Martingale problems for Markov processes

- Levy and Watanabe characterizations
- Ito's formula and martingales associated with solutions of stochastic equations
- Generators and martingale problems for Markov processes
- Equivalence between stochastic equations and martingale problems
- Uniqueness and the Markov property
- Duality
- Dynkin's formula
- Relationship to PDEs

Filtrations and the Markov property

$(\Omega, \mathcal{F}, P)$ a probability space

Available information is modeled by a sub-σ-algebra of $\mathcal{F}$

$\mathcal{F}_t$ information available at time t

$\{\mathcal{F}_t\}$ is a *filtration*. $\quad t < s$ implies $\mathcal{F}_t \subset \mathcal{F}_s$

A stochastic process X is *adapted* to $\{\mathcal{F}_t\}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$.

An E-valued stochastic process X adapted to $\{\mathcal{F}_t\}$ is $\{\mathcal{F}_t\}$-*Markov* if

$$E[f(X(t+r))|\mathcal{F}_t] = E[f(X(t+r))|X(t)], \quad t, r \geq 0, \ f \in B(E)$$

An $\mathbb{R}$-valued stochastic process M adapted to $\{\mathcal{F}_t\}$ is an $\{\mathcal{F}_t\}$-*martingale* if

$$E[M(t+r)|\mathcal{F}_t] = M(t), \quad t, r \geq 0$$

τ is an $\{\mathcal{F}_t\}$-*stopping time* if for each $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$. For a stopping time τ,

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \{\tau \leq t\} \cap A \in \mathcal{F}_t, t \geq 0\}$$

Martingale characterizations

Brownian motion (Levy)

W a continuous $\{\mathcal{F}_t\}$-martingale

$W(t)^2 - t$ an $\{\mathcal{F}_t\}$-martingale

Then W is a standard Brownian motion compatible with $\{\mathcal{F}_t\}$.

Poisson process (Watanabe)

N a counting process adapted to $\{\mathcal{F}_t\}$

$N(t) - \lambda t$ an $\{\mathcal{F}_t\}$-martingale

Then N is a Poisson process with parameter λ compatible with $\{\mathcal{F}_t\}$

Diffusion processes

$\sigma : \mathbb{R}^d \to \mathbb{M}^{d\times m}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$

Consider

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds$$

By Itô's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

where

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x)$$

with $((a_{ij}(x)\ )) = \sigma(x)\sigma(x)^T$.

Markov processes

$X = \{X(t), t \geq 0\}$ is a Markov process if

$$E[f(X(t+s))|\mathcal{F}_t] = E[f(X(t+s))|X(t)]$$

The *generator* of a Markov process determines its short time behavior

$$E[f(X(t+\Delta t)) - f(X(t))|\mathcal{F}_t] \approx Af(X(t))\Delta t$$

$$E[f(X(t+r))-f(X(t))|\mathcal{F}_t] = E[\sum f(X(t_{i+1}))-f(X(t_i))|\mathcal{F}_t] \approx E[\sum Af(t_i)\Delta t_i|\mathcal{F}_t]$$

which suggests

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a martingale.

Example: Continuous time Markov chain

$$P\{X(t+\Delta t)=j|X(t)=i\} \approx q_{ij}\Delta t, \quad j \neq i$$

implies

$$Af(i)=\sum_j q_{ij}(f(j)-f(i)) = q_i \sum_j \frac{q_{ij}}{q_i}(f(j)-f(i))$$

where $q_i = \sum_{j \neq i} q_{ij}$.

Examples of generators

General jump processes:

$$Af(x) = \lambda(x) \int_E (f(y) - f(x))\eta(x, dy)$$

Diffusions:

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum b_i(x) \frac{\partial}{\partial x_i} f(x)$$

Sums of generators should be generators:

$$\begin{aligned} Af(x) &= \lambda(x) \int_E (f(y) - f(x))\eta(x, dy) \\ &= \sum_i \lambda_i(x) \int_E (f(y) - f(x))\eta_i(x, dy) \end{aligned}$$

where $\lambda(x) = \sum_i \lambda_i(x)$ and $\eta(x, dy) = \sum_i \frac{\lambda_i(x)}{\lambda(x)} \eta_i(x, dy)$.

Martingale problem for A

X is a solution for the martingale problem for (A, ν_0), $\nu_0 \in \mathcal{P}(E)$, if $PX(0)^{-1} = \nu_0$ and there exists a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for all $f \in \mathcal{D}(A)$.

Why should a martingale problems characterize a process?

$$E[(f(X(t+s)) - f(X(t)) - \int_t^{t+s} Af(X(r))dr) \prod h_i(X(t_i))] = 0$$

$0 \leq t_1 < \cdots < t_m \leq t < t+s$, $f \in \mathcal{D}(A)$, $h_i \in B(E)$.

Example

$Af = \frac{1}{2}f''$

$f(x) = e^{i\theta x}$, $Af(x) = -\frac{1}{2}\theta^2 f(x)$

$0 \leq t_1 < \cdots < t_m \leq t_{m+1}$

$$
\begin{aligned}
&E[f(X(t_{m+1}))\prod_{i=1}^{m} h_i(X(t_i))] \\
&\qquad = E[f(X(t_m))\prod_{i=1}^{m} h_i(X(t_i))] - \frac{1}{2}\theta^2 \int_{t_m}^{t_{m+1}} E[f(X(r))\prod h_i(X(t_i))]dr
\end{aligned}
$$

so

$$
E[e^{i\theta X(t_{m+1})}\prod_{i=1}^{m} h_i(X(t_i))] = e^{-\frac{1}{2}\theta^2(t_{m+1}-t_m)} E[e^{i\theta X(t_m)}\prod_{i=1}^{m} h_i(X(t_i))]
$$

and the joint distribution of $(X(t_1), \ldots, X(t_m))$ determines the joint distribution of $(X(t_1), \ldots, X(t_{m+1}))$.

General SDE

Let

$$
\begin{aligned}
X(t) = X(0) \quad &+ \quad \int_{U_0\times[0,t]} \sigma(X(s))dW(s) + \int_0^t \beta(X(s))ds \\
&+ \quad \int_{U_1\times[0,t]} \alpha_1(X(s-),u)\tilde{\xi}_1(du\times ds) \\
&+ \quad \int_{U_2\times[0,t]} \alpha_2(X(s-),u)\xi_2(du\times ds)\ .
\end{aligned}
$$

Then

$$
\begin{aligned}
& f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds \\
&\quad = \int_{U_0\times[0,t]} \nabla f(X(s))\cdot\sigma(X(s))dW(s) \\
&\qquad + \int_{U_1} (f(X(s-)+\alpha_1(X(s-),u)) - f(X(s-))\tilde{\xi}_1(du\times ds) \\
&\qquad + \int_{U_2} (f(X(s-)+\alpha_2(X(s-),u)) - f(X(s-))\tilde{\xi}_2(du\times ds)
\end{aligned}
$$

Form of the generator

$((a_{ij}(x)\,)) = \sigma(x)\sigma^T(x)$

$$
\begin{aligned}
Af(x) &= \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x) \\
&\quad + \int_{U_1}(f(x+\alpha_1(x,u)) - f(x) - \alpha_1(x,u)\cdot\nabla f(x))\nu_1(du) \\
&\quad + \int_{U_2}(f(x+\alpha_2(x,u)) - f(x))\nu_2(du)
\end{aligned}
$$

Let $\mathcal{D}(A)$ be a collection of functions for which Af is bounded. Then a solution of the SDE is a solution of the martingale problem for A.

Basic results on martingale problems

Theorem 1 *If any two solutions of the martingale problem for A satisfying $PX_1(0)^{-1} = PX_2(0)^{-1}$ also satisfy $PX_1(t)^{-1} = PX_2(t)^{-1}$ for all $t \geq 0$, then the f.d.d. of a solution X are uniquely determined by $PX(0)^{-1}$.*

If X is a solution of the MGP for A and $Y_a(t) = X(a+t)$, then Y_a is a solution of the MGP for A.

Theorem 2 *If the conclusion of the above theorem holds, then any solution of the martingale problem for A is a Markov process.*

Duality

Two state spaces E_1, E_2

A generator for process in E_1, B generator for process in E_2

$H(x,y)$ on $E_1 \times E_2$ satisfying $AH(x,y) = BH(x,y)$

$X \sim A$, $Y \sim B$ X and Y independent. Then

$$\frac{\partial}{\partial s} E[H(X(s), Y(t-s))] = E[AH(X(s), Y(t-s))] - E[BH(X(s), Y(t-s))] = 0$$

if $Y_y(0) = y$, then

$$E[H(X(t), y)] = E[H(X(0), Y_y(t)]$$

Example

$Af(x) = x(1-x)f''(x)$, $E_1 = [0,1]$

$H(x,y) = x^y$, $E_2 = \{0,1,2,\ldots\}$

$$AH(x,y) = y(y-1)(x^{y-1} - x^y) = BH(x,y)$$

for

$$Bg(y) = y(y-1)(g(y-1) - g(y))$$

Consequently,

$$E[X(t)^y] = E[X(0)^{Y_y(t)}]$$

Dynkin's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

a $\{\mathcal{F}_t\}$-martingale implies

$$E[f(X(t \wedge \tau)] = E[f(X(0))] + E[\int_0^{t\wedge\tau} Af(X(s))ds]$$

for each $\{\mathcal{F}_t\}$-stopping time τ.

For example: X one-dimensional diffusion, $\tau = \inf\{t : X(t) \notin (a,b)\}$, $Af(x) = 0$, $a < x < b$. For $X(0) = x_0 \in (a,b)$,

$$f(x_0) = E[f(X(\tau))] = P\{X(\tau) = a\}f(a) + P\{X(\tau) = b\}f(b)$$

so

$$P\{X(\tau) = b | X(0) = x_0\} = \frac{f(x_0) - f(a)}{f(b) - f(a)}$$

Dirichlet problem

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x)$$

X a solution of the martingale problem

$D \subset \mathbb{R}^d$

$Au(x) = 0$, $x \in D$, $u(x) = f(x)$, $x \in \partial D$

$\tau = \inf\{t : X(t) \notin D\}$

Then

$$\begin{aligned} E[f(X(\tau))|X(0) = x] &= E[u(X(\tau)|X(0) = x] \\ &= u(x) + E[\int_0^\tau Au(X(s))ds|X(0) = x] \\ &= u(x) \end{aligned}$$

Weak solutions of SDEs

$\tilde{X}$ is a *weak solution* of the SDE if there exists a probability space on which are defined X, W, ξ_1, and ξ_2 satisfying the SDE and $\tilde{X}$ has the same distribution as X.

Theorem 3 *Suppose that the a_{ij} and b_i are locally bounded and that for each $f \in C_c^2(\mathbb{R}^d)$*

$$\sup_x \int_{U_1} |f(x + \alpha_1(x,u)) - f(x) - \alpha_1(x,u) \cdot \nabla f(x)| \nu_1(du) < \infty$$

and

$$\sup_x \int_{U_2} |f(x + \alpha_2(x,u)) - f(x)| \nu_2(du) < \infty.$$

Let $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$. Then any solution of the martingale problem for A is a weak solution of the SDE.

Nonsingular diffusions

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + b(X(s))ds$$

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x).$$

Assume that $\sigma(x)$ is invertible for each x and that $|\sigma(x)^{-1}|$ is locally bounded. If $\tilde{X}$ is a solution of the martingale problem for A, then

$$M(t) = \tilde{X}(t) - \int_0^t b(\tilde{X}(s))ds$$

is a local martingale and

$$\tilde{W}(t) = \int_0^t \sigma(\tilde{X}(s))^{-1}dM(s)$$

is a standard Brownian motion compatible with $\{\mathcal{F}_t^{\tilde{X}}\}$. It follows that

$$\tilde{X}(t) = \tilde{X}(0) + \int_0^t \sigma(\tilde{X}(s))d\tilde{W}(s) + \int_0^t b(\tilde{X}(s))ds.$$

Natural form for the jump terms

Generator for a simple pure jump process:

$$Af(x)=\lambda(x)\int_{\mathbb{R}^d}(f(z)-f(x))\mu(x,dz),$$

$\lambda\geq 0$ and $\mu(x,\cdot)$ is a probability measure. There exists $\gamma:\mathbb{R}^d\times[0,1]\to\mathbb{R}^d$ such that $\int_0^1 f(x+\gamma(x,u))du=\int_{\mathbb{R}^d}f(z)\mu(x,dz)$.

$$X(t)=X(0)+\int_{[0,\infty)\times[0,1]\times[0,t]}I_{[0,\lambda(X(s-))]}(v)\gamma(X(s-),u)\xi(dv\times du\times ds),$$

ξ a Poisson random measure on $[0,\infty)\times[0,1]\times[0,\infty)$ with Lebesgue mean measure, is a stochastic differential equation corresponding to A.

$U_2=[0,\infty)\times[0,1]$ and $\alpha_2(x,u,v)=I_{[0,\lambda(x)]}(v)\gamma(x,u)$

$$\begin{aligned}\int_{[0,\infty)\times[0,1]}|\alpha_2(x,u,v)-\alpha_2(y,u,v)|dvdu\leq |\lambda(x)-\lambda(y)|\int_{[0,1]}\gamma(x,u)du\\ +\lambda(y)\int_{[0,1]}|\gamma(x,u)-\gamma(y,u)|du\end{aligned}$$

Reflecting diffusions

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

X has values in D, λ is nondecreasing and increases only when $X(t) \in \partial D$. By Itô's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$
$$= \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x) \qquad Bf(x) = \eta(x)\cdot\nabla f(x)$$

Either take $\mathcal{D}(A) = \{f \in C_c^2(D) : Bf(x) = 0, x \in \partial D\}$ or formulate a *constrained* martingale problem with solution (X, λ) by requiring X to take values in D, λ to be nondecreasing and increase only when $X \in \partial D$, and

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$

to be an $\{\mathcal{F}_t^{X,\lambda}\}$-martingale.

Instantaneous jump conditions

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and W and $N(t)$ is the number of times X has hit the boundary by time t. Then

$$\begin{aligned} f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s-))dN(s) \\ = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s) + \int_0^t (f(X(s-) + \alpha(X(s-), \zeta_{N(s-)+1})) \\ - \int_U f(X(s-) + \alpha(X(s-), u))\nu(du))dN(s) \end{aligned}$$

where

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x)$$

and

$$Bf(x) = \int_U (f(x + \alpha(x,u)) - f(x))\nu(du).$$

More general jump terms

$$X(t) = X(0) + \int_{[0,\infty)\times U\times[0,t]} I_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\xi(dv\times du\times ds)$$

where ξ is a Poisson random measure with mean measure $m\times\nu\times m$. The generator is of the form

$$Af(x) = \int_U \lambda(x,u)(f(x+\gamma(x,u)) - f(x))\nu(du).$$

If ξ is replaced by $\tilde{\xi}$, then

$$Af(x) = \int_U \lambda(x,u)(f(x+\gamma(x,u)) - f(x) - \gamma(x,u)\cdot\nabla f(x))\nu(du).$$

Note that many different choices of λ, γ, and ν will produce the same generator.