A BOUND ON THE EXPONENT OF A PRIMITIVE MATRIX USING BOOLEAN RANK

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A BOUND ON THE EXPONENT OF A PRIMITIVE MATRIX USING BOOLEAN RANK*

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Abstract. We present a bound on the exponent \( \exp(A) \) of an \( n \times n \) primitive matrix \( A \) in terms of its boolean rank \( b = b(A) \); namely \( \exp(A) \leq (b - 1)^2 + 2 \). Further, we show that for each \( 2 \leq b \leq n - 1 \), there is an \( n \times n \) primitive matrix \( A \) with \( b(A) = b \) such that \( \exp(A) = (b - 1)^2 + 2 \), and we explicitly describe all such matrices. The new bound is compared to a well-known bound of Dulmage and Mendelsohn, and to a conjectured bound of Hartwig and Neumann. Several open problems are posed.

1. Introduction. A real \( n \times n \) (entrywise) nonnegative matrix \( A \) is primitive if one of its powers, \( A^k \), has all positive entries for some integer \( k \geq 1 \). The smallest such \( k \) is called the exponent of \( A \), and is denoted by \( \exp(A) \). Since a primitive matrix \( A \) can have no zero line (i.e. row or column), we see that each entry of \( A^m \) is positive whenever \( m \geq \exp(A) \). There is an extensive literature on exponents of primitive matrices, and a good survey of results and references can be found in Braudi and Ryser [1; Section 3.5].

It is clear that neither the primitivity nor the exponent of a primitive matrix depend on the size of the nonzero entries in a nonnegative matrix \( A \); they only depend on the location of the nonzero entries within \( A \). Consequently, it is natural to make the following definition. Two \( n \times n \) nonnegative matrices are said to be combinatorially equivalent if their nonzero entries are in precisely the same locations. Thus any matrix \( B \) which is combinatorially equivalent to a primitive matrix \( A \) is also primitive, and \( \exp(B) = \exp(A) \).

A number of authors have worked on obtaining upper bounds on the exponent of a primitive matrix. The earliest such bound is due to Wielandt [8], and we summarize his result below (a proof can be found in [1]).

Proposition 1.1. If \( A \) is an \( n \times n \) primitive matrix, then

\[
\exp(A) \leq (n - 1)^2 + 1.
\]

Further, equality holds in (1.1) if and only if there is a permutation matrix \( P \) such that \( PAP^t \) is combinatorially equivalent to \( W_n \), where \( W_1 = [1] \), \( W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), and

\[
W_n = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & & \ddots & \ddots & 0 \\
1 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\text{ for } n \geq 3.
\]

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Wielandt’s bound was later generalized by Dulmage and Mendelsohn [2]. They showed that if $A$ is an $n \times n$ primitive matrix, and $s$ is the smallest integer $k \geq 1$ such that $A^k$ has a diagonal entry, then

\[(1.2) \quad \exp(A) \leq n + s(n - 2).\]

We will provide another upper bound on the exponent in terms of the “boolean rank” of a nonnegative matrix, a concept which we now discuss. Given a nonzero $m \times n$ nonnegative matrix $A$, we define its boolean rank, $b(A)$, to be the smallest integer $k$ such that for some nonnegative matrices $F$ and $G$ with $F$ $m \times k$ and $G$ $k \times n$, $A$ is combinatorially equivalent to the product $FG$. The boolean rank of the zero matrix is defined to be 0. (In Section 2, we discuss the notion of boolean rank in more detail).

It follows from Theorem 3.1 below that for any $n \times n$ primitive matrix $A$, if $b(A) = b$ and $1 < b < n$, then

\[(1.3) \quad \exp(A) \leq (b - 1)^2 + 2.\]

In Theorem 3.2, we show that the bound (1.3) can be attained for each such $b$, and we give an explicit characterization of those matrices that do so. That extends Proposition 1.1 above, which characterizes the matrices realizing equality in (1.1) (note that those matrices all have boolean rank $n$).

In Section 4, we compare Dulmage and Mendelsohn’s bound (1.2) with (1.3), showing by example that sometimes it is better and sometimes it is worse. Examples illustrating other points of interest are also presented there. Finally, we suggest some open problems in Section 5.

Throughout, our results are obtained by using the machinery of $(0, 1)$ boolean matrices. In Section 2 below, we give the relevant definitions and basic properties of these matrices.

2. Preliminaries. Given a nonnegative matrix $A$, we define its pattern, $A_+$, to be the $(0, 1)$ matrix which is combinatorially equivalent to $A$. In computing sums and products of $(0, 1)$ matrices and vectors, we will use boolean arithmetic – that is, the usual arithmetic except that $1 + 1 = 1$. It now follows that for any $m \times n$ nonnegative matrices $A$ and $B$, and any $n \times k$ nonnegative matrix $C$, $(A + B)_+ = A_+ + B_+$ and $(AC)_+ = A_+ C_+$. The $(0, 1)$ matrices under this boolean arithmetic are called $(0, 1)$ boolean (or just boolean) matrices. The book of Kim [6] gives an extensive treatment of boolean matrix theory.

The definitions of primitivity and the exponent of a primitive matrix extend quite naturally to $(0, 1)$ boolean matrices. Specifically, if $B$ is an $n \times n$ $(0, 1)$ boolean matrix, then $B$ is primitive if one of its powers, $B^k$, has no zero entry, in which case, its exponent, $\exp(B)$, is the smallest such integer $k$. Thus we see that a nonnegative matrix $A$ is primitive if and only if its pattern $A_+$ (a boolean matrix) is, and that when both are
primitive, \( \exp(A) = \exp(A_+) \). For this reason, we will henceforth confine our investigation of the exponent to \((0,1)\) boolean matrices.

Notice that for any nonnegative matrix \(A\), the boolean ranks of \(A\) and \(A_+\) are the same. Since we will be concerning ourselves with \((0,1)\) boolean matrices (such as \(A_+\)) throughout the sequel, we will take a brief look at the boolean rank of a \((0,1)\) boolean matrix. If \(B\) is an \(m \times n\) \((0,1)\) boolean matrix, it is not difficult to see that \(b(B)\) is the smallest integer \(k\) such that there are \((0,1)\) boolean matrices \(X\) and \(Y\), with \(X\) \(m \times k\) and \(Y\) \(k \times n\), and \(B = XY\). (Indeed, this is the standard definition of boolean rank for a \((0,1)\) boolean matrix; see Kim [6] for example, who calls it “Schein rank”.) If \(b(B) = b\), \(X\) is \(m \times b\), \(Y\) is \(b \times n\) and the product \(XY\) equals \(B\), we will say that \(B = XY\) is a \((boolean) rank factorization of \(B\). The boolean rank of \(B\) can also be thought of as the minimum number of matrices of boolean rank 1 (i.e. outer products of the form \(xy^t\) where \(x\) is a \((0,1)\) boolean \(m\)-vector and \(y\) is a \((0,1)\) boolean \(n\)-vector) that sum to \(B\).

A set of positions \(\{(i_k, j_k) : 1 \leq k \leq l\}\) is said to be isolated (in the matrix \(B\)) if the following conditions hold:

(a) for each \(1 \leq k \leq l\), the \((i_k, j_k)\) entry of \(B\) is 1,

(b) no two positions in the set are in the same row or column of \(B\), and

(c) no two positions in the set are in the same \(2 \times 2\) all ones submatrix of \(B\).

Such a set of positions is referred to as a set of isolated ones (in \(B\)) (see Gregory and Pullman [3]). Using our second interpretation of boolean rank, it follows that \(b(B)\) is always bounded below by the cardinality of any set of isolated ones in \(B\). We will occasionally need this fact in order to estimate the boolean rank from below; in particular, the following result will be useful.

**Proposition 2.1.** Suppose that \(B\) is an \(m \times n\) \((0,1)\) boolean matrix, and that \(b(B) \leq k\). If \(B\) has a \(k\)-element set of isolated ones, then \(b(B) = k\).

Throughout the sequel, we will use the following notation and terminology. Given an \(m \times n\) matrix \(A\), we will denote its entry in the \((i,j)\) spot by \(A_{ij}\), its \(i^{th}\) row by \(A_i\), and its \(j^{th}\) column by \(A_j\). If \(B\) is another \(m \times n\) matrix, we say that \(B\) is dominated by \(A\), and write \(B \leq A\), if \(B_{ij} \leq A_{ij}\) for all \(i\) and \(j\); the same terminology and notation will be used for vectors. We denote the \(m \times n\) all ones matrix by \(J_{m,n}\) (and by \(J_n\) if \(m = n\)), the \(m \times n\) all zeroes matrix by \(0_{m,n}\), the all ones \(n\)-vector by \(j_n\), the \(n \times n\) identity matrix by \(I_n\), and its \(i^{th}\) column by \(e_i(n)\). The subscripts \(m\) and \(n\) will be omitted whenever their values are clear from the context.

3. **A bound on the exponent.** Our bound on the exponent rests on the following observations. The first was noted by Shao [7], and the second will be discussed further in Section 4.
PROPOSITION 3.1. Suppose that $X$ and $Y$ are $n \times m$ and $m \times n$ $(0,1)$ boolean matrices (respectively) and that neither has a zero line. Then

(a) $XY$ is primitive if and only if $YX$ is primitive, and

(b) if $XY$ and $YX$ are primitive, then

$$|\exp(XY) - \exp(YX)| \leq 1.$$ 

Proof. Suppose that $XY$ is primitive, with $\exp(XY) = k$. Then $(YX)^{k+1} = Y(XY)^k X = YJ_n X = J_m$, the last equality following from the fact that neither $X$ nor $Y$ has a zero line. Thus $YX$ is primitive and $\exp(YX) \leq \exp(XY) + 1$. The result follows by exchanging the roles of $X$ and $Y$. □

It follows from Proposition 3.1 that if $X_1, X_2, \ldots, X_d$ are $n_1 \times n_2, n_2 \times n_3, \ldots, n_d \times n_1$ $(0,1)$ boolean matrices (respectively) with no zero lines, then

(a) each of the cyclic products $X_1 X_2 \cdots X_d$, $X_2 X_3 \cdots X_d X_1$, $\ldots$, $X_d X_1 X_2 \cdots X_{d-1}$ is primitive if any one of them is, and

(b) in the case that each is primitive, the exponents of any two differ by at most 1.

These statements also follow from a result of Dulmage and Mendelsohn (see [1; pg 78, exercise 5]).

Now we apply our proposition to get an upper bound on the exponent of a primitive matrix.

THEOREM 3.1. Suppose that $n \geq 2$ and that $B$ is an $n \times n$ primitive $(0,1)$ boolean matrix with $b(B) = b$. Then

$$\exp(B) \leq (b - 1)^2 + 2.$$ 

 Proof. Let $B = XY$ be a boolean rank factorization of $B$. Then $X$ is $n \times b$, $Y$ is $b \times n$, and neither has a zero line. By Proposition 3.1, the $b \times b$ matrix $YX$ is primitive, and $\exp(B) \leq \exp(YX) + 1$. Wielandt’s inequality (1.1) implies that $\exp(YX) \leq (b - 1)^2 + 1$, and so (3.1) follows. □

From (1.1) we see that no matrix of boolean rank $n$ can realize equality in (3.1). Further, since the only $n \times n$ primitive $(0,1)$ boolean matrix of boolean rank 1 is $J_n$, we see that no matrix of boolean rank 1 can realize equality in (3.1). However, we will show that for each $1 < b < n$, there is an $n \times n$ matrix of boolean rank $b$ realizing equality in (3.1). In order to do so, we need to make some observations about the matrix $W_n$ of Proposition 1.1.
PROPOSITION 3.2. If \( n \geq 2 \), the only zero entry in \( W_n^{(n-1)^2} \) occurs in the \((n,n)\) position.

Proof. As always, we are thinking of \( W_n \) as a \((0,1)\) boolean matrix. The result is immediate if \( n = 2 \). For \( n \geq 3 \), a straightforward proof by induction on \( k \) shows that for \( 1 \leq k \leq n-1 \), \( W_n^k = \begin{bmatrix} U & I_{n-k} \\ V & 0_{k,n-k} \end{bmatrix} \), where the \((n-k) \times k\) matrix \( U \) is all 0 except for a 1 in the \((n-k,1)\) position, and the \( k \times k\) matrix \( V \) has 1's on the diagonal and superdiagonal, and 0's elsewhere. It follows that \( W_n^n = W_n + I \), and hence \( W_n^{(n-1)^2} = W_n(W_n + I)^{n-2} = \sum_{k=1}^{n-1} W_n^k \). The formulae for \( W_n^k, 1 \leq k \leq n-1 \) now yield the result. \( \square \)

Next, we give a characterization of the matrices for which equality holds in (3.1).

PROPOSITION 3.3. Suppose that \( B \) is an \( n \times n \) \((0,1)\) boolean matrix with \( 2 \leq b = b(B) \leq n-1 \). Then \( B \) is primitive with \( \exp(B) = (b-1)^2 + 2 \) if and only if \( B \) has a boolean rank factorization \( B = XY \), where \( X \) and \( Y \) have the following properties:

i) \( YX = W_b \), and

ii) some row of \( X \) is \( e_b^1(b) \) and some column of \( Y \) is \( e_b(b) \).

Proof. First, suppose that \( B \) is primitive with \( \exp(B) = (b-1)^2 + 2 \), and that \( B = XY \) is a boolean rank factorization of \( B \). By Proposition 3.1, \( \bar{Y} \bar{X} \) is primitive and \( \exp(\bar{Y} \bar{X}) \geq (b-1)^2 + 1 \). Thus by Proposition 1.1, \( \exp(\bar{Y} \bar{X}) = (b-1)^2 + 1 \), and so there is a permutation matrix \( P \) such that \( P \bar{Y} \bar{X} P^t = W_b \). Letting \( X = \bar{X} P^t \) and \( Y = P \bar{Y} \), we find that \( B = XY \) is a rank factorization of \( B \), and that \( YX = W_b \). Thus \( X \) and \( Y \) satisfy i).

Since \( B \) is primitive, it follows that \( \sum_{i=1}^b X_i = j_n = \sum_{i=1}^b Y_i^t \). Further, since \( \exp(B) = (b-1)^2 + 2 \), \( B^{(b-1)^2+1} \) must have a zero entry, say in the \((p,q)\) position. But by Proposition 3.2, \( B^{(b-1)^2+1} = XW_b^{(b-1)^2}Y = XZY \), where \( Z \) is the \( b \times b \) matrix whose only zero entry is in the \((b,b)\) position. Thus

\[
B^{(b-1)^2+1} = \begin{bmatrix} J_{n,b-1} & \sum_{i=1}^{b-1} X_i Y_i^t & \sum_{i=1}^{b-1} X_i \end{bmatrix} Y = j_n \left( \sum_{i=1}^{b-1} Y_i^t \right) + \left( \sum_{i=1}^{b-1} X_i \right) Y_b,
\]

and hence \( \sum_{i=1}^{b-1} Y_{i q} + \sum_{i=1}^{b-1} X_{pi} Y_{bq} = 0 \). So the first \( b-1 \) entries in \( Y_{-q} \) are zero, and since \( Y \) has no zero lines, we see that \( Y_q \) must equal \( e_b(b) \). Hence the first \( b-1 \) entries of \( X_p \) are zero also, and we find that \( X_p \) must equal \( e_b^1(b) \). Consequently, \( X \) and \( Y \) satisfy ii).

Finally, suppose that \( B = XY \) is a rank factorization of \( B \) and that \( X \) and \( Y \) satisfy i) and ii). Then \( B \) is primitive by Proposition 3.1 (a) and \( \exp(B) \leq (b-1)^2 + 2 \) by
Proposition 3.1 (b) and Theorem 1.1. But it follows from Proposition 3.2 and conditions i) and ii) that $B^{(b-1)^2+1}$ has a zero entry, and so we conclude that $\exp(B) = (b - 1)^2 + 2$. \[\square\]

Our last result of this section will reinterpret conditions i) and ii) of Proposition 3.3 to show that if $B$ yields equality in (3.1), then $B$ is one of 14 basic types of matrices.

**Theorem 3.2.** Suppose $B$ is an $n \times n$ $(0,1)$ boolean matrix with $b(B) = b$, and that $2 \leq b \leq n - 1$. Then $B$ is primitive with $\exp(B) = (b - 1)^2 + 2$ if and only if there is a permutation matrix $Q$ such that $QBQ^t$ has one of the forms displayed in Table 3.1 (if $3 \leq b \leq n - 1$) or Table 3.2 (if $b = 2$).
Table 3.1 (for $b \geq 3$)

$$
M_1 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \\
M_2 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \\
M_3 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \\
M_4 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \\
M_5 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \\
M_6 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}, \\
M_7 = \begin{bmatrix}
0 & J & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & J & 0 & J \\
0 & \ldots & 0 & J & 0 \\
J & 0 & \ldots & 0 & 0 \\
\end{bmatrix}.
$$

Here the rows and columns of $M_1, \ldots, M_7$ are partitioned conformally, so that each diagonal block is square, and the top left hand submatrix common to each has $b$ blocks in its partitioning.
Table 3.2 (for \( b = 2 \))

\[
N_1 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
\end{bmatrix}, \quad N_2 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
\end{bmatrix}, \quad N_3 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
\end{bmatrix},
\]

\[
N_4 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
J & J \\
\end{bmatrix}, \quad N_5 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
J & J \\
\end{bmatrix}, \quad N_6 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
J & J \\
\end{bmatrix},
\]

\[
N_7 = \begin{bmatrix}
0 & J \\
J & 0 \\
J & J \\
J & J \\
\end{bmatrix}.
\]

Again, the rows and columns of \( N_1, \ldots, N_7 \) are partitioned conformally.

**Proof.** Suppose that \( B \) is primitive, \( b \geq 3 \) and \( \exp(B) = (b - 1)^2 + 2 \); by Proposition 3.3, there is a boolean rank factorization of \( B \) as \( B = XY \) such that i) \( YX = W_b \), and ii) some row of \( X \) is \( e^t_b(b) \) and some column of \( Y \) is \( e_b(b) \). Since \( B \) is primitive, \( X \) has no zero row, and so each column of \( Y \) is dominated by a column of \( W_b \). Similarly, each row of \( X \) is dominated by a row of \( W_b \). Thus, each column of \( Y \) is in the set \( \mathcal{C} = \{e_1(b), e_2(b), \ldots, e_b(b), u\} \), where \( u = e_{b-1}(b) + e_b(b) \), and each row of \( X \) is in the set \( \mathcal{R} = \{e^t_1(b), \ldots, e^t_b(b), v^t\} \), where \( v = e_1(b) + e_b(b) \).

Next, we note that for each \( 1 \leq i \leq b \), the outer product \( Y_iX_i \) is dominated by \( W_b \). Since each such \( Y_i \) and \( X_i \) must be in \( \mathcal{C} \) and \( \mathcal{R} \) respectively, we find that \( (Y_i, X_i) \) must be one of the following pairs: \((e_i, e^t_{i+1})\), \(1 \leq i \leq b - 1\), \((e_{b-1}, e^t_1)\), \((e_1, e^t_b)\), \((u, e^t_1)\), and \((e_{b-1}, v^t)\). For each \( 1 \leq i \leq b - 2 \), \( e_i \) is column of \( W_b \) that contains the only nonzero entry of row \( i \). Thus, for each such \( i \), \((e_i, e^t_{i+1}) = (Y_{k_i}, X_{k_i}) \) for some \( k_i \). Further, from ii), some \( X_j \) is \( e^t_b \), so \((e_{b-1}, e^t_b) = (Y_{k_{b-1}}, X_{k_{b-1}}) \) for some \( k_{b-1} \). Similarly, ii) implies that some \( Y_j \) is \( e_b \), so \((e_b, e^t_1) = (Y_{k_b}, X_{k_b}) \) for some \( k_b \). Finally, some outer product \( Y_jX_j \) must have a 1 in the \((b - 1, 1)\) spot, and hence for some \( k_{b+1} \), \((Y_{k_{b+1}}, X_{k_{b+1}}) \) is one of \((e_{b-1}, e^t_1)\), \((u, e^t_1)\), or \((e_{b-1}, v^t)\).

From the above considerations, it follows that there is an \( n \times n \) permutation matrix \( Q \) such that \( YQ^t = [\bar{Y} | \bar{Y}] \) and \( QX = [\bar{X} | \bar{X}] \), where \( \bar{Y} = [e_1j^t_{n_1} | e_2j^t_{n_2} | \cdots | e_bj^t_{n_b}] \) and \( \bar{X} = \).
\[
\begin{bmatrix}
\frac{j_{n_1} e_2^t}{j_{n_2} e_3^t} \\
\frac{j_{n_2} e_3^t}{j_{n_3} e_4^t} \\
\vdots \\
\frac{j_{n_{b-1}} e_b^t}{j_{n_b} e_1^t}
\end{bmatrix}
\]
for some \(n_1, \ldots, n_b \geq 1\), and where each \((\bar{Y}_i, \bar{X}_i)\) is one of \((e_{b-1}, e_1^t)\), \((u, e_1^t)\), or \((e_{b-1}, v^t)\). Thus the pair \(\bar{Y}\) and \(\bar{X}\) can be taken to be one of the following pairs of matrices:

\[
\begin{align*}
\bar{Y}_1 &= e_{b-1} j_{m_1}^t, \bar{X}_1 = j_{m_1} e_1^t \text{ for some } m_1 \geq 1; \\
\bar{Y}_2 &= u j_{m_2}^t, \bar{X}_2 = j_{m_2} e_1^t \text{ for some } m_2 \geq 1; \\
\bar{Y}_3 &= e_{b-1} j_{m_3}^t, \bar{X}_3 = j_{m_3} v^t \text{ for some } m_3 \geq 1; \\
\bar{Y}_4 &= [e_{b-1} j_{m_4}^t | u j_{p_4}^t], \bar{X}_4 = \begin{bmatrix} j_{m_4} e_1^t \\ j_{p_4} e_1^t \end{bmatrix} \text{ for some } m_4, p_4 \geq 1; \\
\bar{Y}_5 &= [e_{b-1} j_{m_5}^t | e_{b-1} j_{p_5}^t], \bar{X}_5 = \begin{bmatrix} j_{m_5} e_1^t \\ j_{p_5} v^t \end{bmatrix} \text{ for some } m_5, p_5 \geq 1; \\
\bar{Y}_6 &= [u j_{m_6}^t | e_{b-1} j_{p_6}^t], \bar{X}_6 = \begin{bmatrix} j_{m_6} e_1^t \\ j_{p_6} v^t \end{bmatrix} \text{ for some } m_6, p_6 \geq 1; \\
\bar{Y}_7 &= [e_{b-1} j_{m_7}^t | u j_{p_7}^t | e_{b-1} j_{q_7}^t], \bar{X}_7 = \begin{bmatrix} j_{m_7} e_1^t \\ j_{p_7} e_1^t \\ j_{q_7} v^t \end{bmatrix} \text{ for some } m_7, p_7, q_7 \geq 1.
\end{align*}
\]

It is now readily verified that \(\begin{bmatrix} \bar{X} \\ \bar{X} \end{bmatrix} + [\bar{Y}] = M_i\) for \(1 \leq i \leq 7\), so that \(QBQ^t\) is one of the matrices in Table 3.1. Minor modifications of this argument show that if \(b = 2\), \(B\) is primitive and \(\exp(B) = (2 - 1)^2 + 2 = 3\), then \(B\) is permutationally similar to one of the matrices in Table 3.2.

Finally, since the rank factorization \(M_i = \begin{bmatrix} \bar{X} \\ \bar{X} \end{bmatrix} + [\bar{Y}]\) satisfies conditions i) and ii) of Proposition 3.3, we see that each \(M_i\) is primitive and \(\exp(M_i) = (b - 1)^2 + 2\); a similar argument applies to the \(N_i\)'s in Table 3.2. \]

4. Comparisons. If \(B\) is an \(n \times n\) \((0, 1)\) boolean matrix and \(b(B) = n\), then obviously Wielandt's bound (1.1) on \(\exp(B)\) is sharper than that of (3.1). However, if \(b(B) \leq n - 1\) we see that (3.1) is an improvement on (1.1). It is natural to wonder how (3.1) compares with Dulmage and Mendelsohn's bound (1.2); the example below shows that they are not comparable in general.
Example 4.1. For any $n \geq 3$, let $A$ be the $n \times n$ matrix
\[
A = \begin{bmatrix}
1 & 1 & \cdots & 1 & 0 \\
1 & 0 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & 1 & 1
\end{bmatrix},
\]
and note that $\exp(A) = 2$. Since $A$ has exactly two distinct rows, we find that $b(A) = 2$. Thus the right hand side of (3.1) is equal to 3. Since $A$ has a one on the diagonal (indeed several), the right hand side of (1.2) is $2n - 2$. Thus for the matrix $A$, (3.1) is a sharper bound than (1.2).

Next, for any $n \geq 5$, let $B$ be the $n \times n$ matrix
\[
B = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 1 & 1 \\
1 & 1 & 0 & \cdots & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]
A straightforward calculation shows that $\exp(B) = 3$. Since the $n-1^{\text{st}}$ and $n-2^{\text{nd}}$ rows of $B$ are the same, $b(B) \leq n - 1$. Now $B$ has a set of isolated ones in positions $(1,1), (2,2), \ldots, (n-2, n-2), (n, n-1)$, and so $b(B) = n - 1$ by Proposition 2.1. Thus the right side of (3.1) is equal to $(n-2)^2 + 2$. However, the right side of (1.2) is equal to $2n - 2$ since $B$ has a one on the diagonal, so we see that for $B$, (1.2) is sharper than (3.1).

Finally, we remark that if $2 \leq n \leq 4$ and $b \leq n - 1$, then $n + s(n-2) \geq 2n - 2 \geq (n-2)^2 + 2 \geq (b-1)^2 + 2$ for any $s \geq 1$, so that for such $n$ and $b$, (1.2) is never tighter than (3.1).

It is interesting to examine the bound (3.1) in connection with some recent work on exponents. In a working paper, Hartwig [4] conjectured that if $M$ is a primitive matrix, then

\begin{equation}
\exp(M) \leq d^2 + 1,
\end{equation}

where $d = d(M)$ is the smallest integer $k$ such that $I + M + M^2 + \cdots + M^k = J$. Now $d$ is the diameter of the directed graph associated with $M$, and since the positions in $M$ corresponding to the arcs of a shortest directed path must be isolated in $M$, it follows that $d(M) \leq b(M)$ for any primitive $M$. By looking at the directed graph associated with the matrix $A$ of Example 4.1, we find that $d(A) = 2$. But $b(A) = 2$ as well, so the bound (3.1) is better than the conjectured bound (4.1) for this example. However, if $M$ is a primitive matrix such that $d(M) < b(M)$, then (4.1) is better than (3.1). The matrix $B$ of Example 4.1 provides an example of such a matrix since $b(B) = n - 1$, while it is easily seen that $d(B) = 3$. 
Recently, Hartwig and Neumann [5] have conjectured that if $M$ is a real primitive matrix, then

\[(4.2) \quad \exp(M) \leq (m - 1)^2 + 1,\]

where $m = m(M)$ is the degree of the minimal polynomial of $M$. Since $M^m_n \leq (I + M + \cdots + M^{m-1})_+$, it follows that $d \leq m - 1$ as long as $M$ is irreducible. (There is no simple inequality relating $m(M)$ and $b(M)$ however). This conjecture is weaker than (4.1) and has been verified in [5] for all but a few cases. As before, (3.1) is sometimes better than (4.2) and sometimes worse. For example, if the $J$ blocks in the matrix $M_1$ of Table 3.1 are all $1 \times 1$ (or even if they are all square and of the same size), then (thinking of $M_1$ as a real (0,1) matrix for the moment) $m(M_1) \leq b(M_1) + 1$. But $d(M_1) = b(M_1)$ here, so it follows that $m(M_1) = b(M_1) + 1$. Thus (3.1) is better than (4.2) in this case. On the other hand, let $M = I + H$ where $H$ is an $n \times n$ skew-Hadamard design; that is, $H$ is a real (0,1) matrix such that $H + H^t = J - I$, and \( HH^t = \left( \frac{n-3}{4} \right) J + \left( \frac{n+1}{4} \right) I \). (Here real matrix arithmetic is used). Then $M$ is normal with three distinct eigenvalues, and so $m = 3$. However, the diagonal 1's of $M_+$ are isolated, so $b = n$. Consequently, the conjectured bound (4.2) is better than (3.1) in this case.

Our last comparison takes its cue from Proposition 3.1 (b). Suppose that $B$ is a primitive (0,1) boolean matrix and that $B = XY$ is a boolean rank factorization of $B$. According to Proposition 3.1(b), $\exp(B) = \exp(YX) + \varepsilon$, where $\varepsilon$ is 1, -1, or 0. The next result shows that all three possible values of $\varepsilon$ can be realized when $\exp(YX)$ is maximum.

**Theorem 4.1.** Suppose that $\varepsilon$ is either 1, -1, or 0 and that $2 \leq b \leq n - 1$. There is a primitive $n \times n$ (0,1) boolean matrix $B$ with boolean rank $b$ and a boolean rank factorization $B = XY$ such that $\exp(YX) = (b - 1)^2 + 1$ and $\exp(B) = \exp(YX) + \varepsilon$.

**Proof.** For $\varepsilon = 1$, consider any matrix $B$ in Table 3.1 (or Table 3.2 if $b = 2$). Then $b(B) = b$ and $\exp(B) = (b - 1)^2 + 2$ by Theorem 3.2. It follows from Propositions 3.1 and 1.1 that if $B = XY$ is any rank factorization of $B$, then $\exp(YX) = (b - 1)^2 + 1$, so that $\exp(B) = \exp(YX) + \varepsilon$ with $\varepsilon = 1$.

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
\vdots \\
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots \\
0 & 1 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots \\
0 & \cdots & 0 \\
1 & \cdots & 1 \\
1 & \cdots & 1 \\
0_{n-b,n-b}
\end{bmatrix}
\]

For $\varepsilon = -1$, consider the $n \times n$ matrix $B = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
\vdots \\
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots \\
0 & 1 & 0 & \cdots & 0
\end{bmatrix}$. Note that $B$ has a set of isolated ones in positions $(b,1)$ and $(i,i+1)$, $1 \leq i \leq b - 1$. Now
let $X$ be the $n \times b$ matrix $X = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$ and let $Y$ be the $b \times n$ matrix

\[
Y = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \vdots \\ 0 \cdots 0 \\ 1 \cdots 1 \\ 1 \cdots 1 \end{bmatrix}
\]

Then $XY = B$, so $b(B) \leq b$ and hence $b(B) = b$ by Proposition 2.1. Further, $YX = W_b$, so $\exp(YX) = (b-1)^2 + 1$ by Proposition 1.1.

Let the principal submatrix of $B$ on its first $b$ rows and columns be $C$. Note that $B$ can be written as

\[
\begin{bmatrix} C & C \epsilon_1 j_{n-b}^t \\ j_{n-b} \epsilon_1^t C & C_{11} J_{n-b} \end{bmatrix}
\]

A straightforward proof by induction on $k$ shows that for any $k \geq 1$, $B^k = \begin{bmatrix} C^k & C^{k-1} \epsilon_1 j_{n-b}^t \\ j_{n-b} \epsilon_1^t C^k & C_{11}^{(k)} J_{n-b} \end{bmatrix}$, where $C_{11}^{(k)}$ is the $(1,1)$ entry of $C^k$. Thus we find that $\exp(B) = \exp(C)$. But $C$ is known to have exponent $(b-1)^2$ (see Brualdi and Ryer, [1; pg. 83]). So we see that in this case, $\exp(B) = \exp(YX) + \varepsilon$ with $\varepsilon = -1$.

For $\varepsilon = 0$, consider the $n \times n$ matrix $\tilde{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0_{n-b,n-b} \end{bmatrix}$.

Again, $\tilde{B}$ has a set of $b$ isolated ones, and $\tilde{B} = \tilde{X}Y$, where $Y$ is as above, and $\tilde{X} = \ldots$
\[
\begin{bmatrix}
I_b \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix}
\]. Thus by Proposition 2.1, \( b(\vec{B}) = b \). An argument similar to the one above shows that the exponent of \( \vec{B} \) is the same as that of its principal submatrix on the first \( b \) rows and columns. But that submatrix is \( W_b \), so we have \( \exp(\vec{B}) = (b - 1)^2 + 1 \). However, \( \vec{Y}X = W_b \), so we see that in this case, \( \exp(\vec{B}) = \exp(\vec{Y}X) + \varepsilon \) with \( \varepsilon = 0 \).

5. Problems. In this section, we suggest a few problems arising from the ideas in the preceding sections.

1. Let \( E_b \) be the set whose members are the exponents of the \( b \times b \) primitive matrices, and let \( E_{n,b} \) be the set whose members are the exponents of the \( n \times n \) primitive matrices of boolean rank \( b \). It is not difficult to show that \( E_{n,b} \subseteq E_{n+1,b} \), and Theorem 3.1 implies that \( \lim_{n \to \infty} E_{n,b} \) exists. Can this limit set be described more explicitly? Perhaps in terms of \( E_b \)?

2. Fix an \( n \geq 3 \), a \( b \) with \( 2 \leq b \leq n - 1 \), and an \( \varepsilon \) equal to either 1, -1, or 0. Given \( k \in E_b \), is there an \( n \times n \) primitive matrix \( B \) with \( b(B) = b \) which has a boolean rank factorization \( B = XY \) such that \( \exp(YX) = k \) and \( \exp(B) = k + \varepsilon \)? Theorem 4.1 answers the question in the affirmative when \( k = (b - 1)^2 + 1 \), but what about other values of \( k \)?

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