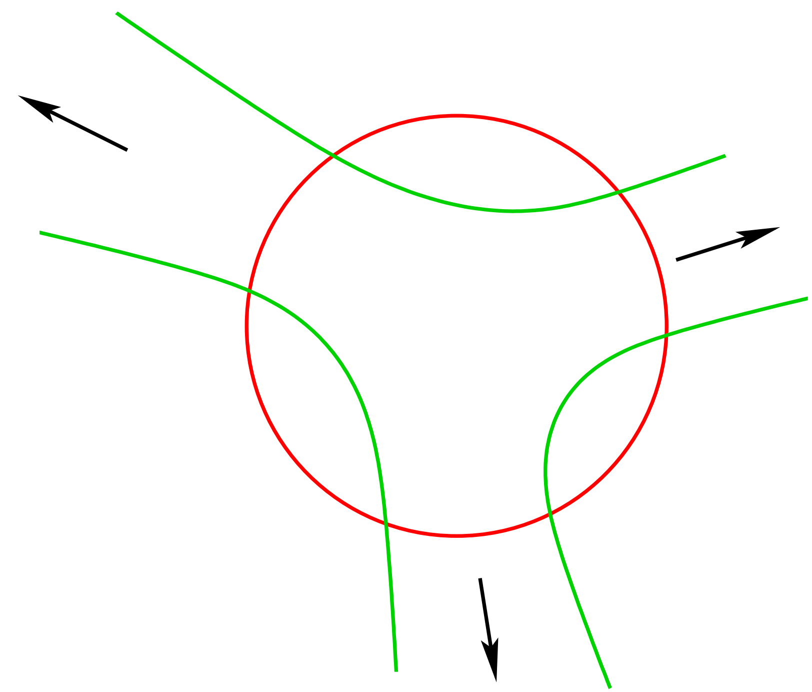
Step 1. Trees as factors

- Label the edges of X. Generate a continuous r.v. $U(e)$ at each edge e in X as a G-factor of $\xi_{2^{m-1}}$:
  - order the elements of G, $g_1 < g_2 < \ldots$,
  - form $U(x, y)$ ($x \sim y$) by concatenating $\xi(x \cdot g_1) \oplus \xi(y \cdot g_1), \xi(x \cdot g_2) \oplus \xi(y \cdot g_2), \ldots$,
  - the $U(e)$ are dependent, but a.s. distinct.



Step 1. Trees as factors

- Use the $U(e)$ and minimal spanning forest to trim to a tree:
  - form the minimal spanning tree inside the red circles
  - identify the red circles points and generate the minimal spanning forest on the resulting graph



This ensures that one of the resulting trees has at least three ends.

From groups to graphs

Consider X a nonamenable transitive graph, $G = \text{Aut}(X)$.
(X is transitive if for $x, y \in X, \exists \gamma \in G$ with $\gamma(x) = y$.)

Question. Does there always exist $m \leq n$ such that there is a G factor from ξ_m to ξ_n on X?
Answer. No. (Though often there is.)

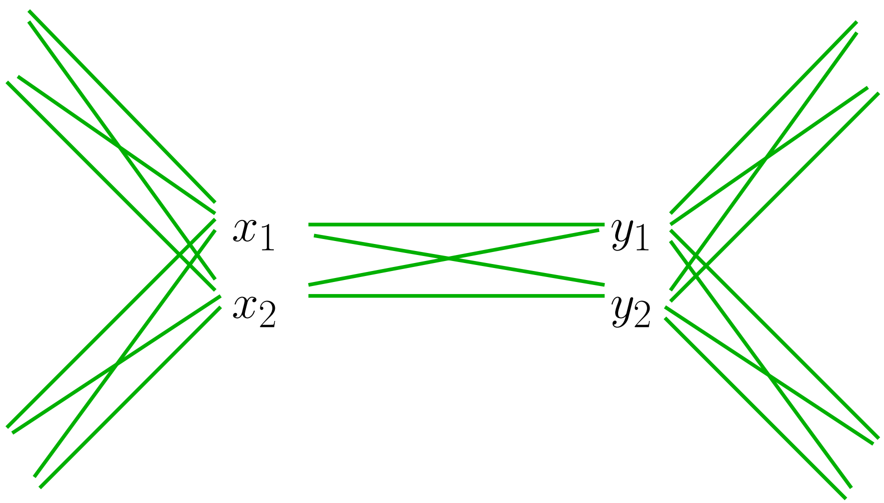
Potential problems with proof outlined above for groups:

- Is $p_c(X) < p_u(X)$?
  (To get a component with at least 3 ends.)
  This is conjectured to be true by Benjamini and Schramm.
- Can we generate distinct edge labels as a G-factor of ξ_m for some m? (To use MSF to trim component trees.)

A counterexample

Let S_3 be the graph formed by taking the regular tree T_3 of degree 3 and:

- replacing each vertex $x \in T_3$ be two vertices x_1, x_2
- replacing each edge $(x, y) \in T_3$ by four edges $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$.



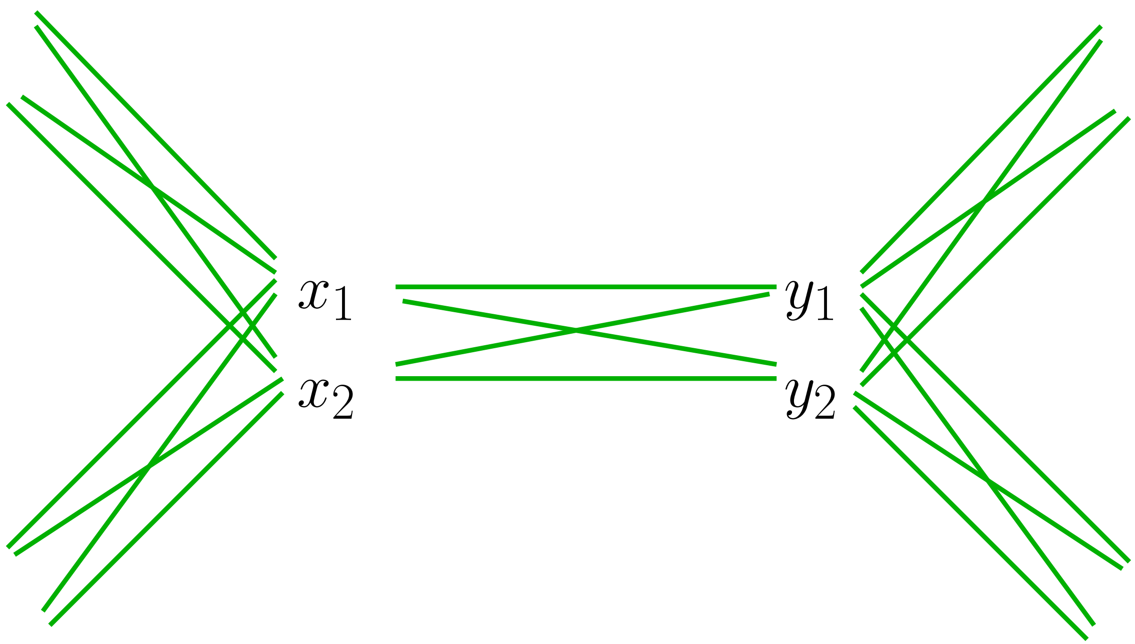
- S_3 is transitive,
- has ∞ many ends,
- and is unimodular.

Note: there is an automorphism γ_x of S_3 which interchanges x_1 and x_2 but fixes the rest of the vertices.

A counterexample

Let G_3 be the full automorphism group of S_3.
Fix m, n and let $F : \xi_m \to \xi_n$ be a G_3-factor. Then:

- If $\xi_m(x_1) = \xi_m(x_2)$, then $(F(\xi_m))(x_1) = (F(\xi_m))(x_2)$



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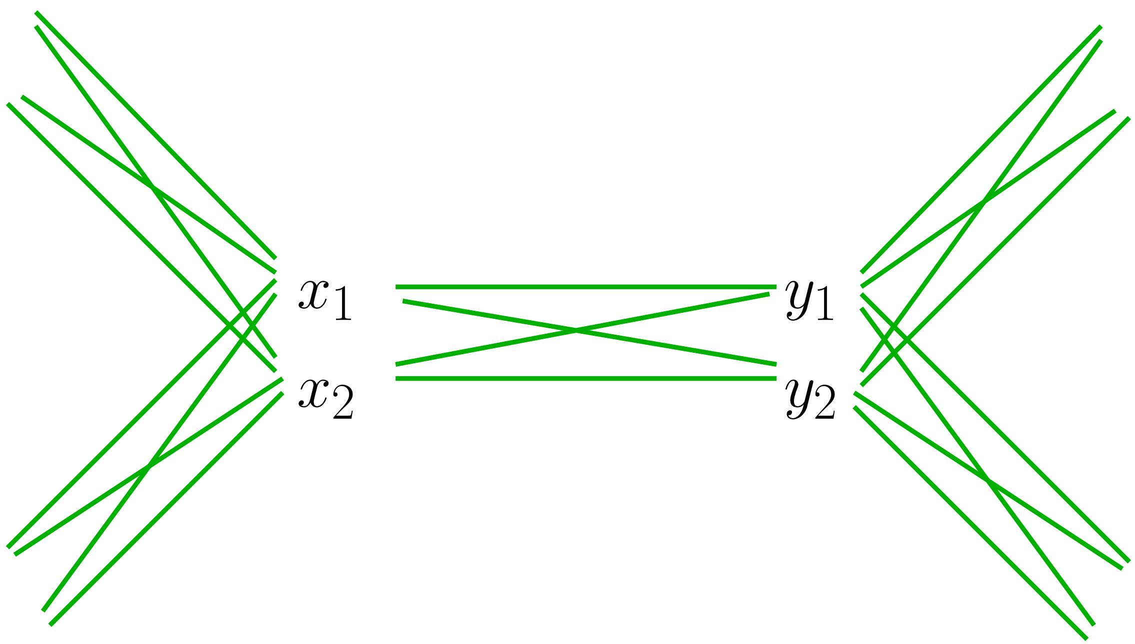
- $P(\xi_m(x_1) = \xi_m(x_2)) = 1/m$ and
  $P(\xi_n(x_1) = \xi_n(x_2)) = 1/n$



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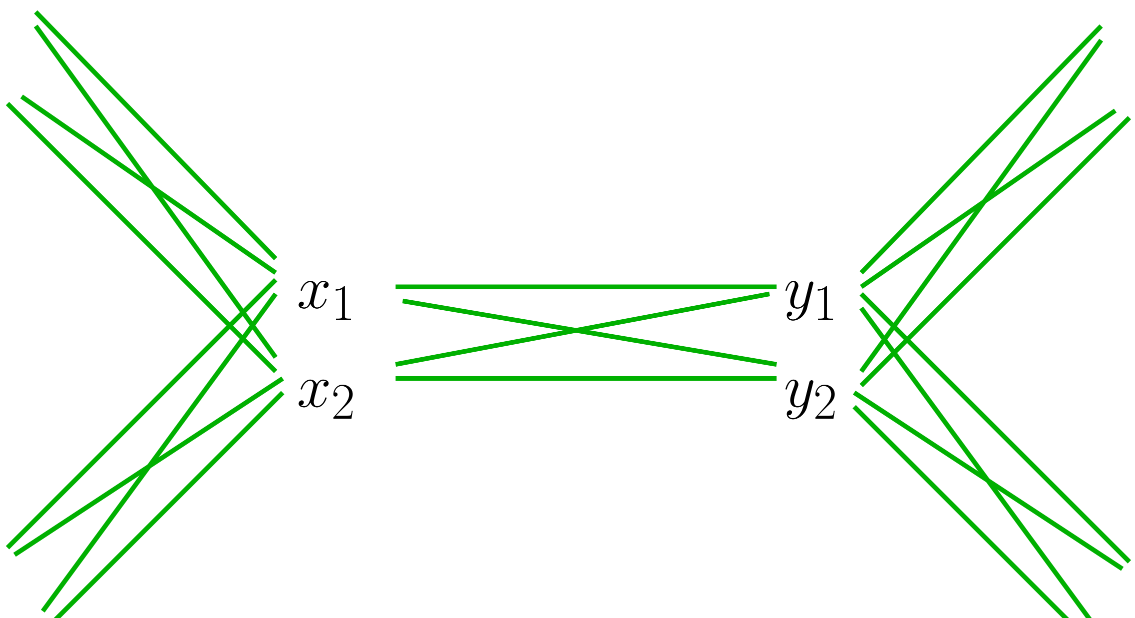
- $P(\xi_m(x_1) = \xi_m(x_2)) = 1/m$ and
  $P(\xi_n(x_1) = \xi_n(x_2)) = 1/n$
- Therefore, $1/m \leq 1/n \Rightarrow m \geq n$.



How to generate distinct edge labels?

"Theorem". Let X be a nonamenable graph with transitive automorphism group G. If

- X has at least 3 ends or $p_c(X) < p_u(X)$, and
- for j large enough, there is a G-factor of ξ_j which puts distinct labels on each edge of X,

then for m sufficiently large, there is a G-factor: $\xi_{2^m} \longrightarrow \xi_{2^{m+1}}$.

Sufficient condition for the second bullet:
There do not exist $x_1, x_2 \in X$ and $\gamma : x_1 \rightarrow x_2$ such that for all but finitely many y,

$$\text{Stab}_G(x_1) \cdot y = \text{Stab}_G(x_2) \cdot \gamma y.$$

Open questions

Question 1. Is there a pair (X, G) and $m \neq n$ such that there is an *invertible* G factor from ξ_m to ξ_n on (X, G)?

Question 2. (R. Lyons) T_4 is the regular tree of degree 4.

- We have seen that $\mathbb{F}_2$ acts on T_4.
  (By giving T_4 the structure of the Cayley graph of $\mathbb{F}_2$.)
- Aut(T_4) is larger than $\mathbb{F}_2$ since it allows a and b edges to be interchanged.

Is there an Aut(T_4)-factor of $\xi_{[0,1]}$ on T_4 which gives T_4 the structure of the Cayley graph of $\mathbb{F}_2$?