THE ANALYSIS OF COATING FLOWS IN A STRIP

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§0. Introduction.

In this paper we consider two-dimensional, stationary coating flows in a strip \( \{0 < y < 1\} \). We shall establish existence, and uniqueness, and regularity of the free boundary, when the boundary data at \( \{y = 1\} \) are perturbation of a uniform flow \( \vec{U} \). The underlying assumptions are that the substrate \( \{y = 0\} \) is moving with velocity \( \vec{U} \), and that the no-slip condition holds. One of the important features of the model is that the free boundary

\[ \Gamma : y = f(x), \quad -\infty < x < 0, \]

which satisfies:

\[ 0 < f(x) < 1 \quad \text{if} \quad -\infty < x < 0, \quad f(0) = 0, \]

in tangent to the moving substrate \( \{y = 0\} \) at \( x = 0 \).

For simplicity we shall deal only with Stokes equation; the analysis for the Navier-Stokes equation will only slightly affect the perturbed system, and can be probably dealt with similarly (see Remark 8.4).

The geometry is described in Figure 1.

\[ \vec{v} = \vec{U} + \varepsilon \vec{z}(x) \]

\[ \Delta \vec{v} = \nabla p \]

\[ \nabla \cdot \vec{v} = 0 \]

\[ (0,0) \quad \vec{v} = \vec{U} = (U, 0), \quad U > 0 \]

Figure 1

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On the free boundary $\Gamma$,

\[ T\vec{n} = \sigma \kappa \vec{n}, \quad \vec{v} \cdot \vec{n} = 0 \]

where $\vec{n}$ is the exterior normal,

\[ \sigma = \frac{1}{Ca}, \quad Ca = \text{capillary number}, \]

$\kappa = \text{curvature}$, and $T$ is the stress tensor

\[ T_{ij} = -p \delta_{ij} + \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right); \]

note that $\text{div}T = 0$.

The analogous problem in the half-space $\{y > 0\}$ was considered in our previous paper [4].

It is well known that the no-slip condition together with (0.1) imply, on physical grounds, that the free boundary must be tangential to the substrate at their contact point (Otherwise the total force will become infinite at that point.) If we linearize the problem about $(0,0)$, we find [4] that

\[ f(x) \sim A(-x)^{2-\rho} \quad \text{near } x = 0 \]

where

\[ 0 < \rho < \frac{1}{2}, \quad \rho = \frac{1}{2\pi i} \log \frac{1-2i\lambda}{1+2i\lambda}, \quad \lambda = -\frac{U}{\sigma}. \]

Here $A$ is a nonnegative constant; if $A = 0$ then (0.2) is to be replaced by $f(x) \sim A_n(-x)^{n-\rho}$ with the smallest positive integer $n$ for which $A_n \neq 0$.

Suppose $f_0(x)$ is any smooth function such that

\[ f_0(x) \sim A(-x)^{2-\rho} \quad \text{near } x = 0, \]

\[ f_0(x) \sim B(-x)^{\beta} \quad \text{near } x = -\infty, \]

\[ f_0(x) > 0 \quad \text{for } -\infty < x < 0, \]

where $0 < \beta < 1$, $A > 0$, $B > 0$. Then, as proved in [4], for any positive and small enough $\epsilon$, the coating flow problem has a solution in $y > 0$ with free boundary of the form

\[ y = \epsilon \tilde{f}(x, \epsilon), \]
where
\[ \| \tilde{f}(\cdot, \epsilon) - f_0(\cdot) \| \to 0 \quad \text{if } \epsilon \to 0; \]
here \( \| \) is a \( C^{4+\alpha} \)-norm with weights \( |x|^{2-\rho} \) and \( |x|^\beta \) near \( x = 0 \) and \( x = -\infty \) respectively.

In the present paper the flow region is a strip, rather than the half-space. This makes the analysis more complicated. However we shall be able to draw heavily upon the methods developed in [4] and, in particular, we shall exploit the Green function constructed in [4] in order to analyze the solution near \((0,0)\).

In Section 1 we linearize the flow problem described in Figure 1 and (0.1) about \( \epsilon = 0 \); this somewhat differs from the linearization, performed in [4], about \( x = 0, y = 0 \).

In Sections 2-5 we study the linearized problem and establish existence and uniqueness (Section 2) and regularity (Sections 3-5). In Section 6 we reformulate the complete coating problem and then, in Sections 7,8, establish existence and uniqueness. The free boundary, generically, has the form
\[ f(x) \sim A(-x)^{m-\rho} \quad (A > 0) \]
near \( x = 0 \) where \( m \) is a positive integer \( \geq 2 \) which depends implicitly on \( \zeta \).

Pileckas ([8] [9]; see also [10]) has studied a problem similar to ours with a somewhat different set of data which, however, are also perturbation of the uniform flow. In his case the free boundary forms a (positive) angle with the substrate at the separation point, i.e., at the initial point of the free boundary; this angle is unknown in advance. The solution therefore has singularity in the sense that the physical force at the separation point is infinite. The perturbation argument of Pileckas is based on sharp weighted Sobolev estimates for the Stokes equation in a fixed unbounded domain which were derived in [13]. By contrast, our estimates for the Stokes equation in a fixed strip are based on the rather explicit integral representation (by means of the Green function) constructed in [4].

§1. The linearized problem.

If \( \epsilon = 0 \) then the solution to the coating flow problem is
\[ \tilde{U} = (U, 0), \quad p = 0, \quad f(x) = 0 \quad (x < 0). \]

We assume a perturbed solution, for any small \( \epsilon > 0 \), having the form:

\[ \text{velocity} = \tilde{v} = \tilde{U} + \epsilon \tilde{G}, \]
\[ \text{pressure} = \epsilon p, \]
\[ \text{free boundary} = \Gamma_\epsilon = \{ y = \epsilon f(x), -\infty < x < 0 \}. \]

Then
\[ \Delta \tilde{G} = \nabla p, \quad \nabla \cdot \tilde{G} = 0 \text{ in the flow region}, \]
and, on the free boundary,

\begin{align}
(1.1) & \quad \epsilon \vec{G} \cdot \vec{n} = -\vec{U} \cdot \vec{n}, \\
(1.2) & \quad \epsilon T(\vec{G}, p) \vec{n} = \sigma \kappa(\epsilon f) \vec{n}
\end{align}

where

\[ T_{ij}(\vec{G}, p) = -p \delta_{ij} + \left( \frac{\partial G_i}{\partial x_j} + \frac{\partial G_j}{\partial x_i} \right). \]

Here

\begin{align}
(1.3) & \quad \vec{n} = \frac{(\epsilon f'(x), -1)}{(1 + \epsilon^2 f'(x)^2)^{1/2}}, \\
(1.4) & \quad \kappa(\epsilon f) = -\frac{\epsilon f''(x)}{(1 + \epsilon^2 f'(x)^2)^{3/2}}.
\end{align}

We shall sometimes write

\[ G_x = G_1, \quad G_y = G_2 \]

and do the same for other vectors.

By (1.3), the lowest order terms in (1.1) give

\[ G_y(x, 0) = U f'(x), \quad x < 0. \]

Next we rewrite (1.2) as two equations:

\[ \tau \cdot T(\vec{G}, p) \vec{n} = 0, \]
\[ \epsilon \vec{n} T(\vec{G}, p) \vec{n} = \sigma \kappa(\epsilon f) \]

where \( \tau \) is obtained by rotating \( \vec{n} \) counterclockwise by 90°. The lowest order terms in the first equation give (using (1.3))

\[ \frac{\partial G_x}{\partial y} + \frac{\partial G_y}{\partial x} = 0, \quad y = 0, x < 0, \]

and in the second equation they give (after using also (1.4))

\[ -p + 2 \frac{\partial G_y}{\partial y} = -\sigma f''(x), \quad y = 0, x < 0. \]

We summarize the linear problem:

\[ \Delta \vec{G} = \nabla p \quad \text{in} \ S = \{-\infty < x < \infty, 0 < y < 1\}, \]
(1.8) \[ \nabla \cdot \vec{G} = 0 \quad \text{in} \ S, \]

(1.9) \[ \vec{G}(x, 1) = \vec{\zeta}(x), \quad -\infty < x < \infty, \]

(1.10) \[ \vec{G}(x, 0) = 0, \quad 0 < x < \infty, \]

(1.11) \[ \frac{\partial G_x}{\partial y} + \frac{\partial G_y}{\partial x} = 0, \quad y = 0, -\infty < x < 0, \]

and, if we use (1.5) to eliminate \( f'' \) from (1.6),

(1.12) \[ -p + 2 \frac{\partial G_y}{\partial y} = \frac{1}{\lambda} \frac{\partial G_y}{\partial x}, \quad y = 0, -\infty < x < 0. \]

The "free boundary" for the linearized problem is given by (1.5), with \( f(0) = 0 \).

In Sections 2-5 we study the linearized problem. Existence and uniqueness of a weak solution are established in Section 2, and regularity is proved in Sections 3-5.

§2. Existence and uniqueness.

For simplicity we assume:

(2.1) \[ \zeta(x) \text{ is a } C^\infty \text{ function with compact support}; \]

the case of more general \( \vec{\zeta} \) is discussed in Remark 8.3.

We wish to replace the inhomogeneous boundary condition at \( \{y = 1\} \) by a homogeneous condition. For this purpose we introduce a function \( \vec{G}_0 \) in \( \bar{S} \) such that

(2.2) \[ \vec{G}_0 = \vec{\zeta}(x) \quad \text{on} \ y = 1, \]

(2.3) \[ \nabla \cdot \vec{G}_0 = 0 \quad \text{in} \ S. \]

We do this by defining

\[ \vec{G}_0 = (-\partial_y h, \partial_x h) \]

where

\[ h(x, y) = (1 - y)\zeta_x(x - c(y)) + \int_{-\infty}^{x - c(y)} \zeta_y(s) ds; \]
here $c(y)$ is a $C^\infty$ function such that $c(y) = 0$ if $\frac{3}{4} < y < 1$. Then, at $y = 1$, 
\[ \partial_x h = \zeta_y, \quad -\partial_y h = \zeta_x, \]
so that (2.2) holds. Clearly (2.3) is also satisfied. We choose $c(y)$ negative and large for 
$0 \leq y \leq \frac{1}{2}$ so that 
\[ (2.4) \quad \tilde{G}_0(x, y) = 0 \quad \text{if} \quad -1 \leq x < \infty, \quad 0 \leq y \leq \frac{1}{2}. \]

Notice finally that $\tilde{G}_0$ has compact support.

We now introduce functions

\[ (2.5) \quad \tilde{W} = \tilde{G} - \tilde{G}_0, \]
\[ (2.6) \quad q = p - \left(2 \frac{\partial G_{0,y}}{\partial y} - \frac{1}{\lambda} \frac{\partial G_{0,y}}{\partial x}\right), \]
\[ (2.7) \quad \tilde{f} = \Delta \tilde{G}_0 - \nabla \left(2 \frac{\partial G_{0,y}}{\partial y} - \frac{1}{\lambda} \frac{\partial G_{0,y}}{\partial x}\right), \]

Then $-\Delta \tilde{W} + \nabla q = \tilde{f}$ and, since $\nabla \cdot \tilde{W} = 0$, we can write

\[ (2.8) \quad \partial_i (\partial_i W_j + \partial_j W_i - q \delta_{ij}) = f_j \quad \text{in} \ S, \]
\[ (2.9) \quad \partial_i W_i = 0 \quad \text{in} \ S. \]

Also

\[ (2.10) \quad \tilde{W}(x, 1) = 0, \quad -\infty < x < \infty, \]
\[ (2.11) \quad \tilde{W}(x, 0) = 0, \quad 0 < x < \infty, \]
\[ (2.12) \quad \frac{\partial W_x}{\partial y} + \frac{\partial W_y}{\partial x} = 0, \quad y = 0, \quad -\infty < x < 0, \]
\[ (2.13) \quad -q + 2 \frac{\partial W_y}{\partial y} = \frac{1}{\lambda} \frac{\partial W_y}{\partial x}, \quad y = 0, \quad -\infty < x < 0. \]

We wish to define a weak version of the problem (2.8)-(2.13). To do this we introduce the space

\[ X_0 = \{ \tilde{\varphi} \in C_0^\infty(\overline{S}); \quad \tilde{\varphi}(x, 1) = 0 \quad \text{if} \quad -\infty < x < \infty, \quad \tilde{\varphi}(x, 0) = 0 \quad \text{if} \quad x > 0, \nabla \cdot \tilde{\varphi} = 0 \text{ in } S \} \]
and its closure

\[ X = \text{closure of } X_0 \text{ in the norm} \]
\[ \| \tilde{w} \| = \left[ \int_S |\nabla \tilde{w}|^2 \right]^{\frac{1}{2}} = \| \tilde{w} \|_X. \]
Take any $\vec{\varphi} = (\varphi_1, \varphi_2) = (\varphi_x, \varphi_y)$, multiply (2.8) by $\varphi_i$, and integrate over $S$. After integrating by parts and using (2.10)-(2.13) we formally arrive at the relation

$$
\frac{1}{2} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)(\partial_i W_j + \partial_j W_i) \, dx \, dy + \frac{1}{\lambda} \int_{-\infty}^{0} \varphi_y(x, 0) \frac{\partial W_y(x, 0)}{\partial x} \, dx
$$

$$
= \int_S f_j \varphi_j \, dx \, dy.
$$

Set

$$
a(\vec{\varphi}, \vec{W}) = \frac{1}{2} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)(\partial_i W_j + \partial_j W_i) \, dx \, dy,
$$

$$
b(\vec{\varphi}, \vec{W}) = \frac{1}{\lambda} \int_{-\infty}^{0} \varphi_y(x, 0) \frac{\partial W_y(x, 0)}{\partial x} \, dx,
$$

$$
A(\vec{\varphi}, \vec{W}) = a(\vec{\varphi}, \vec{W}) + b(\vec{\varphi}, \vec{W}).
$$

**Lemma 2.1.** The bilinear functional $b$ is bounded in $X \times X$, and

$$
b(\vec{\varphi}, \vec{\varphi}) = 0.
$$

**Proof.** For any functions $\chi, \varphi$ in $C_0^\infty(S)$ which vanish on $\{y = 1\}$,

$$
| \int_{-\infty}^{\infty} \chi(x, 0) \frac{\partial \psi(x, 0)}{\partial x} \, dx | = | \int_{-\infty}^{\infty} \widehat{\chi}(\xi, 0)(i\xi)\widehat{\psi}(\xi, 0) \, d\xi |
$$

$$
\leq \left( \int_{-\infty}^{\infty} |\xi||\widehat{\chi}(\xi, 0)|^2 \right)^{1/2} \left( \int_{-\infty}^{\infty} |\xi||\widehat{\psi}(\xi, 0)|^2 \right)^{1/2},
$$

where "\(\widehat{\cdot}\)" refers to the Fourier transform. Consider the function

$$
H(y) = \int_{-\infty}^{\infty} |\xi||\widehat{\chi}(\xi, y)|^2 \, d\xi.
$$

Clearly $H(1) = 0$ and

$$
\left| \frac{dH}{dy} \right| \leq 2 \int_{-\infty}^{\infty} |\xi||\widehat{\chi}(\xi, y)||\widehat{\psi}(\xi, y)| \, d\xi
$$

$$
\leq \int_{-\infty}^{\infty} (|\xi|^2|\widehat{\chi}(\xi, y)|^2 + |\widehat{\psi}(\xi, y)|^2) \, d\xi.
$$
Hence

\[
|H(0)| \leq \int_0^1 \left| \frac{dH}{dy} \right| dy \leq \int_0^1 dy \int_{-\infty}^\infty (|\xi|^2 |\tilde{\chi}(\xi, y)|^2 + |\tilde{\chi}_y(\xi, y)|^2) d\xi
\]

\[
= \int_S \left[ \left( \frac{\partial \chi}{\partial x} \right)^2 + \left( \frac{\partial \chi}{\partial y} \right)^2 \right] dxdy.
\]

A similar estimate holds for the second integral on the right-hand side of (2.19). Therefore,

(2.20) \quad |b(\tilde{\chi}, \tilde{\psi})| \leq \|\tilde{\chi}\|_X \|\tilde{\psi}\|_X

if \( \tilde{\chi} = (0, \chi), \tilde{\psi} = (0, \psi) \). By approximation, this estimate holds also for all \( \tilde{\chi}, \tilde{\psi} \) in \( X \). Finally, if \( \tilde{\varphi} \in X_0 \) then (2.18) is immediate; by approximation (using (2.20)) (2.18) is then valid also for any \( \tilde{\varphi} \) in \( X \).

DEFINITION. A function \( \tilde{W} \) in \( X \) is called a weak solution of (2.8)-(2.13) if it satisfies (2.14) for any \( \tilde{\varphi} \) in \( X \).

Note that, by Lemma 2.1, the second integral on the left-hand side of (2.14) is well defined.

We can rewrite (2.14) in the abbreviated form

(2.21) \quad A(\tilde{\varphi}, \tilde{W}) = \int_S \tilde{f} \cdot \tilde{\varphi} \quad \forall \tilde{\varphi} \in X.

THEOREM 2.2. There exists a unique weak solution.

Proof. by Korn’s inequality [12]

(2.22) \quad a(\tilde{\varphi}, \tilde{\varphi}) \geq \gamma \|\tilde{\varphi}\|^2 \quad \forall \tilde{\varphi} \in X \quad (\gamma > 0).

Recalling also (2.18), we deduce that

(2.23) \quad (A\tilde{\varphi}, \tilde{\varphi}) \geq \gamma \|\tilde{\varphi}\|^2.

Since \( A(\tilde{\varphi}, \tilde{\psi}) \) is also bounded, i.e.,

(2.24) \quad |A(\tilde{\varphi}, \tilde{\psi})| \leq C\|\tilde{\varphi}\|\|\tilde{\psi}\|,

we can apply the Lax-Milgram theorem to deduce that there exists in \( X \) a unique solution \( \tilde{W} \) to (2.21).
§3. Regularity in the interior.

The interior regularity of the weak solution $\tilde{W}$ in $S$ follows from general regularity results for elliptic systems; the boundary regularity at $\{(x,1); -\infty < x < \infty\}$, $\{x,0); 0 < x < \infty\}$ and $\{(x,0); -\infty < x < 0\}$ also follows from general results [2]. It is essential for us, however, to find the precise regularity and the actual form of the weak solution $\tilde{W}$ at the point $(0,0)$ where the boundary conditions undergo abrupt change. To do this we need to develop a modified approach to proving regularity. For the sake of both clarity and completeness, we shall first develop this approach to deduce regularity about any point either in $S$ or on $\partial S \setminus \{0,0\}$. This approach is based on the construction of explicit Green’s function. The Green function needed for the analysis near $(0,0)$ was constructed in [4].

In this section we establish interior regularity; however some of the basic formulas will be used later on also for establishing regularity near the boundary.

Introduce balls $B_\rho(x_0, y_0)$ of center $(x_0, y_0)$ and radius $\rho$. For any $(x_0, y_0) \in \overline{S}$ we construct a $C^\infty$ cutoff function $\xi$ in $\mathbb{R}^2$:

$$
\xi = \begin{cases} 
1 & \text{in } B_{\delta/2}(x_0, y_0) \\
0 & \text{outside } B_{3\delta/4}(x_0, y_0) \quad (0 < \delta < 1);
\end{cases}
$$

if $(x_0, y_0) \in S$ then we choose $\delta$ such that $B_{\delta}(x_0, y_0) \subset S$, and if $x_0 \neq 0, y_0 = 0$, then we choose $\delta < |x_0|$. We now fix $(x_0, y_0) \in \overline{S}$ and a cutoff function $\xi$, and introduce the stream function

$$
\psi(x, y) = \int_{x_0}^{x} W_y(s, y_0) \, ds - \int_{y_0}^{y} W_x(x, t) \, dt.
$$

Set

$$
\tilde{\psi} = \psi \xi,
$$

$$
\tilde{W}_x = -\frac{\partial \tilde{\psi}}{\partial y}, \quad \tilde{W}_y = \frac{\partial \tilde{\psi}}{\partial x}, \quad \tilde{W} = (\tilde{W}_x, \tilde{W}_y).
$$

Then $\text{div} \tilde{W} = 0$.

Introducing

$$
\epsilon_{ij} = \begin{cases} 
+1 & \text{if } i < j \\
-1 & \text{if } i > j \\
0 & \text{if } i = j,
\end{cases}
$$

we can write

$$
(3.1) \quad \tilde{W}_j = W_j \xi + \epsilon_{ij} \psi \partial_i \xi.
$$
Clearly $\tilde{W}_j \in H^1(S)$. For any $\xi \in X$ we easily compute that

$$A(\xi, \tilde{W}) = \frac{1}{\lambda} b(\xi, \tilde{W}) + T_1 + T_2 + T_3 + T_4$$

where

$$T_1 = \frac{1}{2} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)(\partial_i W_j + \partial_j W_i) \xi = \frac{1}{2} \int_S [\partial_i(\xi \varphi_j) + \partial_j(\xi \varphi_i)](\partial_i W_j + \partial_j W_i)$$

$$- \frac{1}{2} \int_S [(\partial_i \xi) \varphi_j + (\partial_j \xi) \varphi_i](\partial_i W_j + \partial_j W_i),$$

$$T_2 = \frac{1}{2} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)(W_j \partial_i \xi + W_i \partial_j \xi) = \frac{1}{2} \int_S (\varphi_j n_i + \varphi_i n_j)(W_j \partial_i \xi + W_i \partial_j \xi)$$

$$- \frac{1}{2} \int_S [\varphi_j \partial_i(W_j \partial_i \xi + W_i \partial_j \xi) + \varphi_i \partial_j(W_j \partial_i \xi + W_i \partial_j \xi)],$$

$$T_3 = \frac{1}{2} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)[\epsilon_{ij} \partial_i \psi \partial_i \xi + \epsilon_{ij} \partial_j \partial_i \xi]$$

$$= \frac{1}{2} \int_S (\varphi_j n_i + \varphi_i n_j)[\epsilon_{ij} \partial_i \psi \partial_i \xi + \epsilon_{ij} \partial_j \partial_i \xi]$$

$$- \frac{1}{2} \int_S \{\varphi_j \partial_i[\epsilon_{ij} \partial_i \psi \partial_i \xi + \epsilon_{ij} \partial_j \partial_i \xi] + \varphi_i \partial_j[\epsilon_{ij} \partial_i \psi \partial_i \xi + \epsilon_{ij} \partial_j \partial_i \xi]\},$$

$$T_4 = \frac{1}{2} \int_S (\partial_i \varphi_j + \partial_j \varphi_i)[\epsilon_{ij} \partial_i \partial_j \xi + \epsilon_{ij} \partial_j \partial_i \xi] \psi$$

$$= \frac{1}{2} \int_S (\varphi_j n_i + \varphi_i n_j)[\epsilon_{ij} \partial_i \partial_j \xi + \epsilon_{ij} \partial_j \partial_i \xi] \psi$$

$$- \frac{1}{2} \int_S \varphi_j \partial_i[(\epsilon_{ij} \partial_i \partial_j \xi + \epsilon_{ij} \partial_j \partial_i \xi) \psi] + \varphi_i \partial_j[(\epsilon_{ij} \partial_i \partial_j \xi + \epsilon_{ij} \partial_j \partial_i \xi) \psi];$$

here $\vec{n} = (n_i)$ is the outward normal to $S$. Using (2.14) we can transform $T_1$ into

$$T_1 = -\frac{1}{\lambda} b(\xi, \tilde{W}) + \int_S f_j \varphi_j - \frac{1}{2} \int_S [(\partial_i \xi) \varphi_j + (\partial_j \xi) \varphi_i](\partial_i W_j + \partial_j W_i).$$

Observing also that

$$b(\xi, \tilde{W}) - b(\xi, \tilde{W}) = \int_{-\infty}^0 \varphi_2(x, 0)|2W_2(x, 0)\partial_x \xi(x, 0) + \psi(x, 0)\partial_x \xi(x, 0)|,$$
we conclude from (3.2) that

\[ A(\varphi, \vec{W}) = \int_{-\infty}^{0} \varphi_j(x, 0) M_j(\xi, \vec{W}, \psi)(x, 0)dx \]

\[(3.3)\]

\[ + \int_{S} L_j(\xi, \vec{W}, \psi, \vec{f}) \varphi_j dx dy \]

where \( M_j(\xi, \vec{W}, \psi) \) is linear in \( \vec{W}, \psi \) and \( L_j(\xi, \vec{W}, \psi, \vec{f}) \) is linear in \( \vec{W}, \psi, \vec{f} \) and \( \partial_i \vec{W} \).

Since \( \vec{W} \in W^{1,2}(S) \), its restriction to \( \{y = 0\} \) is in \( H^{1/2}(-\infty, \infty) \).

We shall use (3.3) to prove regularity of \( W \) in the interior. Taking \((x_0, y_0) \in S\), the boundary integral in (3.3) drops out, and we get

\[(3.4)\]

\[ a(\varphi, \vec{W}) = \int_{S} L_j(\xi, \vec{W}, \psi, \vec{f}) \varphi_j dx dy. \]

Taking \( \varphi \) such that \( \varphi_j = \delta_{j\ell} \) in \( B_\delta(x_0, y_0) \) we deduce that

\[(3.5)\]

\[ \int_{S} L_i(\xi, \vec{W}, \psi, \vec{f}) dx dy = 0. \]

We now introduce Green’s function in \( \mathbb{R}^2 \) for the Stokes equation [11; p. 60]:

\[(3.6)\]

\[ G_{ij} = -\delta_{ij} \log r + \frac{x_i x_j}{r^2}, \]

\[ p_i = 2 \frac{x_i}{r^2}. \]

Setting \( X = (x_1, x_2), Y = (y_1, y_2) \) we form the Stokes potentials

\[(3.7)\]

\[ Q_j(X) = \int_{\mathbb{R}^2} G_{ij}(X - Y)L_i(Y)dy, \]

where \( L_i = L_i(\xi, \vec{W}, \psi, \vec{f}) \).

One can easily verify that

\[ G_{ij}(X - Y) = -\delta_{ij} \log |X| + \frac{x_i x_j}{|X|^2} + O\left(\frac{1}{|X|}\right), \]

\[ \nabla G_{ij}(X - Y) = -\delta_{ij} \frac{X}{|X|^2} + \nabla \left(\frac{x_i x_j}{|X|^2}\right) + O\left(\frac{1}{|X|^2}\right), \]

\[ \nabla^2 G_{ij}(X - Y) = -\delta_{ij} \frac{1}{|X|^2} + \nabla^2 \left(\frac{x_i x_j}{|X|^2}\right) + O\left(\frac{1}{|X|^3}\right), \]
for $Y$ in a bounded set and $|X| \to \infty$. Recalling (3.5), it follows that

$$|Q_j(X)| = O\left(\frac{1}{|X|}\right), \quad |\nabla Q_j(X)| = O\left(\frac{1}{|X|^2}\right)$$

as $|X| \to \infty$. Also, by standard $L^2$ estimates,

$$\tilde{Q} \in W^{1,2} \text{ if } \tilde{L} \in L^2, \text{ supp } \tilde{L} \text{ bounded.} \tag{3.9}$$

If the $L_i$ are smooth with compact support then by integration by parts (or Green’s formula)

$$\frac{1}{2} \int (\partial_i \varphi_j + \partial_j \varphi_i)(\partial_i Q_j + \partial_j Q_i) = \int L_j \varphi_j \tag{3.10}$$

for any $\varphi \in W^{1,2}(\mathbb{R}^2)$ such that $\text{div}\varphi = 0$ and $\varphi$ satisfies the same estimates as in (3.8), as $|X| \to \infty$. (For then the boundary integrals disappear.) By approximation, this is true also for the $L_i = L_i(\xi, \tilde{W}, \psi, \tilde{f})$ (which have compact support and belong to $L^2$). In particular, (3.10) is valid for $\varphi = \tilde{W}_i - Q_i$, since $\tilde{W}_i$ has compact support.

Notice that (3.4) also holds for such $\varphi_i$ (since $\tilde{W}$ has compact support).

Subtracting (3.10) from (3.4) for $\varphi = \tilde{W}_i - Q_i$, we get,

$$a(\tilde{W} - \tilde{Q}, \tilde{W} - \tilde{Q}) = 0. \tag{3.11}$$

By Korn’s inequality we then have

$$\nabla(\tilde{W}_i - Q_i) \equiv 0, \quad \text{or } (\tilde{W}_i - Q_i)(X) = \text{const.}$$

Taking $|X| \to \infty$ we see that the constant is zero, i.e.,

$$\tilde{W}_i = Q_i.$$

Since by $L^2$ elliptic estimates [2] $Q_i \in W^{2,2}$, the same holds for $\tilde{W}_i$ or, locally, for $W_i$.

This allows us to repeat the previous argument with $L_i$ in $W^{1,2}$, and conclude that $W_i$ is locally in $W^{3,2}$.

By bootstrap argument we conclude:

**Theorem 3.1.** The weak solution $\tilde{W}$ belongs to $C^\infty(S)$.

**4. Regularity on** $\{y = 0, x > 0\}$.

In this section we prove regularity near a boundary point $(x_0, 0)$ with $x_0 > 0$; the same proof applies also to boundary points $(x_0, 1), -\infty < x_0 < \infty$. We follow the same
procedure as in Section 3, but introduce Green’s function for the half space \( \mathbb{R}^2_+ \equiv \{ x_2 > 0 \} \) (see [11; p. 93]):

\[
G(X, Y) = \Phi(X - Y) - \Phi(X - Y^*) + 2y_2^2 G^D(X - Y^*) - 2y_2 G^{SD}(X - Y^*)
\]

where

\[
X = (x_1, x_2), \quad Y = (y_1, y_2), \quad Y^* = (y_1, -y_2),
\]

and \( \Phi = (\Phi_{ij}) \) where

\[
\Phi_{ij}(X) = -\delta_{ij} \log |X| + \frac{x_i x_j}{|X|^2},
\]

\[
G^D_{ij}(X) = \pm \left( \frac{\delta_{ij}}{|X|^2} - 2 \frac{x_i x_j}{|X|^4} \right),
\]

\[
G^{SD}_{ij}(X) = x_2 G^D_{ij}(X) \pm \frac{\delta_{j2} x_i - \delta_{i2} x_j}{|X|^2};
\]

if \( j = 1 \) the sign is +, and if \( j = 2 \) the sign is −. The pressure is defined in a similar way but, as in Section 2, it will not be needed.

We introduce the potentials

\[
(4.1) \quad Q_j(X) = \int_{\mathbb{R}^2_+} G_{ij}(X - Y) L_j(Y) dY,
\]

where \( L_j(Y) = L_j(\xi, \bar{W}, \psi, \bar{f}) \).

If \( Y \) varies in a bounded set then

\[
|\Phi(X - Y) - \Phi(X - Y^*)| \leq C \frac{|\log X|}{|X|}
\]

and therefore

\[
(4.2) \quad |G(X, Y)| \leq C \frac{|\log X|}{|X|}
\]

as \( |X| \to \infty \). Similarly,

\[
(4.3) \quad |\nabla G(X, Y)| \leq C \frac{|\log X|}{|X|^2}
\]

if \( Y \) varies in a bounded set and \( |X| \to \infty \). Since the \( L_i(Y) \) have compact support, we conclude that

\[
(4.4) \quad |Q_j(X)| = O \left( \frac{|\log |X||}{|X|} \right), |\nabla Q_j(X)| = O \left( \frac{|\log |X||}{|X|^2} \right)
\]
as \(|X| \to \infty\). These estimates allows us to deduce, by integration by parts, that (3.10) is valid, where the integration is over \(\mathbb{R}_+^2\), for any \(\varphi\) in \(H^1(\mathbb{R}_+^2)\) such that \(\text{div}\varphi = 0\), \(\varphi = 0\) on \(\{y = 0\}\), and \(\varphi\) satisfies the same bounds at \(\infty\) as the \(Q_j\) in (4.4).

Combining this version of (3.10) with (3.4) and choosing \(\varphi = \tilde{W} - \tilde{Q}\) (this function satisfies the same estimates as \(\tilde{Q}\) in (4.4) since \(\tilde{W}\) has compact support), we obtain the relation (3.11). We can now proceed as in Section 3, making use of boundary \(L^2\) estimates for potentials of the form (4.1) [2]. We summarize:

**Theorem 4.1.** The weak solution \(\tilde{W}\) belongs to \(C^\infty\) in \(S \cup \{(x,0); x > 0\}\) and, similarly, in \(S \cup \{(x, 1), -\infty < x < \infty\}\).

§5. Regularity near \((0, 0)\).

In this section we study the behavior of \(\tilde{W}\) near \(\{(x,0), -\infty < x \leq 0\}\). We concentrate on the case \(x = 0\); the case \(x < 0\) is much simpler.

We shall use the same approach as in Sections 3, 4. The corresponding Stokes potentials will be constructed by using results from [4]. In that paper we worked with the stream function and, in particular, derived integral representation for a solution \(\psi\) of the system

\[
\begin{align*}
\Delta^2 \psi &= 0 \quad \text{in} \quad \{y > 0\}, \\
\psi_x(x,0) &= \psi_y(x,0) = 0 \quad \text{if} \quad x > 0, \\
(\psi_{yy} - \psi_{xx})(x,0) &= k_1(x) \quad \text{if} \quad x < 0, \\
\lambda(3\psi_{xxy} + \psi_{yy})(x,0) - \psi_{xxx}(x,0) &= k_2(x) \quad \text{if} \quad x < 0.
\end{align*}
\]

The representation is as follows [4; Th. 5.1]: \(\psi = \psi_1 + \psi_2\) so that

\[
\begin{align*}
D_x \psi &= D_x \psi_1 + D_x \psi_2, \quad D_y \psi = D_y \psi_1 + D_y \psi_2,
\end{align*}
\]

and

\[
\begin{align*}
D_x \psi_1(x, y) &= -\frac{3\lambda}{\pi}(1 - e^{-2\pi \rho i})^{-1} \int_{-\infty}^{0} d\xi \Phi(\xi)(-\xi)\rho \{[xG(x - iy, \xi) \\
&- xG(x + iy, \xi)] + [Q(x - iy, \xi) - Q(x + iy, \xi)]\}, \\
D_y \psi_1(x, y) &= -\frac{3\lambda}{\pi}(1 - e^{-2\pi \rho i})^{-1} \int_{-\infty}^{0} d\xi \Phi(\xi)(-\xi)\rho y[G(x - iy, \xi) - G(x + iy, \xi)]
\end{align*}
\]
where

\[ \Phi(q) = \frac{1}{1 + 9\lambda^2} (\lambda D^2 h_2(q) + k_2(q)), \]

\[ D_x h_2(q) = \frac{-q}{2\pi} \int_{-\infty}^{0} \frac{k_1(\xi)}{\sqrt{-\xi} (q - \xi)} d\xi, \]

\[ G(x - iy, \xi) - G(x + iy, \xi) = \int_{x + iy}^{x - iy} \frac{(-q)^{-p}}{q - \xi} dq, \]

\[ Q(x - iy, \xi) - Q(x + iy, \xi) = -\int_{x + iy}^{x - iy} \frac{(-q)^p q}{q - \xi} dq, \]

the \( q \)-integrals are along the arc of the circle with center 0 connecting (clockwise) \( x + iy \) to \( x - iy \), and

\[ D_x \psi_2(x, y) = \int_{-\infty}^{0} d\xi \frac{k_1(\xi)}{\sqrt{-\xi}} [Z'(x - iy, \xi) - Z'(x + iy, \xi)], \]

\[ D_y \psi_2(x, y) = \int_{-\infty}^{0} d\xi \frac{k_1(\xi)}{\sqrt{-\xi}} \{[Z(x - iy, \xi) - Z(x + iy, \xi)] \]

\[ - i y [Z'(x + iy, \xi) + Z'(x - iy, \xi)] \}

where

\[ Z(x - iy, \xi) - Z(x + iy, \xi) = \frac{1}{4\pi} \int_{x + iy}^{x - iy} \frac{dq}{\sqrt{-q} q - \xi}, \]

\[ Z'(\gamma, \xi) = \frac{1}{4\pi} \frac{\sqrt{-\gamma}}{\gamma - \xi}. \]

We want to construct a special solution \( \bar{\Omega} = (\Omega_j) \) to (3.3); more precisely, a solution such that

\[ \frac{1}{2} \int_{\mathbb{R}^2} (\partial_i \varphi_j + \partial_j \varphi_i)(\partial_i \Omega_j + \partial_j \Omega_i) dx dy + \frac{1}{\lambda} b(\varphi, \bar{\Omega}) \]

\[ = \int_{-\infty}^{0} \varphi_j(x, 0) M_j(\xi, \bar{W}, \psi)(x, 0) dx + \int_{\mathbb{R}^2} L_j(\xi, \bar{W}, \psi, \bar{f}) \varphi_j dx dy. \]
for any $\varphi$ in $H^1(\mathbb{R}_+^2)$ such that $\text{div} \varphi = 0$, $\varphi(x,0) = 0$ if $x > 0$, and $\varphi$ and $\bar{\Omega}$ decay "appropriately" at $\infty$ ($\varphi$ need not have compact support).

Observe that (5.10) is a weak form of the system

(5.11) $-\Delta \bar{\Omega} + \nabla p = \tilde{L}$, $\nabla \cdot \bar{\Omega} = 0$ in $\mathbb{R}_+^2$,
(5.12) $\Omega_x(x,0) = \Omega_y(x,0) = 0$ if $x > 0$,
(5.13) $\frac{\partial \Omega_x}{\partial y} + \frac{\partial \Omega_y}{\partial x} = M_1$ at $y = 0, x < 0$,
(5.14) $-p + 2\frac{\partial \Omega_y}{\partial y} - \frac{1}{\lambda} \frac{\partial \Omega_y}{\partial x} = M_2$ at $y = 0, x < 0$.

As in [4] we introduce the stream function $\tilde{\varphi}$,

(5.15) $\bar{\Omega} = (\frac{\partial \tilde{\varphi}}{\partial y}, \frac{\partial \tilde{\varphi}}{\partial x})$

and then arrive at the following problem:

(5.16) $\Delta^2 \tilde{\varphi} = \frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial x}$ in $\mathbb{R}_+^2$,
(5.17) $\tilde{\varphi}_x(x,0) = \tilde{\varphi}_y(x,0) = 0$ if $x > 0$,
(5.18) $(\tilde{\varphi}_{xx} - \tilde{\varphi}_{yy})(x,0) = M_1(x)$ if $x < 0$,

and

(5.19) $\lambda(3\tilde{\varphi}_{xxy} + \tilde{\varphi}_{yy}) - \tilde{\varphi}_{xxx} = \lambda(L_1 + \frac{\partial M_2}{\partial x})$ if $y = 0, x < 0$;

The last equation is obtained by differentiating (5.14) with respect to $x$ and eliminating $\partial p/\partial x$ by using (5.11).

We shall construct $\tilde{\varphi}$ in several steps. We begin with the special solution of (5.16), defined as

(5.20) $g_1(\xi) = K \int_{\mathbb{R}_+^2} |\xi - q|^2 \log |\xi - q| \left( \frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial x} \right)(q) dq$, $K > 0$.

Recalling that $\partial L_1/\partial y - \partial L_2/\partial x$ is compactly supported, we can argue as in [4; §6] to deduce that

$$g_1(\xi) = C_1 |\xi|^2 \log |\xi| + C_2 |\xi|^2 + C_3 |\xi| \log |\xi| + C_4 |\xi| + O\left( \frac{\log |\xi|}{|\xi|} \right) \quad \text{as} \ |\xi| \to \infty$$
where the $C_i$ are constants bounded (together with the “$O$” term) by the $H^{-1}$-norm of
\( \partial L_1/\partial y - \partial L_2/\partial x \); hence they are bounded by the $L^2$ norm of the $L_i$. Actually, the proof
here is simpler than in [4;§6], where we derived estimates on the $(4 + \alpha)$-norm.

We next subtract from $g_1(\xi)$ the pieces
\[
C_1|\xi|^2 \log |\xi|, \quad C_2|\xi|^2, \quad C_3|\xi| \log |\xi|, \quad C_4|\xi|
\]
by subtracting special solutions (see [4; formula (6.42)]). This provides us with a solution
\[
\overline{g}_1(\xi) = g_1(\xi) - K_1|\xi - \xi_0|^2 \log |\xi - \xi_0| - K_2|\xi - \xi_0|^2 - K_3 \cdot (\xi - \xi_0) \log |\xi - \xi_0|
- K_4 \cdot (\xi - \xi_0) - K_5 \log |\xi - \xi_0| \quad (\xi_0 = (0, -1))
\]
of (5.16) which satisfies
\[
|\overline{g}_1(\xi)| = O(\log |\xi|/|\xi|), \quad |\nabla \overline{g}_1(\xi)| = O \left( \frac{\log |\xi|}{|\xi|^2} \right) \quad \text{if } |\xi| \to \infty,
\]
and has the same regularity as the function $g_1(\xi)$; here $K_3, K_4$ are vectors and the other
$K_j$ are scalars.

The next step is to construct a solution $g_2$ to
\[
\Delta^2 g_2 = 0 \quad \text{in } \mathbb{R}^2_+
\]
with
\[
\frac{\partial g_2}{\partial x} = \frac{\partial \overline{g}_1}{\partial x}, \quad \frac{\partial g_2}{\partial y} = \frac{\partial \overline{g}_1}{\partial y} \quad \text{at } y = 0, x > 0.
\]
By [4; §4] such a solution can be given by
\[
\frac{\partial g_2}{\partial x} = 2 \int_{-\infty}^{\infty} \frac{y^3}{((x - s)^2 + y^2)^2} \frac{\partial \overline{g}_1(s, 0)}{\partial x} ds - 2 \int_{-\infty}^{\infty} \frac{(x - s)y^2}{((x - s)^2 + y^2)^2} \frac{\partial \overline{g}_1(s, 0)}{\partial y} ds,
\]
\[
\frac{\partial g_2}{\partial y} = -2 \int_{-\infty}^{\infty} \frac{(x - s)y^2}{((x - s)^2 + y^2)^2} \frac{\partial \overline{g}_1(s, 0)}{\partial x} ds + 2 \int_{-\infty}^{\infty} \frac{(x - s)^2 y}{((x - s)^2 + y^2)^2} \frac{\partial \overline{g}_1(s, 0)}{\partial y} ds.
\]
To estimate $\partial g_2/\partial x$ we write
\[
\frac{\partial g_2}{\partial x} = 2 \int_{-\infty}^{\infty} \left[ \frac{y^3}{((x - s)^2 + y^2)^2} - \frac{y^3}{(x^2 + y^2)^2} \right] \frac{\partial \overline{g}_1(s, 0)}{\partial s} ds
\]
\[
-2 \int_{-\infty}^{\infty} \left[ \frac{(x - s)y^2}{((x - s)^2 + y^2)^2} - \frac{x y^2}{(x^2 + y^2)^2} \right] \frac{\partial \overline{g}_1(s, 0)}{\partial y} ds
\]
\[
-2 \frac{xy^2}{(x^2 + y^2)^2} \int_{-\infty}^{\infty} \frac{\partial \overline{g}_1(s, 0)}{\partial y} ds = I_1 + I_2 + I_3
\]
(note that $\int_{-\infty}^{\infty} (\partial \tilde{g}_1(s,0)/\partial s) ds = 0$). By (5.21),

$$
|I_1| \leq C \frac{y^3}{(x^2 + y^2)^2} \int_{-\infty}^{\infty} \frac{(x^2 + y^2)^2}{((x-s)^2 + y^2)^2} - 1 \frac{\log(2 + |s|)}{(1 + |s|)^2} ds.
$$

To estimate $I_1$ as $|X| \to \infty$, note that if $0 < x < y^\theta$ for some $0 < \theta < 1$, $\theta$ near 1, then the integral on the right-hand side of (5.25) is bounded by

$$
C \int_{0}^{2y^\theta} \frac{1}{y^{2(1-\theta)}} \frac{\log(2 + s)}{(1 + s)^2} ds + C \int_{2y^\theta}^{\infty} \frac{\log(2 + s)}{(1 + s)^2} ds,
$$

so that

$$
|I_1| \leq \frac{C}{|X|^{1+\delta}} \text{ for some } \delta > 0.
$$

The same estimate is obtained for any $x, y$ if we integrate only over $|s| < x^\theta$. Thus it remains to estimate

$$
\tilde{I}_1 = \int_{x^\theta}^{\infty} \frac{y^3}{((x-s)^2 + y^2)^2} \frac{\log(2 + s)}{(1 + s)^2} ds + \int_{x^\theta}^{\infty} \frac{y^3}{(x^2 + y^2)^2} \frac{\log(2 + s)}{(1 + s)^2} ds.
$$

$$
\equiv I_{11} + I_{12}, \text{ for } x > y^\theta.
$$

Clearly $I_{12}$ is bounded by the right-hand side of (5.26). To estimate $I_{11}$ we write

$$
I_{11} \leq \frac{\log(2 + x^\theta)}{(1 + x^\theta)^2} \int_{x^\theta}^{\infty} \frac{y^3}{((x-s)^2 + y^2)^2} ds
$$

and note that the integral is bounded independently of $(x, y)$. Taking $\theta$ near 1 we conclude that (5.26) is valid in all cases.

A similar estimate hold for $I_2$. Hence

$$
\frac{\partial g_2}{\partial x} = -\frac{2}{\pi} \frac{xy^2}{(x^2 + y^2)^2} \int_{-\infty}^{\infty} \frac{\partial \tilde{g}_1(s,0)}{\partial y} ds + O \left( \frac{1}{|X|^{1+\delta}} \right).
$$

Similarly

$$
\frac{\partial g_2}{\partial y} = \frac{2}{\pi} \frac{x^2 y}{(x^2 + y^2)^2} \int_{-\infty}^{\infty} \frac{\partial \tilde{g}_1(s,0)}{\partial y} ds + O \left( \frac{1}{|X|^{1+\delta}} \right).
$$
Introducing the functions

\[(5.29) \quad \hat{h}_2 = \frac{1}{2\pi} \frac{x^2 - y^2}{x^2 + y^2} \int_{-\infty}^{0} \frac{\partial \bar{g}_1(s, 0)}{\partial y} ds\]

and

\[(5.30) \quad \bar{g}_2 = \bar{g}_1 - g_2 - \hat{h}_2\]

we then have:

\[(5.31) \quad \Delta \bar{g}_2 = \frac{\partial L_1}{\partial y} - \frac{\partial L_2}{\partial x} \quad \text{in } \mathbb{R}_+^2,\]

\[(5.32) \quad \partial_x \bar{g}_2(x, 0) = \partial_y \bar{g}_2(x, 0) = 0, \quad -\infty < x < \infty,\]

and

\[(5.33) \quad D^k \bar{g}_2 = O \left( \frac{1}{r^{k+\delta}} \right) \quad \text{as } r = |X| \to \infty\]

for \(k = 1\); one can similarly establish the estimate (5.33) for \(k = 2, 3, \ldots\).

The final step is to construct functions \(g_3, h_3\) satisfying:

\[(5.34) \quad \Delta^2 g_3 = 0, \quad \Delta^2 h_3 = 0 \quad \text{in } \mathbb{R}_+^2,\]

\[(5.35) \quad \partial_x g_3 = \partial_y g_3 = 0, \quad \partial_x h_3 = \partial_y h_3 = 0 \quad \text{at } y = 0, x > 0,\]

and

\[(5.36) \quad \partial_x^2 g_3 - \partial_y^2 g_3 = -\partial_x^2 \bar{g}_2 + \partial_x^2 \bar{g}_2 \equiv A_1,\]

\[\lambda(3\partial_x^2 \partial_y + \partial_y^3)g_2 - \partial_x^2 g_2 = -\lambda(3\partial_x^2 \partial_y + \partial_y^3)\bar{g}_2 + \partial_x^2 \bar{g}_2 \equiv A_2 \quad \text{at } y = 0, x < 0,\]

\[\lambda(3\partial_x^2 \partial_y + \partial_y^3)h_3 - \partial_x^3 h_3 = \lambda(L_1 + \partial_x M_2) \quad \text{at } y = 0, x < 0.\]

We wish to write \(g_3 = \psi_1 + \psi_2\) and use the representation (5.6)-(5.9) with \(k_i = A_i\). By (5.33),

\[(5.38) \quad |A_1| \leq \frac{C}{r^{2+\delta}}, \quad |A_2| \leq \frac{C}{r^{3+\delta}} \quad \text{for } r \text{ large.}\]

Hence

\[|\Phi(x)| = \left| \frac{1}{1 + 9\lambda^2} A_2 \right| \leq \frac{C}{|x|^{3+\delta}}.\]
Similarly one shows that

\begin{equation}
|D^k \Phi(x)| \leq \frac{C_k}{|x|^{3+k+\delta}} \quad \text{if } |x| > 1, \quad k = 0, 1, 2.
\end{equation}

We can now proceed as in [4; §8] and obtain (cf. [4; (8.20) and (8.21)])

\[ g_3 = \sum r^{1-\rho} Q_j + \sum r^{1/2} \mu_j + \tilde{g}_3 \]

where

\begin{equation}
|D^k \tilde{g}_3(z)| \leq \frac{C}{|z|^p+k} \quad \text{if } |z| \to \infty \quad (k = 0, 1, 2);
\end{equation}

here we used the notation \( z = x + iy \) (instead of \( X = (x, y) \)) as in [4; §8], and \( Q_j, \mu_j \) are polynomials in \( z \) of degree \( j \) such that \( r^{1-\rho} Q_j, r^{1/2} \mu_j \) form solutions of the homogeneous system (5.1) - (5.4) (i.e., with \( k_1 \equiv k_2 \equiv 0 \)).

Similarly

\[ h_3 = \sum r^{-\rho} \tilde{Q}_j + \sum r^{1/2} \tilde{\mu}_j + \hat{h}_3 \]

where \( r^{-\rho} \tilde{Q}_j, r^{1/2} \tilde{\mu}_j \) are solutions of the homogeneous system (5.1) - (5.4) and \( \hat{h}_3 \) satisfies (5.40).

The function

\begin{equation}
\tilde{\varphi} = \bar{g}_2 + \tilde{g}_3 + \hat{h}_3
\end{equation}

is then a solution of (5.16) - (5.19) and

\begin{equation}
|D^k \tilde{\varphi}| \leq \frac{C}{|z|^{k+\delta}} \quad \text{if } |z| \to \infty \quad (k = 0, 1, 2).
\end{equation}

We shall write the boundary conditions (5.18), (5.19) in the form

\begin{align}
\tilde{B}_1(D_x, D_y)\tilde{\varphi} &= M_1 \quad \text{if } y = 0, x < 0, \\
\tilde{B}_2(D_x, D_y)\tilde{\varphi} &= \lambda(L_1 + \partial_x M_2) \quad \text{if } y = 0, x < 0.
\end{align}

From the structure of \( M_j, L_j \) (see the sentence containing (3.3)),

\[ M_j \in H^{1/2}(-\infty, \infty), \quad L_j \in L^2(\mathbb{R}_+^2). \]

It will be proved in Appendix (§9) that

\begin{equation}
\tilde{\varphi} \in H^2(\mathbb{R}_+^2).
\end{equation}

(It will actually be proved that \( \tilde{\varphi} \in H^{\frac{3}{2}-\delta}(\mathbb{R}_+^2) \) for any \( \delta > 0 \).) From (5.15) we then deduce that

\[ \int_{\mathbb{R}_+^2} |D^k \tilde{\varphi}|^2 < \infty \quad (k = 0, 1). \]

Using this estimate we can show, as in Section 4, that (3.11) is valid with \( \tilde{Q} = \tilde{\varphi} \) so that \( \tilde{W} = \tilde{\varphi} \) in \( \mathbb{R}_+^2 \)-neighborhood of the origin. (The proof is based on verifying ,by using (5.42), that (3.10) and (5.10) hold for \( \tilde{\varphi} = \tilde{W} - \tilde{\Omega} \).) We summarize:
Theorem 5.1. There exists an $\mathbb{R}^2_+$-neighborhood $D$ of $(0,0)$ such that

\[ \widetilde{W} = \widetilde{\Omega} \quad \text{in} \ D, \]

where $\widetilde{\Omega}$ is defined by (5.15).

It remains to study the regularity of $\widetilde{\varphi}$ in $D$. By (5.15), (5.41), we need only study the behavior of $\widetilde{g}_2, \widetilde{g}_3$ and $\widetilde{h}_3$ in $D$. Comparing (2.8)-(2.13) with (5.11) - (5.14) and recalling (5.46), we find that

\[ \widetilde{L} = \tilde{f} \ast f \quad \text{and} \quad M_1 = M_2 = 0 \]

in (5.11), (5.13), (5.14). It follows that

\[ \Delta^2 \widetilde{\varphi} = \tilde{f} \quad \text{in} \ \mathbb{R}^2_+, \]

\[ \tilde{B}_1(D_x, D_y)\widetilde{\varphi} = k_1, \quad \tilde{B}_2(D_x, D_y)\widetilde{\varphi} = k_2 \quad \text{if} \ y = 0, x < 0 \]

where $\tilde{f}$ and $k_i$ are $C^\infty$ functions with compact support; $\tilde{f} = 0$ in $D$ and $k_i(x) = 0$ if $|x|$ is small.

A review of the structure of $\widetilde{g}_2$ then shows that

\[ \widetilde{g}_2 \text{ is analytic in a neighborhood of } (0,0). \]

We next study the function $g_3$ using the representation (5.5) - (5.9). In view of (5.50) and (5.36),

\[ \text{the functions } k_1 \equiv A_1, \ k_2 \equiv A_2 \quad \text{(and then also the corresponding } \Phi) \]

are analytic in $x$, for $x$ near 0.

We begin with the function

\[ \int_{-\infty}^{0} d\xi \Phi(\xi)(-\xi)^{\rho} M_1(z, \xi) \]

where

\[ M_1(z, \xi) = K \frac{z + \bar{z}}{2} \int_{\xi}^{z} \frac{(-q)^{-\rho}}{q - \xi} dq, \]

which comes from (5.6).
Introduce a cutoff function

\[ \eta(\xi) = [\tanh(-\xi)]^A \quad (\xi < 0) \]

where \( A \) is a large positive integer. Note that

- \( \eta(\xi) \sim (-\xi)^A \) if \( \xi \to 0 \),
- \( \eta(\xi) \sim 1 - 2Ae^\xi \) if \( \xi \to -\infty \).

One can easily check that

\[ |\Phi(\xi)(-\xi)\eta(\xi)| \leq C|\xi|^{A-1}. \]

We can then use the splitting argument, as in [4; §7], to get

\[ \int_{-\infty}^{0} d\xi \Phi(\xi)(-\xi)\eta(\xi)M_1(z, \xi) - \sum_{j=0}^{A-1} \nu_j(z) = O(|z|^{A-\rho}) \tag{5.54} \]

for \( |z| < 1 \), where \( \nu_j(z) = |z|^{-\rho}\tilde{\nu}_j(z) \) and \( \tilde{\nu}_j(z) \) is homogeneous polynomial of degree \( j \) in \((x, y)\); the factor \( |z|^{-\rho} \) arises from the factor \((-q)^{-\rho}\) in \( M_1(z, \xi) \).

For any positive integer \( M \),

\[ |\Phi(\xi) - P_M(\xi)| \leq C|\xi|^{M+1} \]

where \( P_M \) is a polynomial of degree \( M \). Set

\[ \tilde{\Phi} = \Phi - P_M(z). \]

Proceeding as in the derivation of (5.54) we find that

\[ \int_{-\infty}^{0} d\xi \tilde{\Phi}(\xi)(-\xi)^{\rho}(1 - \eta(\xi))M_1(z, \xi) - |z|^{-\rho}\tilde{P}_M(z) = O(|z|^{M+\delta}) \tag{5.55} \]

for some \( \delta > 0 \), where \( \tilde{P}_M \) is a polynomial of degree \( M \). Finally it remains to evaluate

\[ \int_{-\infty}^{0} d\xi P_M(\xi)(-\xi)^{\rho}(1 - \eta(\xi))M_1(z, \xi) \tag{5.56} \]

for \( |z| \) small. A typical term in \( P_M \) contributes

\[ \int_{-\infty}^{0} d\xi(q(-q)^{-\rho}) \left[ \int_{-\infty}^{0} (-\xi)^{\rho+k}(1 - \eta(\xi)) \frac{d\xi}{q - \xi} \right] \frac{K(z + \bar{z})}{2} \quad (K \text{ constant}). \tag{5.57} \]
We use the formula

\[(5.58) \quad \frac{1}{x-a} = -\frac{1}{a} \sum_{n=0}^{m} \left( \frac{x}{a} \right)^n + \left( \frac{x}{a} \right)^{m+1} \frac{1}{x-a} \]

to rewrite the expression in the brackets in the form

\[(5.59) \quad \int_{-\infty}^{0} (-\xi)^{\rho+k}(1-\eta(\xi)) \left[ -\frac{1}{\xi} \sum_{n=0}^{B} \left( \frac{q}{\xi} \right)^B + \left( \frac{q}{\xi} \right)^{B+1} \frac{1}{q-\xi} \right] d\xi \]

where \(B\) is chosen so that

\[(5.60) \quad B < \rho + k < B + 1.\]

Each term in \(\sum_n\) contributes a homogeneous polynomial times \(r^{-\rho}\), after substituting into (5.57). It remains to consider the last term:

\[q^{B+1} \int_{-\infty}^{0} (-\xi)^{\rho+k}(1-\eta(\xi)) \frac{1}{\xi^{B+1}} \frac{d\xi}{q-\xi} \]

\[= q^{B+1} \left\{ \int_{-\infty}^{0} (-\xi)^{\rho+k} \frac{1}{\xi^{B+1}} \frac{d\xi}{q-\xi} - q^{B+1} \int_{-\infty}^{0} (-\xi)^{\rho+k} \frac{\eta(\xi)}{\xi^{B+1}} \frac{d\xi}{q-\xi} \right\} \]

\[= q^{B+1} (I_1 - I_2).\]

Suppose \(Imq > 0\). Substituting in \(I_1 \quad q = r\omega, \xi = ru\) we get

\[I_1 = r^{\rho+k+1} \omega^{B+1} \int_{-\infty}^{0} (-u)^{\rho+k} \frac{du}{u^{B+1}(\omega-u)};\]

recall that \(|q| = |z| = r\) so that \(|\omega| = 1\). We can deform the \(u\)-contour into \(\Gamma\) as shown in Figure 2.

![Figure 2](image)
Then $|u - \omega|$ remains uniformly positive and by (5.60) the last integral is uniformly convergent. To determine $q^{B+1} I_1$ more sharply, we substitute in the last integral $u = \lambda \omega$. We get

$$q^{B+1} I_1 = q^{\rho+k+1+B+1} \int_{\Gamma_q} \frac{(-1)^{\rho+k}}{\lambda^{B+1}(1 - \lambda)} d\lambda$$

where $\Gamma_q$ is obtained from $\Gamma$ by multiplying by $1/\omega$. Note that $\Gamma_q$ goes from $0$ to $\infty$ above the $x$-axis and it avoids the point $1$ as long as $Imq > 0$. Hence we can modify $\Gamma_q$ so as to obtain a contour independent of $q$. Consequently

(5.61)

$$I_1 = cq^{\rho+k+1+B+1}, \quad c \text{ constant.}$$

To evaluate $I_2$ we again use (5.58):

$$I_2 = \int_{-\infty}^{0} (-\xi)^{\rho+k} \frac{\eta(\xi)}{\xi^{B+1}} \left[ \frac{1}{\xi} \sum_{j=0}^{L} \left( \frac{q}{\xi} \right)^{j} - \left( \frac{q}{\xi} \right)^{L+1} \frac{1}{q - \xi} \right] d\xi$$

where

$$L < A + \rho + k - (B + 1) < L + 1.$$ 

The terms in $\sum_j$ give homogeneous polynomials of degree $j$. In the last term we can deform the contour as before (recall that $\eta$ is holomorphic); this term is then bounded by $O(q^{L+1})$ and it is an error term if $L$ is taken large enough.

The above analysis applied also to $Imq < 0$ (with corresponding deformation of the contour). Substituting the results for $Imq > 0$ and $Imq < 0$ into (5.57) we obtain, up to an error term of large order,

$$\sum_j \mu_j(x,y)$$

where $\mu_j$ are homogeneous polynomials in $(x, y)$ of degree $j$.

Combining this with (5.54), (5.55), we conclude that the function (5.52) is equal to a sum

(5.62)

$$r^{-\rho} \sum_{j=0}^{N_1} Q_j(x,y) + \sum_{j=0}^{N_0} \mu_j(x,y)$$

plus error term $O(r^{N_1+\rho+\delta} + r^{N_0+\delta})$, $\delta > 0$, for any positive integers $N_1, N_0$, where $Q_j, \mu_j$ are polynomials of degree $j$.

The same analysis applies when $M_1$ is replaced by

$$K \frac{z + \bar{z}}{z} \int_{\bar{z}} q(-q)^{\rho} \frac{q}{q - \xi} dq,$$

24
and this completes the evaluation of $D_x \psi_1$.

The same procedure can be used to evaluate the derivatives of $D_x \psi_1$. This was carried out (in slightly different context) for the $C^{4+\gamma}$-norm in [4]. However, it actually applies to any norm $C^{k+\gamma}$. Thus, in particular,

$$D^k[D_x \psi_1 - r^{-\rho} \sum_{j=0}^{N_1} Q_j - \sum_{j=0}^{N_0} \mu_j] \leq C r^{-N_1-\rho-k+\gamma} + C r^{-N_0-k+\gamma}$$

for $k = 0, 1, \ldots, \bar{N}$, for any $\bar{N} < \min(N_0, N_1 - \rho), r < 1$.

A similar result can be established for $D_x \psi_2$. We can also obtain similar results for $D_x \psi_2, D_x \psi_2$; $r^{-\rho}$ is to be replaced by $r^{-1/2}$. We conclude that

$$D^k[g_3 - r^{-\rho} \sum_{j=0}^{N_1} Q_j - r^{-1/2} \sum_{j=0}^{N_2} \tilde{Q}_j - \sum_{j=0}^{N_0} P_j] \leq C(r^{-N_1-\rho-k+\gamma} + r^{-N_2-1/2-k+\gamma} + r^{-N_0-k+\gamma})$$

for $0 \leq k < \bar{N}$, where $Q_j, \tilde{Q}_j, P_j$ are polynomials of degree $j$, and $\bar{N} < \min(N_0, N_1 - \rho, N_2 - \frac{1}{2})$.

The analysis for $h_3$ is much simpler since the data on $y = 0, x < 0$ vanish for $|x|$ small. This leads to

$$D^k[h_3 - \sum_{j=0}^{N_0} \tilde{P}_j] \leq C r^{-N_0+1},$$

instead of (6.4).

From the representation of $\tilde{\varphi}$ in (5.41), and from (5.50) and the estimates (5.50), (6.4), (6.5) we conclude that $\tilde{\varphi}$ satisfies the same estimates as $g_3$ in (6.4):

$$D^k[\tilde{\varphi} - r^{-\rho} \sum_{j=0}^{N_1} Q_j - r^{-1/2} \sum_{j=0}^{N_2} \tilde{Q}_j - \sum_{j=0}^{N_0} P_j] \leq C(r^{-N_1-\rho-k+\gamma} + r^{-N_2-1/2-k+\gamma} + r^{-N_0-k+\gamma})$$

with different polynomials $Q_i, \tilde{Q}_j, P_j$.

By (5.45) the terms $r^{-\rho} Q_j \ (j = 0, 1, 2)$ and $r^{-1/2} \tilde{Q}_j \ (j = 0, 1)$ must vanish. Moreover, the blow-up argument as used, for instance, in the proof of Lemma 6.4 in [4], shows, that each of the terms

$$r^{-\rho} Q_j, r^{-1/2} \tilde{Q}_j, P_j,$$

must be a solution of the homogeneous problem (i.e., (5.1)-(5.4) with $k_1 \equiv k_2 \equiv 0$). In view of [4; Th. 2.2] the $P_j$ must then vanish.

We have proved:
**Theorem 5.2.** The weak solution $\tilde{W}$, in a small $\mathbb{R}^{2+}$ neighborhood of $(0,0)$, has the form $\tilde{W} = (-\partial_y \tilde{\varphi}, \partial_x \tilde{\varphi})$ where

\begin{equation}
\tilde{\varphi} = 2Re \sum_{j=1}^{2} \left[ \Psi_j(z) + \Phi_j(z) \frac{\bar{z}}{z} \right]
\end{equation}

where

\begin{equation}
\Phi_1(z) = z^{1/2} \sum_{n=2}^{\infty} \Gamma_n z^n, \quad \Phi_2(z) = iz^{-\rho} \sum_{n=2}^{\infty} \Gamma_n z^n \quad (\Gamma_n \in \mathbb{R})
\end{equation}

and

\begin{equation}
\Psi_1(z) = \Phi_1(z), \quad \frac{\partial}{\partial z} \Psi_2(z) = 2\frac{\Phi_2(z)}{z} - \frac{\partial}{\partial z} \Phi_2(z).
\end{equation}

The series expansion in (5.67) should be understood in the sense of asymptotic series, as described in (5.66). The expressions for $\Psi_j, \Phi_j$ come from the form of the solutions $r^{-\rho}Q_j, r^{-1/2}Q_j$ as established in [4; §2].

The free boundary $y = f(x)$ corresponding to the solution $\tilde{W}$ has the following form near $x = 0$ (cf. [4; §2]):

\begin{equation}
f(x) = \frac{2}{U} Re(\Phi_1 + \Phi_2 + \Psi_1 + \Psi_2)
\end{equation}

\begin{equation}
= \frac{2}{U} Re \sum_{n=2}^{\infty} 2i\Gamma_n \frac{z^{n-\rho}}{n - \rho}.
\end{equation}

**§6. The perturbed problem.**

We now consider the complete coating flow problem as a perturbation of the linearized problem studied in the previous sections. We seek

\begin{equation}
\text{velocity} = \vec{v} = \vec{U} + \varepsilon \vec{G} \\
\quad = \vec{G}(x, y, \varepsilon) \quad (\vec{G} = G(x, y, \varepsilon)),
\end{equation}

\begin{equation}
\text{pressure} = p = \varepsilon \tilde{p} \quad (\tilde{p} = \tilde{p}(x, y, \varepsilon)),
\end{equation}

and free boundary

\begin{equation}
\Gamma_\varepsilon : y = \varepsilon \tilde{f}(x, \varepsilon) \quad \text{for} \quad x < 0, \quad \tilde{f}(0, \varepsilon) = 0
\end{equation}

with $\tilde{f}(x, \varepsilon)$ uniformly bounded in $(x, \varepsilon)$ ($\varepsilon$ small), such that in the fluid region

\begin{equation}
\Omega_\varepsilon = \{(x, y); -\infty < x < \infty, \quad 0 < y < 1 \quad \text{if} \quad x > 0 \quad \text{and} \quad \varepsilon \tilde{f}(x, \varepsilon) < y < 1 \quad \text{if} \quad x \leq 0\}
\end{equation}
the following equations hold:

\begin{align}
\Delta \tilde{v} &= \nabla p \quad \text{in} \quad \Omega_\varepsilon, \\
\nabla \cdot \tilde{v} &= 0 \quad \text{in} \quad \Omega_\varepsilon; \\
\end{align}

furthermore, on the fixed boundaries:

\begin{align}
\tilde{v}(x, 1) &= \tilde{U} + \varepsilon \zeta(x), \quad -\infty < x < \infty. \\
\tilde{v}(x, 0) &= \tilde{U}, \quad 0 < x < \infty, \\
\end{align}

and on the free boundary:

\begin{align}
T \tilde{n} &= \sigma \kappa \tilde{n} \quad \text{on} \quad \Gamma_\varepsilon \\
\nabla \cdot \tilde{n} &= 0 \quad \text{on} \quad \Gamma_\varepsilon; \\
\end{align}

\(T, \tilde{n}, \sigma\) and \(\kappa\) were defined in §0. Recall that \(\tilde{U} = (U, 0), \ U > 0\).

We refer to the system (6.1)–(6.9) as Problem (C).

From the calculations in §1 we have:

\begin{align}
\Delta \tilde{G} &= \nabla \tilde{p} \quad \text{in} \quad \Omega_\varepsilon, \\
\nabla \cdot \tilde{G} &= 0 \quad \text{in} \quad \Omega_\varepsilon, \\
\tilde{G}(x, 1) &= \tilde{\zeta}(x), \quad -\infty < x < \infty, \\
\tilde{G}(x, 0) &= 0 \quad 0 < x < \infty. \\
\end{align}

Next, (6.9) reduces to

\begin{align}
G_y(x, \varepsilon \tilde{f}) = U \tilde{f} + \varepsilon \tilde{f}^\prime G_x(x, \varepsilon \tilde{f}). \\
\end{align}

Finally, taking the scalar product of (6.8) with the tangent \(\tilde{\tau}\) and normal \(\tilde{n}\), respectively, we arrive after simple calculation, at the following equations:

\begin{align}
\frac{\partial G_y}{\partial x} + \frac{\partial G_y}{\partial x} = \frac{4 \varepsilon \tilde{f}^\prime}{1 - \varepsilon^2 (\tilde{f}^\prime)^2} \frac{\partial G_x}{\partial x} \quad \text{at} \quad (x, \varepsilon \tilde{f}(x, \varepsilon)), \quad x < 0, \\
\end{align}
\begin{align*}
(6.16) \quad -\tilde{p} + \frac{1}{1 + \varepsilon^2(\tilde{f})^2} \left[ 2 \frac{\partial G_y}{\partial y} + 2\varepsilon^2(\tilde{f})^2 \frac{\partial G_x}{\partial x} - 2\varepsilon^2 \tilde{f}' \left( \frac{\partial G_x}{\partial y} + \frac{\partial G_y}{\partial x} \right) \right] \\
= -\frac{\sigma \tilde{f}''}{[1 + \varepsilon^2(\tilde{f})^2]^{3/2}} \quad \text{at} \quad (x, \varepsilon \tilde{f}(x, \varepsilon)), \quad x < 0.
\end{align*}

For simplicity we assume that

\begin{align*}
(6.17) \quad \zeta(x) \text{ is } C^\infty \text{ with compact support;}
\end{align*}

the case of general \( \zeta \) will be considered in Remark 8.3.

To solve problem (C), or (6.10)–(6.16), we first map the fluid region \( \Omega_\varepsilon \) onto the strip \( S = \{(x, y); -\infty < x < \infty, \ 0 < y < 1\} \). It is convenient to introduce the stream function \( \psi \):

\[ \tilde{\Gamma} = (-\psi_y, \psi_x). \]

Note that

\begin{align*}
(6.17) \quad \Delta^2 \psi = 0 \quad \text{in} \quad \Omega_\varepsilon.
\end{align*}

Introduce a function \( \eta(t) \) as in [4; (3.14)] and define a new stream function in \( S \) by

\begin{align*}
(6.18) \quad \tilde{\psi}(x, y) &= \psi(x, y + \varepsilon \tilde{f}(x, \varepsilon) \eta(x, y)) \quad \text{if} \quad x < 0, \\
\tilde{\psi}(x, y) &= \psi(x, y) \quad \text{if} \quad x \geq 0,
\end{align*}

where

\[ \eta(x, y) = \eta \left( \frac{y}{|x|} \right) \quad \text{if} \quad -2 \leq x < 0 \]

\[ = 1 \quad \text{if} \quad x < -2, \ 0 < y < \delta_0 \]

\[ = 0 \quad \text{if} \quad x < -2, \ 1 - \delta_0 < y < 1, \]

\( 0 \leq \eta \leq 1 \) elsewhere, and \( \eta(x, y) \) is \( C^\infty \) for \( x \leq -1 \); here \( \delta_0 \) is any small positive number that will be fixed from now on.

As in [4; (3.18)],

\[ \Delta^2 \tilde{\psi} = \varepsilon H_4[\tilde{\psi}, \tilde{f}] \quad \text{in} \quad S \]

where the right-hand side is a perturbation term. However, it will be more convenient to consider the transformed problem not for \( \tilde{\psi} \) but rather for the pair \( (\tilde{\Gamma}, \tilde{p}) \) defined by

\[ \tilde{\Gamma} = (-\tilde{\psi}_y, \tilde{\psi}_x), \]

\[ \tilde{p}(x, y) = \tilde{p}(x, y + \varepsilon \tilde{f}(x, \varepsilon) \eta(x, y), \varepsilon). \]
Observe that

\[ \nabla \cdot \overrightarrow{G} = 0 \quad \text{in} \quad S. \]

We shall rewrite (6.10), (6.12)–(6.16) in terms of \( \overrightarrow{G} \) and \( \overline{p} \).

As in [4; (3.16)], if \(-2 \leq x < 0\) then

\[ G_y = \partial_x \overline{\psi} - \partial_x \psi + \varepsilon \partial_y \psi \left[ \tilde{f}' \eta + \frac{y \tilde{f}}{|x|^2} \eta' \right], \]

\[ -G_x = \partial_y \overline{\psi} = \partial_y \psi + \varepsilon \partial_y \psi \frac{f}{(-x)} \eta' \quad \left( \text{where} \quad \eta = \eta \left( \frac{y}{|x|} \right) \right) \]

and

\[ \frac{\partial \overline{p}}{\partial x} = \frac{\partial \overline{p}}{\partial x} + \varepsilon \partial_y \overline{p} \left[ \tilde{f}' \eta + \frac{y \tilde{f}}{|x|^2} \eta' \right], \]

\[ \frac{\partial \overline{p}}{\partial y} = \frac{\partial \overline{p}}{\partial y} + \varepsilon \partial_y \overline{p} \frac{f}{(-x)} \eta'. \]

Analogous formulas hold for \( x < -2 \). Using the relation \( \Delta (-\partial_y \psi) = \Delta G_x = \partial \overline{p}/\partial x \), we get

\[ \Delta \overline{G}_x = \Delta (-\partial_y \psi) - \Delta \left[ \varepsilon \partial_y \psi \frac{f}{(-x)} \eta' \right] \]

\[ = \frac{\partial \overline{p}}{\partial x} - \varepsilon \Delta \left[ \partial_y \psi \frac{f}{(-x)} \eta' \right] \]

(6.20)

\[ = \frac{\partial \overline{p}}{\partial x} - \varepsilon (\partial_y \overline{p}) \left[ \tilde{f}' \eta + \frac{y \tilde{f}}{|x|^2} \eta' \right] - \varepsilon \Delta \left[ \partial_y \psi \frac{\tilde{f}}{(-x)} \eta' \right] \]

\[ \equiv \frac{\partial \overline{p}}{\partial x} + \varepsilon L_1[x, y, \nabla \overline{p}, \overrightarrow{G}, \nabla \overrightarrow{G}, \nabla^2 \overrightarrow{G}, \tilde{f}, \tilde{f}'] \]

if \(-2 \leq x < 0\), and similarly

(6.21)

\[ \Delta \overline{G}_y = \frac{\partial \overline{p}}{\partial y} + \varepsilon L_2[\cdots] \]

where \( L_2 \) has the same structure as \( L_1 \). Similar equations hold for \( x < -2 \). Equations (6.19)–(6.21) hold in \( S \). The boundary conditions are:

(6.22)

\[ \overrightarrow{G}(x, 1) = \overline{\zeta}(x), \quad -\infty < x < \infty, \]

(6.23)

\[ \overrightarrow{G}(x, 0) = 0, \quad 0 < x < \infty, \]
\[ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = \varepsilon Q_1[\nabla \overline{G}, \tilde{f}, \tilde{f}'] \text{ at } (x,0), \quad -\infty < x < 0 , \]

\[ -\overline{\rho} + 2 \frac{\partial G}{\partial y} + \frac{\sigma}{U} \frac{\partial G}{\partial x} = \varepsilon Q_2[\nabla \overline{G}, \tilde{f}, \tilde{f}'] \text{ at } (x,0), \quad -\infty < x < 0 . \]

Here

\[ Q_1 = \partial_y \psi \left\{ -\frac{\tilde{f}}{(-x)} \eta' + \tilde{f}' \eta + \frac{y \tilde{f}}{|x|^2} \eta' \right\}, \quad \eta = \eta \left( \frac{y}{|x|} \right) \]

if \(-2 < x < 0\); for \(x < -2\) the expression for \(Q_2\) is slightly different. \(Q_2[\cdots]\) has the same structure as \(Q_1[\cdots]\). Finally, the free boundary \(\tilde{f}(x,\varepsilon)\) is determined by

\[ \tilde{f}'(x,\varepsilon) = \frac{1}{U} \overline{G}_y(x,0) + \varepsilon Q_3[\overline{G}, \tilde{f}'], \quad -\infty < x < 0 \]

where

\[ Q_3[\overline{G}, \tilde{f}'] = \tilde{f} \overline{G}_x . \]

So far we have reduced the coating problem \((C)\) to the system \((6.20)-(6.27)\) in the fixed domain \(S\). Denote the weak solution of the linearized problem (established in Theorem 2.2) by \((U + \varepsilon \overline{G}_0, \varepsilon p_0, \varepsilon f_0)\). Set

\[ \tilde{W}(x,y) = \overline{G}(x,y) - \tilde{G}_0(x,y) , \]

\[ q(x,y) = \overline{p}(x,y) - p_0(x,y) , \]

\[ \ell(x) = \tilde{f}(x,\varepsilon) - f_0(x) . \]

Then

\[ \Delta \tilde{W} = \nabla q + \varepsilon \tilde{L} \quad \text{in } S , \]

\[ \nabla \cdot \tilde{W} = 0 \quad \text{in } S \]

where \(\tilde{L} = (L_1, L_2)\) and

\[ L_i = L_i[x,y, \nabla p_0 + \nabla q, \tilde{G}_0 + \tilde{W}, \nabla \tilde{G}_0 + \nabla \tilde{W}, \nabla^2 \tilde{G}_0 + \nabla^2 \tilde{W}, f_0 + \ell, f_0' + \ell'] . \]

The boundary conditions are

\[ \tilde{W}(x,1) = 0 , \quad -\infty < x < \infty , \]

\[ \tilde{W}(x,0) = 0 , \quad 0 < x < \infty , \]
\[
(6.33) \quad \frac{\partial W_x}{\partial y} + \frac{\partial W_y}{\partial x} = \varepsilon Q_1[\nabla \tilde{G}_0 + \nabla \tilde{W}, f_0 + \ell, f_0' + \ell'] \quad \text{at} \quad (x,0), \quad -\infty < x < 0 ,
\]

\[
(6.34) \quad -g + 2\frac{\partial W_y}{\partial y} - \frac{1}{\lambda} \frac{\partial W_y}{\partial x} = \varepsilon Q_2[\nabla \tilde{G}_0 + \nabla \tilde{W}, f_0 + \ell, f_0' + \ell'] \quad \text{at} \quad (x,0), \quad -\infty < x < 0 .
\]

Finally,

\[
(6.35) \quad \ell'(x) = \frac{1}{U} W_y(x,0) + \varepsilon Q_3[\tilde{G}_0 + \tilde{W}, f_0' + \ell'] .
\]

In the next two section we shall establish existence and uniqueness for Problem (C) in its formulation (6.28)--(6.35).

\section{Solution to the coating problem.}

We shall assume that

\[
(7.1) \quad f_0(x) \approx A(-x)^{2-\rho} \quad \text{as} \quad x \uparrow 0 , \quad A > 0 ;
\]

the case \( A \leq 0 \) will be considered in Remark 8.1.

In this and the next section we follow closely the notation of [4], but replace the half plane \( \Omega = \{(x,y), -\infty < x < \infty, y > 0\} \) by the strip \( S \). In particular, we shall work with norms

\[
\|\varphi\|_{S,m+\gamma}^q = \sup_S \left\{ \sum_{|k| \leq m} \frac{r^{|k|}}{g(r)} |D^k \varphi(X)| + \sum_{|k|=m} \frac{r^{m+\gamma}}{g(r)} |D^k \varphi(X)| \right\}
\]

where \( \gamma \) is a small positive number, \( r = |X| \), and \( g(r) \) is a positive smooth function for \( 0 < r < \infty \), such that

\[
g(r) = r^\alpha \quad \text{if} \quad r < \frac{1}{2} , \quad g(r) = 1 \quad \text{if} \quad r > 2 ,
\]

and

\[
(7.2) \quad \alpha = 2 - \rho .
\]

Similarly we define the norm \( \|f\|_{m+\gamma}^q \) for functions \( f(x), \quad -\infty < x \leq 0 \).

We shall need the following Phragmén–Lindelöf type results:
**Lemma 7.1.** Suppose \( \varphi(x, y) \) is a bounded solution of

\[
\Delta^2 \varphi = 0 \quad \text{in} \quad S \cap \{ x > A \}, \quad A > 0,
\]

\[
\varphi(x, 1) = \varphi_y(x, 1) = 0 \quad \text{if} \quad x > A,
\]

\[
\varphi(x, 0) = \varphi_y(x, 0) = 0 \quad \text{if} \quad x > A.
\]

Then there exist positive constants \( C, \theta \) such that

\[
|\varphi(x, y)| \leq C e^{-\theta x} \quad \text{in} \quad S \cap \{ x > A \}.
\]

**Lemma 7.2.** Suppose \( \varphi(x, y) \) is a bounded solution of

\[
\Delta^2 \varphi = 0 \quad \text{in} \quad S \cap \{ x < -A \}, \quad A > 0,
\]

\[
\varphi(x, 1) = \varphi_y(x, 1) = 0 \quad \text{if} \quad x < -A,
\]

\[
(\varphi_{yy} - \varphi_{xx})(x, 0) = 0 \quad \text{if} \quad x < -A,
\]

\[
\lambda (3\varphi_{xyy} + \varphi_{yyy}) - \varphi_{xxx} = 0 \quad \text{at} \quad (x, 0), \quad x < -A.
\]

Then there exist positive constants \( C, \theta \) such that

\[
|\varphi(x, y)| \leq C e^{-\theta|x|} \quad \text{in} \quad S \cap \{ x < -A \}.
\]

Both lemmas follow from an abstract Phragmén–Lindelöf theorem due to Lax [5]; the case of Dirichlet data, as in Lemma 7.1, is explicitly stated in [5], whereas the case of the boundary data as in Lemma 7.2 follows by the same arguments as in [5] provided we use the elliptic estimates of [1]. Using a priori estimates, we can also extend the exponential decay bounds to any derivative of \( \varphi(x, y) \).

Applying Lemmas 7.1, 7.2 to \( \vec{G}_0 \), or rather to the stream function \( \varphi^0 \) \( (\vec{G}_0 = (-\varphi_y^0, \varphi_x^0)) \) we conclude:

**Lemma 7.3.** The function \( \vec{G}_0 \) satisfies:

\[
(7.3) \quad |\vec{G}_0(x, y)| \leq C e^{-\theta|x|} \quad \text{in} \quad \Omega,
\]

where \( C, \theta \) are some positive constants.

Introduce a class \( K \) of functions \((\vec{W}, q, f)\):

\[
K = \left\{ (\vec{W}, q, f); \| \vec{W} \|_{5,3+\gamma}^q \leq B, \| q \|_{5,2+\gamma}^q \leq B, \| f \|_{4+\gamma}^p \leq B \right\}
\]

32
where $f = f(x)$, $-\infty < x \leq 0$. We shall further restrict this class by requiring that

\[
\sum_{|k| \leq 3} |D^k \tilde{W}(X)| + \sum_{|k| = 3} |D^{k+\gamma} \tilde{W}(X)| \leq B_0 e^{-\mu |x|}, \quad |x| > 1,
\]
\[
\sum_{|k| \leq 2} |D^k q(X)| + \sum_{|k| = 2} |D^{k+\gamma} q(X)| \leq B_0 e^{-\mu |x|}, \quad |x| > 1,
\]
\[
\sum_{k=0}^{3} |D^k f'(x)| + |D^{3+\gamma} f'(x)| \leq B_0 e^{-\mu |x|}, \quad x < -1;
\]
(7.4)

here $D^\gamma h(X)$ refers to Hölder coefficient of $h$ (corresponding to Hölder exponent $\gamma$) taken over pairs of points whose distance to $X$ is $\leq 1/2$.

**Definition 7.1.** We denote by $K_0$ the subclass of functions in $K$ which satisfy (7.4).

We shall henceforth fix the positive constants $B, B_0$ and $\mu$, taking

\[
0 < \mu < \theta, \quad \theta \text{ as in (7.3)}.
\]
(7.5)

Given $(\tilde{W}, \tilde{q}, \tilde{f})$ in $K_0$, denote by

\[
\tilde{L}, \tilde{Q}_1, \tilde{Q}_2 \quad \text{and} \quad \tilde{Q}_3
\]

the function $\tilde{L}, Q_1, Q_2, Q_3$ defined in (6.28) (or (6.30)), (6.33), (6.34) and (6.35), respectively, when $\tilde{W}, q, \ell$ are replaced by $\tilde{W}, \tilde{q}, \tilde{f}$ respectively. Consider the system

\[
\Delta \tilde{W} = \nabla q + \varepsilon \tilde{L} \quad \text{in} \quad S,
\]
(7.6)
\[
\nabla \cdot \tilde{W} = 0 \quad \text{in} \quad S
\]
(7.7)

\[
\tilde{W}(x,1) = 0, \quad -\infty < x < \infty,
\]
(7.8)
\[
\tilde{W}(x,0) = 0, \quad 0 < x < \infty,
\]
(7.9)

\[
\frac{\partial W_x}{\partial y} + \frac{\partial W_y}{\partial x} = \varepsilon \tilde{Q}_1, \quad -\infty < x < 0, \quad y = 0,
\]
(7.10)

\[
-q + 2 \frac{\partial W_y}{\partial y} - \frac{1}{\lambda} \frac{\partial W_y}{\partial x} = \varepsilon \tilde{Q}_2, \quad -\infty < x < 0, \quad y = 0,
\]
(7.11)
and define \( f(x) \) by \( f(0) = 0 \) and

\[(7.12) \quad f'(x) = \frac{1}{U} W_y(x, 0) + \varepsilon \tilde{Q}_3, \quad -\infty < x < 0.\]

Our goal is to show that this system has a unique bounded solution which belongs to the class \( K_0 \), and that the mapping \( T \) defined by

\[ T(\tilde{W}, \tilde{q}, \tilde{f}) = (\tilde{W}, q, f) \]

is a contraction, provided \( \varepsilon \) is small enough. This will establish existence and uniqueness to problem \( (C) \).

\[ \S 8. \text{ Solution to the coating problem (continued).} \]

To study the system \((7.6) - (7.11)\) we introduce a stream function \( \varphi : \tilde{W} = (-\partial_y \varphi, \partial_x \varphi) \). Then the system takes the form

\[ (8.1) \quad \Delta^2 \varphi = \varepsilon \beta_0 \text{ in } S, \]
\[ (8.2) \quad \varphi(x, 1) = \varphi_y(x, 1) = 0, \quad -\infty < x < \infty, \]
\[ (8.3) \quad \varphi(x, 0) = \varphi_y(x, 1) = 0, \quad 0 < x < \infty, \]
\[ (8.4) \quad (\varphi_{yy} - \varphi_{xx})(x, 0) = \varepsilon \beta_1, \quad -\infty < x < 0, \]
\[ (8.5) \quad \lambda(3\varphi_{xxy} + \varphi_{yyy}) - \varphi_{xxx} = \varepsilon \beta_2 \text{ at } (x, 0), \quad -\infty < x < 0 \]

where the \( \beta_i \) depend on \( \hat{L} \) and \( \hat{Q}_1, \hat{Q}_2 \) (cf \((5.10) - (5.19))\); in particular,

\[ (8.6) \quad |\beta_i| \leq C e^{-\mu|x|} \text{ if } |x| > 1 \text{ (} 0 \leq i \leq 2 \text{).} \]

The same estimate holds for the \( 2 + \gamma \) derivative of \( \beta_2 \), the \( 1 + \gamma \) derivative of \( \beta_3 \) and the \( \gamma \) derivative of \( \beta_0 \). (Recall that "\( \gamma \) derivative" at \((x, y)\) means the \( \gamma \)-Hölder coefficient computed for pair points whose distance from \((x, y)\) is \( \leq 1/2 \).)

We also have, as in [4],

\[ (8.7) \quad |\beta_0| \leq C r^\alpha, \quad |\beta_1| \leq C r^{\alpha-2}, \quad |\beta_2| \leq C r^{\alpha-3} \quad (r < 1) \]

with similar estimates on higher order derivative (the first derivative of \( \beta_1 \) is bounded by \( C r^{\alpha-3} \), etc.).

We wish to find a solution \( \varphi^* \) of

\[ (8.8) \quad \varphi_{yy} - \varphi_{xx} = \varepsilon \beta_1, \quad \lambda(3\varphi_{xxy} + \varphi_{yyy}) - \varphi_{xxx} = \varepsilon \beta_2 \text{ at } (x, 0), \quad -\infty < x < 0 \]

34
which is supported in the region \( \hat{\Sigma}_{\theta_0} \) bounded by the negative \( x \)-axis, the segment \( y = -\theta_0 x \) for \(-\theta_0/4 < x < 0 \) and \( \left\{ y = \frac{1}{4} \right\} \) \( (\theta_0 \) is any small positive number), and satisfies:

\[
\|\varphi^*\|_{s, 4+\gamma}^q \leq C\varepsilon.
\]

To do this we introduce the Poisson kernel

\[
P(x, y) = \frac{y}{x^2 + y^2},
\]

and form the function

\[
Q(x, y) = c_1 y^2 (P \ast \beta_1)(x, y) + y^3 \left\{ P \ast [c_2 H(\partial_x \beta_1) + c_3 \beta_2] \right\} (x, y)
\]

where \( H \) is the Hilbert transform. The \( c_i \) are positive constants chosen so that the Fourier transform (in \( x \)) \( \hat{Q} \) of \( Q \) satisfies:

\[
\hat{Q}(\xi, \eta) = -\frac{1}{2} \hat{\beta_1}(\xi)y^2 e^{-|\xi|y} + \frac{1}{6}(|\xi|\hat{\beta_1}(\xi) + \hat{\beta_2}(\xi))y^3 e^{-|\xi|y}.
\]

It follows that

\[
\hat{Q} = \hat{Q}_y = 0, \quad \hat{Q}_{yy} = \hat{\beta_1}, \quad \hat{Q}_{yyy} = \hat{\beta_2} \quad \text{at} \quad y = 0,
\]

so that

\[
Q = Q_y = 0, \quad Q_{yy} = \beta_1, \quad Q_{yyy} = \beta_2 \quad \text{at} \quad y = 0.
\]

Hence \( \varepsilon Q \) satisfies (8.8).

We proceed to estimate \( \varepsilon Q \) by the splitting argument of [4]. We write

\[
\beta_j(x) = \sum_{n=-\infty}^{\infty} \beta_j^n(x) \quad (j = 1, 2)
\]

where \( \beta_j^n \) is supported in the interval

\[
\left[ -2 \left( \frac{1}{2} \right)^n, - \left( \frac{1}{2} \right)^n \right] \quad \text{if} \quad n \geq 0 \quad \text{and} \quad \left[ n - 1, n \right] \quad \text{if} \quad n \leq -1.
\]

We split each convolution integral in (8.11) into series whose \( n \)th term corresponds to \( \beta_j^n \). Each term with \( n \geq 0 \) we rescale by a factor \( 2^n \) and then apply the Schauder estimates to get, for the corresponding term, \( Q_n \),

\[
\|Q_n\|_{4+\gamma} \leq C(\|\beta_1^n\|_{2+\gamma} + \|\beta_2^n\|_{1+\gamma}).
\]

35
Indeed, by the Schauder boundary estimates, the $2 + \gamma$ norm of $P \ast \beta$ can be estimates by the $2 + \gamma$ norm of $\beta$; the $4 + \gamma$ norm of $y^2(P \ast \beta)$ can be estimates by the $2 + \gamma$ norm of $\beta$, if we use also interior Schauder estimates. We have also used here the fact that the $1 + \gamma$ norm of $H(\partial_x \beta)$ is bounded by the $2 + \gamma$ norm of $\beta$.

For $n \leq -1$ there is no need to rescale, and we obtain a similar $4 + \gamma$ estimate. Summing over all $n$, we get
\[
\|Q\|_{5,4+\gamma} \leq C\|\beta_1\|_{2+\gamma} + C\|\beta_2\|_{1+\gamma}.
\]

Taking $\varphi^* = \varepsilon \xi_0(x,y)Q(x,y)$ where $\xi_0$ is a suitable cut-off function, we obtain the desired solution of (8.8). In view of the exponential decay established in (8.6),
\[
|\varphi^*(x,y)| \leq C\varepsilon e^{-\mu|x|} \quad \text{if} \quad |x| > 1,
\]
and the same estimate holds also for the $4 + \gamma$ derivatives of $\varphi^*$ (using again the Schauder estimates).

We now define $\tilde{W}^* = (W_x^*, W_y^*)$ by
\[
W_x^* = -\partial_y \varphi^*, \quad W_y^* = \partial_x \varphi^*
\]
and introduce
\[
\tilde{R} = \tilde{W} - \tilde{W}^*, \quad h = q + c^* \quad (c^* \text{ is a constant}).
\]
Then (7.6)–(7.11) take the form
\[
\Delta \tilde{R} = \nabla h + \varepsilon \tilde{Q}_0 \quad \text{in} \quad S,
\]
\[
\nabla \cdot \tilde{R} = 0 \quad \text{in} \quad S,
\]
\[
\tilde{R}(x,1) = 0, \quad -\infty < x < \infty,
\]
\[
\tilde{R}(x,0) = 0, \quad 0 < x < \infty,
\]
\[
\frac{\partial R_x}{\partial y} + \frac{\partial R_y}{\partial x} = 0, \quad -\infty < x < 0,
\]
\[
-h + 2 \frac{\partial R_y}{\partial y} - \frac{1}{\lambda} \frac{\partial R_y}{\partial x} = 0, \quad -\infty < x < 0,
\]
provided $c^*$ is chosen so that (8.18) is satisfied at one point, say at $x = -1$. Furthermore, $\hat{Q}_0$ is supported in $\tilde{S}_{\delta_0}$,
\[
|\hat{Q}_0(x,y)| \leq C|x|^\alpha \quad \text{for} \quad -1 \leq x < 0,
\]
\[
|\hat{Q}_0(x,y)| \leq C e^{-\mu|x|} \quad \text{for} \quad x \leq -1,
\]
and a similar bound holds for the $\gamma$-derivative of $\hat{Q}_0$. 

36
By the proof of Theorem 2.2 there exists a unique solution $\tilde{R}$ of (8.13)-(8.18) satisfying

\begin{equation}
\int_{S} |\nabla \tilde{R}|^{2} \leq C \int_{S} |\tilde{Q}_{0}|^{2}.
\end{equation}

The regularity results of Section 5 imply that if we set

$$\tilde{R} = (-\partial_{y} \psi, \partial_{x} \psi)$$

then

\begin{equation}
\|\psi\|_{S^{4+\gamma}}^{2} \leq C \varepsilon.
\end{equation}

We need to derive a decay estimate for $\psi$ as $|x| \to \infty$. Note that

\begin{equation}
\Delta^{2} \psi = \varepsilon \hat{H}
\end{equation}

where $\hat{H}$ is supported in $\hat{\Sigma}_{\theta_{0}}$ and

\begin{equation}
|\hat{H}(x, y)| \leq Ce^{-\mu|x|}.
\end{equation}

We claim that

\begin{equation}
|\psi(x, y)| \leq C \varepsilon e^{-\mu|x|} \quad \text{if} \quad |x| > 1.
\end{equation}

For $x > -1$ this follows from Lemma 7.1. To prove it for $x < -1$ we introduce the semigroup $V$ of translations, $\xi \to V(\xi)$, defined by

$$V(\xi)u(x, y) = u(x + \xi, y) \quad (\xi < 0)$$

for functions defined for $x < -1$, $0 < y < 1$ and satisfying the (homogeneous) boundary conditions of Lemma 7.2. By the proof of Lemmas 7.2 (see [5])

\begin{equation}
\|V(\xi)u(x, \cdot)\| \leq Ce^{-\theta|\xi|}\|u(x, \cdot)\|
\end{equation}

where the norm $\| \cdot \|$ can be taken the $L^{\infty}$-norm. Recalling (8.22) we can write

\begin{equation}
\psi(x, y) = \varepsilon \left[ \int_{-1}^{x} V(x - \xi) \hat{H}(\xi, \cdot) d\xi \right] (y) + V(x) \psi(-1, y)
\end{equation}

and using (8.23), (8.25) and the fact that $|\psi(-1, y)| \leq C \varepsilon$, we easily find upon recalling (7.5), that (8.24) is valid for $x \leq -1$. 

37
Similarly we can estimate the $4 + \gamma$ derivatives of $\psi$ by the right-hand side of (8.24), for $|x| \geq 1$ with another constant $C$.

As in [4] we can also derive an estimate

$$\|\psi\|_{S,4+\gamma}^g \leq C \varepsilon,$$

using the splitting argument.

Combining all these estimates on $\psi$, one can easily check that

(8.27) \hspace{1cm} T \text{ maps } K_0 \text{ into itself.}

Similarly one can prove (cf. [4]) that $T$ is a contraction in $K_0$ with respect to the norm, say,

$$\|\tilde{W}\|_{S,3+\gamma}^g + \|q\|_{S,2+\gamma}^g.$$

Consequently $T$ has a unique fixed point.

**Definition 8.1.** A regular solution of Problem (C) is a solution which belongs to class $K_0$.

We have proved:

**Theorem 8.1.** If (7.1) holds then, for any sufficiently small $\varepsilon > 0$, there exists a unique regular solution to Problem (C).

Note that the free boundary is given by

(8.28) \hspace{1cm} y = f(x, \varepsilon) = \varepsilon f_0(x) + \varepsilon^2 f_1(x, \varepsilon)

where

(8.29) \hspace{1cm} |x|^{\alpha+j} |D^j f_1(x, \varepsilon)| \leq C \quad \text{if} \quad -1 \leq x < 0

and

(8.30) \hspace{1cm} |D^j f_0(x)| + |D^j f_1(x, \varepsilon)| \leq C e^{-\mu |x|} \quad \text{if} \quad x \leq -1

for $j = 1, 2, 3, 4, 4 + \gamma$; (8.29) holds also for $j = 0$.

**Remark 8.1.** By Theorem 5.1 the linearized free boundary $f_0(x)$ has, generically, the asymptotic behavior

(8.31) \hspace{1cm} f_0(x) \approx A(-x)^{m-\rho} \quad \text{as} \quad x \uparrow 0,
with \( m = 2 \). If \( A = 0 \) then the generic subcase becomes (8.31) with \( m = 3 \), etc. Consider the case where (8.31) holds for some positive integer \( m \geq 2 \), where \( A \) is a positive constant. Setting
\[
\alpha = m - \rho
\]
the proof of Theorem 8.1 then proceeds exactly as before. If (8.31) holds with \( m = 2 \) (or with \( m \geq 3 \)) and \( A < 0 \) then the free boundary will go into \( \{ y < 0 \} \), but all our results can still be extended with minor changes.

**Remark 8.2.** There is a relation between the free boundary \( y = \varepsilon f(x) \) of the linearized problem at \( x = -\infty \) and the perturbation term \( \tilde{\zeta}(x) \), namely
\[
(8.32) \quad f(-\infty) = -\int_{-\infty}^{\infty} \zeta_y(x) dx.
\]

Indeed, integrating \( \nabla \cdot \vec{G} = 0 \) over \( S \) and using the decay of \( \vec{G} \) as \( |x| \to \infty \) we get, after using (1.5),
\[
0 = \varepsilon \int_{-\infty}^{\infty} \zeta_y(x) dx - U \varepsilon \int_{-\infty}^{0} f'(x) dx,
\]
from which (8.32) follows.

**Remark 8.3.** So far we have assumed that the data \( \tilde{\zeta}(x) \) is \( C^\infty \) with compact support. The results immediately extend to the case where \( \tilde{\zeta}(x) \) is only assumed to belong to \( C^{3+\gamma} \). Consider next the case where \( \tilde{\zeta}(x) \) does not have compact support and is instead bounded by
\[
|D^j \tilde{\zeta}(x)| \leq \frac{C}{(1 + |x|)^{N+j}} \quad (N > 1)
\]
for \( j = 0, 1, 2, 3, 3 + \gamma \). Instead of [5] we can now use results of [7] (see also [3; Chap. 3]) to deduce that
\[
\|V(\xi)\| \leq \frac{C}{(1 + |\xi|)^M}, \quad |\vec{G}_0(x, y)| \leq \frac{C}{(1 + |x|)^M}
\]
for some \( M > 1 \) (if \( N \) is large enough); a similar bound holds for the appropriate derivatives. We can now proceed as before, with only minor modifications, to extend the proof of Theorem 8.1. We thus establish the existence of a unique regular solution to Problem (C), with the exponential decay replaced by appropriate polynomial decay as \( |x| \to \infty \).

**Remark 8.4.** Theorem 8.1 can probably be extended to the Navier-Stokes equation; cf. Remark 9.5 in [4]. The main change occurs in the linearized problem (2.7)-(2.10) of [4], where the term \(-\lambda U \psi_{xy}\) now appears on the left-hand side of (2.10). This will require a new analysis of the eigenvalue problem.
Remark 8.5. The results of this paper can be extended to coating flow which separates from \{y = 1\}, say at a point \((a, 0)\). That means that the linearized problem has two free boundaries:

\[ y = f_{01}(x), \quad -\infty < x < 0 \quad \text{with} \quad f_{02}(0) = 0 \]

and

\[ y = f_{02}(x), \quad a < x < \infty \quad \text{with} \quad f_{02}(a) = 1. \]

Here we assume that \(\zeta(x) \equiv 0\) if \(x \geq a - \delta\) for some \(\delta > 0\).

Remark 8.6. All the results of the paper extend to the case \(U < 0\); in this case \(-\frac{1}{2} < \rho < 0\).


Lemma 9.1. If \(\psi\) satisfies (5.1)–(5.4) with \(k_1 \in H^{1/2}(-\infty, 0), \ k_2 \in H^{-1/2}(-\infty, 0)\), and if the supports of \(k_1, k_2\) are bounded, then

\[
\int_{\mathbb{R}^2 \cap \{|X| < M\}} |\nabla^2 \psi|^2 < \infty \quad \forall \ M > 0.
\]

Proof. By Theorem 4.3 of [4]

\[
(9.2) \quad \psi_x(x, 0) = h_1(x), \quad \psi_y(x, 0) = h_2(x) \quad \text{for} \quad x < 0,
\]

where

\[
(9.3) \quad h_2(x) = \int_x^0 \frac{\sqrt{-\eta}}{2\pi} d\eta \int_{-\infty}^0 \frac{k_1(\xi)}{\sqrt{-\xi(\eta - \xi)}} d\xi,
\]

\[
(9.4) \quad h_1(x) = \frac{1}{1 + 9\lambda^2} \int_x^0 d\eta \int_{-\infty}^\eta k_2(q) dq - \frac{3\lambda}{(1 + 9\lambda^2)\pi} \int_x^0 d\eta \int_{-\infty}^\eta (-q)^{-\rho} dq \int_{-\infty}^0 \frac{(-\xi)^\rho}{q - \xi} k_2(\xi) d\xi.
\]

We shall prove that if we extend the \(h_j(x)\) by 0 to \(x > 0\) then, for any \(M > 0\),

\[
(9.5) \quad \|h_j\|_{H^{1-\xi}_{loc}(-M, M)} \leq C, \quad C = C(M) \quad (j = 1, 2)
\]
where \( \delta \) is any positive number (\( C(M) \) depends on \( \delta \)). To prove (9.5) we may assume that
the \( k_j \) are in \( C_0^\infty(-\infty,0) \), since \( \mathcal{D}(-\infty,0) \) is dense in \( H^s \) for any \( s \leq \frac{1}{2} \) (see Th. 11.1, p. 60 in [6]).

Let \( g(x) \) be a \( C_0^\infty(-2,1) \) function such that \( g(x) = 1 \) if \(-1 \leq x \leq 0\) and set

\[
Q(x) = \frac{g(x)}{\sqrt{-x}}, \quad -\infty < x < 0.
\]

In the Fourier transform of \( Q \),

\[
\hat{Q}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{g(\xi)}{\sqrt{-\xi}} e^{-i\xi k} d\xi,
\]

we substitute \( \xi = u/k \) and easily find that

\[
|\hat{Q}(k)| \leq \frac{C}{\sqrt{|k|}}.
\]

Extending \( k_1(x) \) by zero to \( x > 0 \) we then have

\[
|\hat{k_1}Q| = |\hat{k_1} * \hat{Q}| \leq \int_{-\infty}^{\infty} |\hat{k_1}(v)| \frac{dv}{1 + |k - v|^{1/2}}.
\]

The extended function \( k_1 \) belongs to \( H^{\frac{1}{2} - \varepsilon}(-\infty,\infty) \) for any \( \varepsilon > 0 \) (see Theorem 11.4, p. 66 in [6]); therefore

\[
\int_{-\infty}^{\infty} |\hat{k_1}(v)|^2 (1 + |v|)^{1-2\varepsilon} dv < \infty.
\]

It follows that

\[
\int_{-\infty}^{\infty} (1 + |k|)^{-\delta} |\hat{k_1}Q(k)|^2 dk \leq C||k_1||_{H^{\frac{1}{2} - \varepsilon}({\mathbb{R}})}
\]

\[
\times \int_{-\infty}^{\infty} \frac{dk}{(1 + |k|)^{\delta}} \int_{-\infty}^{\infty} \frac{dv}{(1 + |v|)^{1-2\varepsilon}(1 + |k - v|)}.
\]

Substituting \( v = kx \) and splitting the inner integral according to \( |1 - x| \geq \frac{1}{k} \), we get the bound

\[
C||k_1||_{H^{\frac{1}{2} - \varepsilon}({\mathbb{R}})} \int_{-\infty}^{\infty} \frac{dk}{(1 + |k|)^{\delta}} \frac{|\log(|k|)|}{|k|^{1-2\varepsilon}}.
\]
Hence

\[(9.6) \quad \| k_1 Q \|_{H^{-\delta}(\mathbb{R})} \leq C \]

and the same holds for \( k_1(\xi)/\sqrt{-\xi} \). The Hilbert transform

\[ \int_{-\infty}^{0} \frac{k_1(\xi)}{\sqrt{-\xi}} \frac{1}{(\eta - \xi)} \]

maps \( H^{-\delta} \) into \( H^{-\delta} \), and multiplication by \( \sqrt{-\eta} \) maps \( H^{\delta}_{\text{loc}} \) into \( H^{\delta}_{\text{loc}} \). Finally, the integral\( \int_{z}^{0} \) takes \( H^{\delta}_{\text{loc}} \) into \( H^{1-\delta}_{\text{loc}} \). Therefore, from (9.3) we conclude that (9.5) is satisfied for \( j = 2 \).

The proof of (9.5) for \( j = 1 \) is similar. Indeed, the first term on the right-hand side of (9.4) is estimated precisely as before. As for the second term, the inner integral is the Hilbert transform of \( (-\xi)^{\rho} k_2 \), which belongs to \( H^{-1/2}(-\infty, 0) \) (since \( k_2 \) belongs to \( H^{-1/2}(-\infty, 0) \) and has bounded support). Hence this inner integral is in \( H^{-1/2}(-\infty, 0) \).

Since \( 0 < \rho < \frac{1}{2} \), multiplication by \( (-q)^{-\rho} \) maps the inner integral into \( H^{1-\delta}_{\text{loc}} \). (Here we use the same argument as in the proof of (9.6)). Finally the two integrations in the last term of (9.4) give a function in \( H^{1-\delta}_{\text{loc}} \). This concludes the proof of (9.5) for \( j = 1 \).

We now apply elliptic regularity [5; p. 47] to \( \psi \), using the boundary conditions (5.2) and (9.2). We obtain

\[(9.7) \quad \| \nabla^2 \psi \|_{H^{\frac{3}{2} - \delta}(\mathbb{R}^2_{+} \cap \{ ||x|| < M \})} \leq C, \]

and this of course implies (9.1).

To complete the proof of (5.45) we use (5.41): \( \bar{g}_2 \in H^2(\mathbb{R}^2_{+}) \) by interior elliptic estimates in \( H^s \) spaces with \( s \) real and negative, and \( \hat{g}_3, \hat{h}_3 \) belong to \( H^2(\mathbb{R}^2_{+}) \) by Lemma 9.1.

The above proof shows that, actually, \( \bar{\psi} \in H^{\frac{3}{2} - \delta}(\mathbb{R}^2_{+}) \) for any \( \delta > 0 \).

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REFERENCES


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<th>Title</th>
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</thead>
<tbody>
<tr>
<td>1050</td>
<td>J.E. Dunn &amp; Roger Fosdick</td>
<td>The Weierstrass condition for a special class of elastic materials</td>
</tr>
<tr>
<td>1051</td>
<td>Bei Hu &amp; Jianhua Zhang</td>
<td>A free boundary problem arising in the modeling of internal oxidation of binary alloys</td>
</tr>
<tr>
<td>1052</td>
<td>Eduard Feireisl &amp; Enrique Zuazua</td>
<td>Global attractors for semilinear wave equations with locally distributed nonlinear damping and critical exponent</td>
</tr>
<tr>
<td>1053</td>
<td>I-Heng McComb &amp; Chjan C. Lim</td>
<td>Stability of equilibria for a class of time-reversible, $D_{x}x\otimes(2)$-symmetric homogeneous vector fields</td>
</tr>
<tr>
<td>1054</td>
<td>Ruben D. Spies</td>
<td>A state-space approach to a one-dimensional mathematical model for the dynamics of phase transitions in pseudoelastic materials</td>
</tr>
<tr>
<td>1055</td>
<td>H.S. Dumas, F. Golse, and P. Lochak</td>
<td>Multiphase averaging for generalized flows on manifolds</td>
</tr>
<tr>
<td>1056</td>
<td>Bei Hu &amp; Hong-Ming Yin</td>
<td>Global solutions and quenching to a class of quasilinear parabolic equations</td>
</tr>
<tr>
<td>1057</td>
<td>Zhangxin Chen</td>
<td>Projection finite element methods for semiconductor device equations</td>
</tr>
<tr>
<td>1058</td>
<td>Peter Guttorp</td>
<td>Statistical analysis of biological monitoring data</td>
</tr>
<tr>
<td>1059</td>
<td>Wensheng Liu &amp; Héctor J. Sussmann</td>
<td>Abnormal sub-Riemannian minimizers</td>
</tr>
<tr>
<td>1060</td>
<td>Chjan C. Lim</td>
<td>A combinatorial perturbation method and Arnold's whiskered Tori in vortex dynamics</td>
</tr>
<tr>
<td>1061</td>
<td>Yong Liu</td>
<td>Axially symmetric jet flows arising from high speed fiber coating</td>
</tr>
<tr>
<td>1062</td>
<td>Li Qiu &amp; Tongwen Chen</td>
<td>$H_{2}$ and $H_{\infty}$ designs of multirate sampled-data systems</td>
</tr>
<tr>
<td>1063</td>
<td>Eduardo Casas &amp; Jiongmin Yong</td>
<td>Maximum principle for state-constrained optimal control problems governed by quasilinear elliptic equations</td>
</tr>
<tr>
<td>1064</td>
<td>Suzanne M. Lenhart &amp; Jiongmin Yong</td>
<td>Optimal control for degenerate parabolic equations with logistic growth</td>
</tr>
<tr>
<td>1065</td>
<td>Suzanne Lenhart</td>
<td>Optimal control of a convective-diffusive fluid problem</td>
</tr>
<tr>
<td>1066</td>
<td>Enrique Zuazua</td>
<td>Weakly nonlinear large time behavior in scalar convection-diffusion equations</td>
</tr>
<tr>
<td>1067</td>
<td>Caroline Fabre, Jean-Pierre Puel &amp; Enrike Zuazua</td>
<td>Approximate controllability of the semilinear heat equation</td>
</tr>
<tr>
<td>1068</td>
<td>M. Escobedo, J.L. Vazquez &amp; Enrike Zuazua</td>
<td>Entropy solutions for diffusion-convection equations with partial diffusivity</td>
</tr>
<tr>
<td>1069</td>
<td>M. Escobedo, J.L. Vazquez &amp; Enrike Zuazua</td>
<td>A diffusion-convection equation in several space dimensions</td>
</tr>
<tr>
<td>1070</td>
<td>F. Fagnani &amp; J.C. Willems</td>
<td>Symmetries of differential systems</td>
</tr>
<tr>
<td>1071</td>
<td>Zhangxin Chen, Bernardo Cockburn, Joseph W. Jerome &amp; Chi-Wang Shu</td>
<td>Mixed-RKDG finite element methods for the 2-D hydrodynamic model for semiconductor device simulation</td>
</tr>
<tr>
<td>1072</td>
<td>M.E. Bradley &amp; Suzanne Lenhart</td>
<td>Bilinear optimal control of a Kirchhoff plate</td>
</tr>
<tr>
<td>1073</td>
<td>Héctor J. Sussmann</td>
<td>A cornucopia of abnormal subriemannian minimizers. Part I: The four-dimensional case</td>
</tr>
<tr>
<td>1074</td>
<td>Marek Rakowski</td>
<td>Transfer function approach to disturbance decoupling problem</td>
</tr>
<tr>
<td>1075</td>
<td>Yuncheng You</td>
<td>Optimal control of Ginzburg-Landau equation for superconductivity</td>
</tr>
<tr>
<td>1076</td>
<td>Yuncheng You</td>
<td>Global dynamics of dissipative modified Korteweg-de Vries equations</td>
</tr>
<tr>
<td>1077</td>
<td>Mario Taboada &amp; Yuncheng You</td>
<td>Nonuniformly attracting inertial manifolds and stabilization of beam equations with structural and Balakrishnan-Taylor damping</td>
</tr>
<tr>
<td>1078</td>
<td>Michael Böhm &amp; Mario Taboada</td>
<td>Global existence and regularity of solutions of the nonlinear string equation</td>
</tr>
<tr>
<td>1079</td>
<td>Zhangxin Chen</td>
<td>BDM mixed methods for a nonlinear elliptic problem</td>
</tr>
<tr>
<td>1080</td>
<td>J.J.L. Velázquez</td>
<td>On the dynamics of a closed thermostyphon</td>
</tr>
<tr>
<td>1081</td>
<td>Frédéric Bonnans &amp; Eduardo Casas</td>
<td>Some stability concepts and their applications in optimal control problems</td>
</tr>
<tr>
<td>1082</td>
<td>Hong-Ming Yin, $L^{2,\mu}(Q)$-estimates for parabolic equations and applications</td>
<td>Smoothing and decay properties of solutions of the Korteweg-de Vries equation on a periodic domain with point dissipation</td>
</tr>
<tr>
<td>1083</td>
<td>David L. Russell &amp; Bing-Yu Zhang</td>
<td>Fluids of differential type: Critical review and thermodynamic analysis</td>
</tr>
<tr>
<td>1084</td>
<td>J.E. Dunn &amp; K.R. Rajagopal</td>
<td>Global stabilization of the von Kármán plate with boundary feedback acting via bending moments only</td>
</tr>
<tr>
<td>1085</td>
<td>Mary Elizabeth Bradley &amp; Mary Ann Horn</td>
<td>Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback</td>
</tr>
<tr>
<td>1086</td>
<td>Mary Ann Horn &amp; Irena Lasiecka</td>
<td>Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback</td>
</tr>
<tr>
<td>1087</td>
<td>Vilmos Komornik</td>
<td>Decay estimates for a petrovski system with a nonlinear distributed feedback</td>
</tr>
<tr>
<td>1088</td>
<td>Jesse L. Barlow</td>
<td>Perturbation results for nearly uncoupled Markov chains with applications to iterative methods</td>
</tr>
<tr>
<td>1089</td>
<td>Jong-Shenq Guo</td>
<td>Large time behavior of solutions of a fast diffusion equation with source</td>
</tr>
<tr>
<td>1090</td>
<td>Tongwen Chen &amp; Li Qiu</td>
<td>$H_{\infty}$ design of general multirate sampled-data control systems</td>
</tr>
<tr>
<td>1091</td>
<td>Satyanad Kichenassamy &amp; Walter Littman</td>
<td>Blow-up surfaces for nonlinear wave equations, I</td>
</tr>
<tr>
<td>1092</td>
<td>Nahum Shimkin</td>
<td>Asymptotically efficient adaptive strategies in repeated games, Part I: certainty equivalence strategies</td>
</tr>
<tr>
<td>1093</td>
<td>Caroline Fabre, Jean-Pierre Puel &amp; Enrike Zuazua</td>
<td>On the density of the range of the semigroup for semilinear heat equations</td>
</tr>
</tbody>
</table>
Robert F. Stengel, Laura R. Ray & Christopher I. Marrison, Probabilistic evaluation of control system robustness

H.O. Fattorini & S.S. Sritharan, Optimal chattering controls for viscous flow

Kathryn E. Lenz, Properties of certain optimal weighted sensitivity and weighted mixed sensitivity designs

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Nahum Shimkin, Extremal large deviations in controlled I.I.D. processes with applications to hypothesis testing

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Yuan Wang & Eduardo D. Sontag, Orders of input/output differential equations and state space dimensions

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