ON THE NONSMOOTH VERIFICATION TECHNIQUE
FOR THE DYNAMIC PROGRAMMING OF VISCOUS FLOW

By

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On The Nonsmooth Verification Technique for The Dynamic Programming Of Viscous Flow

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Abstract

In this paper we continue our investigation of the dynamic programming technique for the Navier-Stokes equations. We establish in particular a relationship between the notion of viscosity solution in the sense of Crandall and Lions and the generalized solution in the sense of Clarke for the infinite dimensional Hamilton-Jacobi-Bellman equation. This result is then used to clarify the verification technique for the feedback control problem.

1 Introduction

Optimal control theory of viscous fluid motion has many important applications in engineering science. In [16, 18, 17, 19, 9] a number of major

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results were established for the feedback control of large class of viscous flow problems. Main results were the existence theorem for optimal control, Pontryagin maximum principle for the necessary conditions and the verification theorem for the dynamic programming to provide the sufficient conditions. In this paper, we will elaborate the nonsmooth analytic aspects of the problem to bring more light in to the picture.

In section (2) we define two notions of generalized solutions for the infinite dimensional Hamilton-Jacobi-Bellman equation and establish a connection between them.

In section (3) we formulate an infinite dimensional nonlinear optimal control problem with unbounded nonlinearity and cost functional. This general problem covers a large class of viscous flow control problems. As an example we discuss an optimal control problem in exterior hydrodynamics. We then state the existence theorem for optimal control and the Pontryagin maximum principle for the necessary conditions which were proved in our earlier papers. A "nonsmooth" version of the maximum principle is also derived.

In section (4) we use the results of the two previous sections to clarify feedback synthesis using the Hamilton-Jacobi-Bellman equation satisfied by the value function. The concepts of smooth and nonsmooth verification functions are explained. The central theorem of this section is that, optimality is verified if there exists a locally Lipschitz function satisfying the Hamilton-Jacobi-Bellman in the "nonsmooth" sense of Clarke.

2 Nonsmooth solutions to the Hamilton-Jacobi-Bellman equation

Let $S, X$ be separable Hilbert spaces with the embedding $S \subset X$ continuous and dense. Identifying the dual $X^* = L(X; R)$ with $X$ using the Riesz representation theorem we get,

$$S \subset X = X^* \subset S^*$$

with all embeddings continuous and dense. Let us begin with a definition for the Hamiltonian.

Definition 1 A function $\mathcal{H} : [0,T] \times S \times X \to R$ will be called an admissible Hamiltonian if it satisfies the following two conditions.
(i) Sequentially weakly lower semicontinuous in the following sense: if \( t^n \to t \), \( y^n \to y \) strongly in \( S \) and \( p^n \to p \) weakly in \( X \) then
\[
\liminf_{n \to \infty} \mathcal{H}(t^n, y^n, p^n) \geq \mathcal{H}(t, y, p).
\]

(ii) \( \mathcal{H}(t, y, \cdot) : X \to \mathbb{R} \) is convex:
\[
\mathcal{H}(\cdot, \cdot, \lambda p + (1 - \lambda)q) \leq \lambda \mathcal{H}(\cdot, \cdot, p) + (1 - \lambda)\mathcal{H}(\cdot, \cdot, q), \quad \forall \lambda \in (0, 1), \forall p, q \in X.
\]

We are interested in verifying a locally Lipschitz solution \( \mathcal{V} : [0, T] \times X \to \mathbb{R} \) to the Hamilton-Jacobi-Bellman equation,
\[
\partial_t \mathcal{V} - \mathcal{H}(t, y, \partial_y \mathcal{V}) = 0, \quad \forall (t, y) \in (0, T) \times S \tag{1}
\]
with
\[
\mathcal{V}(T, y) = \Phi_0(y), \quad \forall y \in X.
\]

Here \( \Phi_0(\cdot) : X \to \mathbb{R} \) is a continuous function.

The following recent result by Preiss [13] sheds more light into the situation.

**Proposition 1** Let \( Y \) be a separable Hilbert space and \( f : Y \to \mathbb{R} \) be locally Lipschitz. Then \( f \) is Frechet differentiable in a dense set \( \Sigma_Y \subset Y \).

This result provides us a hope for satisfying the equation (1) classically by a locally Lipschitz function in a dense set of \((0, T) \times S\). For the verification theorem however, we need a characterization of the solution in all of \((0, T) \times S\). In what follows the concept of viscosity solution [7, 6] and the notion of Clarke generalized solution [4, 5] are generalized to infinite dimensions and then used to resolve the above problem. We will begin by recalling the characterizations of various notions of generalized derivatives for a locally Lipschitz function [4, 6, 1].

**Definition 2** Let \( Y \) be a separable Hilbert space and \( f : Y \to \mathbb{R} \) be locally Lipschitz function.

(i) The superdifferential of \( f \) at a point \( x \in Y \) is the subset of \( Y^* \) defined as
\[
\partial^+ f(x) = \left\{ \zeta \in Y^* : \limsup_{y \to x} \left[ \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \right] \leq 0 \right\}. \tag{2}
\]
(ii) The subdifferential of \( f \) at a point \( x \in Y \) is the subset of \( Y^* \) defined as

\[
\partial^- f(x) = \left\{ \zeta \in Y^*; \liminf_{y \to x} \left[ \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \right] \geq 0 \right\}.
\] (3)

(iii) The Clarke generalized gradient of \( f \) at a point \( x \in Y \) is the subset of \( Y^* \) defined as

\[
\partial f(x) = \left\{ \zeta \in Y^*; f^0(x; v) \geq \langle \zeta, v \rangle, \quad \forall v \in Y \right\},
\] (4)

where \( f^0(x; v) \) denotes the directional derivative,

\[
f^0(x; v) = \limsup_{t \to 0, y \to x} \frac{f(y + tv) - f(y)}{t}.
\] (5)

Let us now list some useful properties of these derivatives.

**Proposition 2:** Let \( Y \) be a separable Hilbert space and \( f : Y \to R \) be a locally Lipschitz function. Then for any point \( x \in Y \),

(I.i) the sets \( \partial^+ f(x) \), \( \partial^- f(x) \) and \( \partial f(x) \) are closed convex,

\[
(I.ii) \quad -\partial^- (-f)(x) = \partial^+ f(x),
\] (6)

\[
(I.iii) \quad -\partial^- f(x) = \partial f(x),
\] (7)

and \( (I.iv) \quad \partial^- f(x) \cup \partial^+ f(x) \subseteq \partial f(x) \subseteq Y^*. \) (8)

Moreover,

\[
(II) \quad \partial f(x) = \bigcap_{s > 0} \overline{\partial} \{ \nabla f(y); \quad y \in B_Y(x; s) \cap \Sigma_Y \}
\] (9)

where \( B_Y(x; s) \) is the ball in \( Y \) of radius \( s \) centered at \( x \). The closure here is taken in the weak-star topology.

**Proof:** Properties (I.i), (I.ii) and (I.iii) were proved in [6, 4]. Property (II) was proved in [13]. Let us establish property (I.iv). Consider an arbitrary element \( \zeta \in \partial^- f(x) \). Then by definition,

\[
\sup_{\epsilon > 0} \inf_{\|y - x\| < \epsilon} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0.
\] (10)
The special choice \( y = x + tv, t > 0, v \in Y \) will not decrease the left hand side of (10) and hence,

\[
\sup_{c > 0} \inf_{0 < t < c} \frac{f(x + tv) - f(x)}{t} \geq \langle \zeta, v \rangle, \quad \forall v \in Y.
\]

Hence also

\[
\inf_{c > 0} \sup_{0 < t < c} \frac{f(x + tv) - f(x)}{t} \geq \langle \zeta, v \rangle, \quad \forall v \in Y.
\]  \( (11) \)

But the directional derivative (see (5)) of \( f \) given by

\[
f^0(x; v) = \inf_{c > 0} \inf_{\delta > 0} \sup_{0 < t < \delta} \sup_{||z - x|| < \delta} \frac{f(z + tv) - f(z)}{t}
\]

is not smaller than the left hand side of (11) and hence

\[
f^0(x; v) \geq \langle \zeta, v \rangle, \quad \forall v \in Y.
\]

This implies, by the definition of the Clarke generalized gradient \( \zeta \in \partial f(x) \) and hence \( \partial^- f(x) \subseteq \partial f(x) \). Now,

\[
\partial^+ f(x) = -\partial^-( -f)(x) \subseteq -\partial(-f)(x) = \partial f(x).
\]

Thus \( \partial^- f(x) \cup \partial^+ f(x) \subseteq \partial f(x) \).

\[ \hfill \]

We recall here that the classical subdifferential in the sense of convex analysis [2] is defined as

\[
\partial^{CV} f(x) = \{ \zeta \in Y^*; f(y) - f(x) - \langle \zeta, y - x \rangle \geq 0, \ \forall y \in Y \}.
\]

It is evident that \( \partial^{CV} f(x) \subseteq \partial^- f(x) \). In fact Clarke [4] shows that when \( f(\cdot) \) is convex and locally Lipschitz \( \partial^{CV} f(x) = \partial f(x) \) and hence we have in this case

\[
\partial^{CV} f(x) = \partial^- f(x) = \partial f(x).
\]

Example:
(i) (upward corner) Let \( f(\cdot) : Y \to R \) be defined as \( f(x) = \|x\|_Y \). Then,
\[
\partial^{cv} f(0) = \partial^- f(0) = \partial f(0) = \bar{B}_{Y^*}(0, 1),
\]
where \( \bar{B}_{Y^*}(0, 1) \) is the closed unit ball in \( Y^* \) centered at the origin; and \( \partial^+ f(0) = \emptyset \).

(ii) (downward corner) If we define \( g(\cdot) : Y \to R \) as \( g(x) = -\|x\|_Y \), then
\[
\partial^+ g(0) = \partial g(0) = \bar{B}_{Y^*}(0, 1)
\]
and
\[
\partial^{cv} g(0) = \partial^- g(0) = \emptyset.
\]
This example will also be useful in understanding the verification theorem in section (4).

We now return to the Hamilton-Jacobi-Bellman equation (1) and define generalized solutions.

**Definition 3 (Viscosity solutions)** Let \( V : [0, T] \times X \to R \) be a locally Lipschitz function. Then \( V \) is called a viscosity subsolution to the equation (1) if for each \( (t, y) \in (0, T) \times S \),
\[
- \zeta + \mathcal{H}(t, y, \xi) \leq 0, \quad \forall (\zeta, \xi) \in \partial^+ V(t, y) \tag{12}
\]
and viscosity supersolution if for each \( (t, y) \in (0, T) \times S \),
\[
- \zeta + \mathcal{H}(t, y, \xi) \geq 0, \quad \forall (\zeta, \xi) \in \partial^- V(t, y). \tag{13}
\]
If \( V \) satisfies (12) and (13) then it is called a viscosity solution.

**Definition 4 (Clarke generalized solutions)** Let \( V : [0, T] \times X \to R \) be a locally Lipschitz function. Then \( V \) is called a Clarke generalized solution to the equation (1) if for each \( (t, y) \in (0, T) \times S \),
\[
\max \{-\zeta + \mathcal{H}(t, y, \xi); \quad (\zeta, \xi) \in \partial V(t, y)\} = 0. \tag{14}
\]
Let us now establish a connection between the above two types of solutions. This result is inspired by a similar result for the finite dimensional case by Frankowska [10].
Theorem 1 A locally Lipschitz function $\mathcal{V} : [0,T] \times X \to R$ is a viscosity solution if and only if it is a Clarke generalized solution and

$$ -\zeta + \mathcal{H}(t,y,\xi) = 0, \quad \forall (t,y) \in (0,T) \times S, \quad \forall (\zeta,\xi) \in \partial^{-}\mathcal{V}(t,y). \quad (15) $$

Proof: Let $\mathcal{V}$ be a Clarke generalized solution of (1), then (14) implies that

$$ -\zeta + \mathcal{H}(t,y,\xi) \leq 0, \quad \forall (t,y) \in (0,T) \times S, \quad \forall (\zeta,\xi) \in \partial\mathcal{V}(t,y). \quad (16) $$

This result and the inclusion $\partial^{+}\mathcal{V}(t,y) \subseteq \partial\mathcal{V}(t,y)$ imply that $\mathcal{V}$ is a viscosity subsolution to (1). Now the results (15), (16) and the inclusion $\partial^{-}\mathcal{V}(t,y) \subseteq \partial\mathcal{V}(t,y)$ imply that $\mathcal{V}$ should also be a viscosity supersolution and hence is a viscosity solution to (1).

Let us now prove the converse. Let $(t,y) \in (0,T) \times S$ be an arbitrary point. Then, given $\epsilon > 0$, $\exists$ an element $(t^n,y^n) \in (0,T) \times S$ such that

$$ |t - t^n| + \|y - y^n\|_S \leq \epsilon $$

and the Frechet derivative of $\mathcal{V}$ satisfies,

$$ |\nabla_t\mathcal{V}(t^n,y^n)| + \|\nabla_y\mathcal{V}(t^n,y^n)\|_X \leq L(\|y\|_X, \epsilon). $$

Here $L(\cdot)$ is the Lipschitz constant of $\mathcal{V}$. This result is a consequence of the Proposition (1). In fact, since $\mathcal{V}$ Lipschitz in $X$ (and hence also in $S$), it is Frechet differentiable in a dense set $\Sigma_X \subset X$ (and also in $\Sigma_S \subset S$). We can take $y^n \in \Sigma_X \cap \Sigma_S$. Note also that almost differentiability of $\mathcal{V}$ in $t$ follows from the Rademacher's theorem. We thus obtain a sequence $\nabla\mathcal{V}(t^n,y^n)$ which converges weakly to some element $(\tilde{\zeta},\tilde{\xi})$ in $R \times X$. By the property (II) of Proposition (2) we deduce that $(\tilde{\zeta},\tilde{\xi}) \in \partial\mathcal{V}(t,y)$. Note also that, since $\mathcal{V}$ is a viscosity solution (12) and (13) imply that

$$ -\nabla_t\mathcal{V}(t^n,y^n) + \mathcal{H}(t^n,y^n,\nabla_y\mathcal{V}(t^n,y^n)) = 0, \quad \forall n. \quad (17) $$

Taking the limit $n \to \infty$ and noting the lowersemicontinuity property (Definition (1)) of the Hamiltonian, we get

$$ -\tilde{\zeta} + \mathcal{H}(t,y,\tilde{\xi}) \leq 0 \quad \text{with} \quad (\tilde{\zeta},\tilde{\xi}) \in \partial\mathcal{V}(t,y). \quad (18) $$

Now, let $(\zeta,\xi)$ be an arbitrary element of $\partial\mathcal{V}(t,y)$. Then, by the property (II) of Proposition (2), $(\zeta,\xi)$ can be expressed as a finite convex combination.
of elements of the type \((\bar{\zeta}, \bar{\xi})\) which are obtained as limits of gradients. That is \(\exists N > 0\), numbers \(\mu_i > 0, i = 1, \ldots, N\) and elements \(\{\bar{\zeta}_i, \bar{\xi}_i\}_{i=1}^N \in \partial \mathcal{V}(t, y)\) such that

\[
\sum_{i=1}^N \mu_i = 1,
\]

\[-\bar{\zeta}_i + \mathcal{H}(t, y, \bar{\xi}_i) \leq 0, \ i = 1, \ldots, N
\]  

(19)

and

\[
(\zeta, \xi) = \sum_{i=1}^N \mu_i (\bar{\zeta}_i, \bar{\xi}_i).
\]

Hence, multiplying (19) by \(\mu_i\) and summing over \(i\), we get

\[-\zeta + \sum_{i=1}^N \mu_i \mathcal{H}(t, y, \bar{\xi}_i) \leq 0.
\]

Now, using the convexity of the Hamiltonian we get

\[-\zeta + \mathcal{H}(t, y, \zeta) \leq 0, \ \forall (\zeta, \xi) \in \partial \mathcal{V}(t, y).
\]  

(20)

But, since \(\mathcal{V}\) is a viscosity solution and \(\partial^- \mathcal{V} \subseteq \partial \mathcal{V}\), comparing (20) with (13), we deduce that,

\[-\zeta + \mathcal{H}(t, y, \zeta) = 0, \ \forall (\zeta, \xi) \in \partial^- \mathcal{V}(t, y).
\]  

(21)

This proves (15). Now taking the supremum of (20) and noting that the maximum is achieved for some element of \(\partial \mathcal{V}\) (equation (21)) we get (14)), which proves that \(\mathcal{V}\) is a Clarke generalized solution.

\[\clubsuit\]

3 Optimal control of viscous flow

In this section we will formulate a general class of viscous flow control problem. We will first define a general infinite dimensional nonlinear control problem in the Hilbert space \(X\) and then show how such problems arise in fluid mechanics. Let \(\mathcal{A}\) be a self adjoint, regularly accretive operator [20] with domain \(D(\mathcal{A}) = S\). We will denote \(D(\mathcal{A}^\alpha) := S_\alpha\) for \(\alpha \in R\). Let \(\mathcal{N}(\cdot)\)
be a continuous locally bounded and Frechet differentiable nonlinear map such that
\[ N(\cdot) : S_{1/2} \to S_{-\epsilon}, \text{ for some } \epsilon \text{ with } 0 < \epsilon < 1. \]

Let \( F \) be a separable Hilbert space with \( U \subseteq F \) a closed convex set. Finally let \( B \in \mathcal{L}(U; X) \). We now consider the nonlinear dynamical system
\[ y_t + Ay + N(y) = BU, \quad t \in (\tau, T) \quad (22) \]
and
\[ y(\tau) = \zeta \in X. \]

Here \( T > \tau \) is a given number and \( U(\cdot) : [\tau, T] \to U \) is a strongly measurable \( U - \) valued function: \( U(\cdot) \in \mathcal{M}(\tau, T; U) \). If \( U \subseteq F \) is unbounded we will take \( U(\cdot) \in \mathcal{M}(\tau, T; U) \cap L^2(\tau, T; U) \). We will denote the above class of controls as admissible controls \( \mathcal{M}_{ad}(\tau, T; U) \).

Let us pose additional hypothesis on the nonlinearity \( N(\cdot) \) so that our theory would be more specialized towards fluid mechanics.

**Definition 5:** (Admissible nonlinearity) A locally bounded and Frechet differentiable map \( N(\cdot) : S_{1/2} \to S_{-\epsilon} \) for some \( \epsilon \) with \( 0 < \epsilon < 1 \) is called admissible if \( N(\cdot) : S \to X \) is strongly continuous: if \( y^n \to y \) then \( N(y^n) \to N(y) \) strongly in \( X \); and also if \( N(\cdot) \) satisfies the following three estimates:

(i) \[ | < N(y), y > | \leq C_1\|y\|_X^2, \quad \forall y \in S_{1/2}, \quad (23) \]

(ii) \[ | < N(y) - N(z), y - z > | \]
\[ \leq C_2\|y - z\|_X\|y - z\|_{1/2} \left\{ \|y\|_{1/2} + \|z\|_{1/2} \right\}, \quad \forall y, z \in S_{1/2}, \quad (24) \]

and

(iii) \[ |(N(y), Ay)_X| \leq C_3\|y\|^{1/2}_X\|y\|_{1/2}\|y\|_X \left\{ \|y\|_X + \|y\|_S \right\}^{1/2}, \quad \forall y \in S. \quad (25) \]

Here we denote \( \|A^a z\|_X := \|z\|_a \).

It has been shown in [16, 18, 17, 19, 9] that viscous flow control problems in two dimensional (bounded and unbounded ) domains as well as three dimensional thin (bounded) domains produce admissible nonlinearities of the above type. The following theorem is proved in [9].
Theorem 2 Solvability theorem Let $\zeta \in X$, $U \in \mathcal{M}_{ad}(\tau, T; U)$ and $N(\cdot)$ be an admissible nonlinearity. Then, $\exists$ a unique solution $y \in L^2(\tau, T; S_{1/2}) \cap C([\tau, T]; X)$ to the problem (22) such that $y_t \in L^2(\tau, T; S_{-1/2})$. If in addition $\zeta \in S_{1/2}$ then $y \in L^2(\tau, T; S) \cap C([\tau, T]; S_{1/2})$ with $y_t \in L^2(\tau, T; X)$.

This theorem defines a trajectory $y(\cdot, \tau; \zeta; U)$ for the dynamical system (22) corresponding to the initial data $\zeta$ and control $U(\cdot)$.

Let $Y \subseteq X$ be a given closed set (not necessarily convex). Target condition for the trajectory is prescribed as

$$y(T, \tau; \zeta; U) \in Y.$$  \hspace{1cm} (26)

Cost functional for the control problem is defined in the Bolza form:

$$C(T, \tau; \zeta; U) = \Phi_0(y(T)) + \int_{\tau}^{T} [\Theta(t, y(t)) + \mathcal{Y}(U(t))] dt \rightarrow \text{infimum.} \hspace{1cm} (27)$$

We will call $L(t, y, U) := \Theta(t, y) + \mathcal{Y}(U)$ the Lagrangian. In this paper we will consider functionals that are slightly special than in our earlier works. Namely,

(I) $\Phi_0(\cdot) : X \rightarrow R$ is continuous, locally bounded and Frechet differentiable.

(II) $\Theta(t, y)$ is continuous in $t$ and for each $t \in [0, T]$ defined as $\Theta(t, y) = a(t; y, y)$. Here the bilinear form $a(t; \cdot, \cdot) : S_{1/2} \times S_{1/2} \rightarrow R$ is continuous,

$$|a(t; y, z)| \leq \alpha_1 \|y\|_{1/2} \|z\|_{1/2}, \ \forall y, z \in S_{1/2} \hspace{1cm} (28)$$

and coercive

$$a(t; y, y) = \Theta(t, y) \geq \alpha_2 \|y\|_{1/2}^2, \ \forall y \in S_{1/2} \hspace{1cm} (29)$$

for some $\alpha_1, \alpha_2 > 0$. Finally the functional

(III) $\mathcal{Y}(\cdot) : U \subseteq F \rightarrow R$ is lowersemicontinuous convex and satisfies

$$\beta_1 \|U\|_{P}^2 \leq \mathcal{Y}(U) \leq \beta_2 \|U\|_{P}^2, \ \forall U \in U \hspace{1cm} (30)$$

for some $\beta_1, \beta_2 > 0$.

Remark: Observe that since $\mathcal{Y}(\cdot)$ is lowersemicontinuous, if a sequence $\{U^n\} \in U$ converges strongly to $U(\cdot) \in U$ in $F$ then we have by Fatou's lemma

$$\int_{\tau}^{T} \mathcal{Y}(U(t)) dt \leq \int_{\tau}^{T} \liminf \mathcal{Y}(U^n(t)) dt \leq \liminf \int_{\tau}^{T} \mathcal{Y}(U^n(t)) dt.$$
Now, the functional $U(\cdot) \rightarrow \int_\tau^T \mathcal{Y}(U(t))dt$ is convex (since $\mathcal{Y}(\cdot)$ is convex) and lowersemi-continuous and hence also weakly sequentially lowersemi-continuous. Moreover $\mathcal{U}$ is closed convex and hence closed in the weak topology of $\mathbf{F}$. These two properties are crucial in the proof for the existence theorem (Theorem (3) stated below) for optimal control. If these two convexities are absent (especially the convexity of $\mathcal{Y}(\cdot)$) then we need to use relaxed (Young measure-valued) optimal controls [22, 21, 11] and this issue will be addressed in a future paper.

The Optimal control problem is to find $U(\cdot) \in \mathcal{M}_{ad}(\tau, T; U)$ such that the pair $(y(\cdot), U(\cdot))$ satisfies the dynamical system (22), the final state $y(T)$ belongs to the target set $\mathcal{Y}$ (condition (26)) and the cost functional $\mathcal{C}(T, \tau; \zeta; U)$ achieves a global minimum.

Before proceeding to the mathematical results we will discuss a specific viscous flow control problem.

Example: (Optimal maneuvering of an obstacle.) The model we will discuss is a slight generalization of the optimal control problem formulated first in [16]. Consider the motion of an obstacle in $\mathbb{R}^n$, $n = 2, 3$, started from rest and translating with speed $-l(t)$ along the coordinate axis $x_1$. Then the governing equations in the noninertial frame fixed on the body can be described in the following way. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ denote the (noncompact) domain exterior to the obstacle. If $(u, p) : \Omega \times (\tau, T) \rightarrow \mathbb{R}^n \times \mathbb{R}$ denote the velocity and pressure measured in the above coordinate system then the Navier-Stokes equations are

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + \bar{e}_1 l(t), \ \text{in} \ \Omega \times (\tau, T), \quad (31)$$

$$\nabla \cdot u = 0, \ \text{in} \ \Omega \times (\tau, T), \quad (32)$$

$$u \rightarrow l(t)\bar{e}_1 \ \text{as} \ |x| \rightarrow \infty, \quad (33)$$

$$u(x, \tau) = 0, \ x \in \Omega, \quad (34)$$

and

$$u(x, t) = \phi(x)k(t) + \psi(x)r(t), \ x \in \partial \Omega, t \in (\tau, T). \quad (35)$$

Here $\phi$ and $\psi$ satisfy the following conditions.

$$\phi(x) \cdot \tau = 0, \ \forall x \in \partial \Omega \ \text{and} \ \int_{\partial \Omega} \phi \cdot n dS = 0;$$

and $\psi(x) \cdot n = 0, \ \forall x \in \partial \Omega.$
In the above $\tau, n$ denote respectively the local tangent and normal at a point $x$ on the boundary $\partial \Omega$ and $\vec{e}_1$ is the unit vector in the $x_1$-coordinate direction.

The initial condition (34) corresponds to the case where the obstacle starts from rest ($l(\tau) = 0$). The boundary condition (35) represents various boundary velocity prescriptions. The distribution $\phi(\cdot) : \partial \Omega \to \mathbb{R}^n$ represents suction and blowing on the boundary and $\psi(\cdot) : \partial \Omega \to \mathbb{R}^n$ represents tangential velocity prescription. For example, if the obstacle is an ellipsoid then rotations about the axis of revolution can be represented by a surface vectorfield of the type $\psi(\cdot)$. Thus the system (31)-(35) describe in particular, viscous flow past a revolving ogive shaped vehicle translating along its axis of revolution.

**Remark:** Unique solvability theorem for exterior hydrodynamics for the case of $\phi(\cdot) = \psi(\cdot) = 0$ is available in the literature. The two dimensional problem is uniquely solvable for arbitrarily large times [16]. The three dimensional problem is uniquely solvable up to a maximal time [3].

Let us now consider the case $\phi(x) \equiv 0$. We will write down the (formal) expression for energy expenditure for this case. Let $\sigma_{ij}$ denotes the stress tensor defined by

$$\sigma_{ij} = -p\delta_{ij} + 2\nu[\text{def}u]_{ij}, \quad \text{where} \quad \text{def}u = [\nabla u + (\nabla u)^T]/2.$$ 

Then the rate of work done on the surface of the vehicle $\partial \Omega$ is

$$- \int_{\partial \Omega} \sigma_{ij} (u_i - l(t)\delta_{i1}) n_j dS.$$

Here $u - l\vec{e}_1$ is the boundary velocity of the fluid at a point on $\partial \Omega$ in the inertial coordinate system. It has been shown in [16] that under suitable decay conditions at infinity the total energy expenditure is given by

$$\int_{\tau}^{T} \int_{\partial \Omega} \sigma_{ij} (-u_i + l(t)\delta_{i1}) n_j dS dt = \frac{1}{2} \int_{\Omega} |u(x, T) - l(T)\vec{e}_1|^2 dx$$

$$+ 2\nu \int_{\tau}^{T} \int_{\Omega} [\text{def}u : \text{def}u] dx dt. \quad (36)$$

In fact the solvability theorems quoted above provide us with generalized solutions for which the right hand side of (36) is finite.
We will now formulate the control problem. Let $U \subset \mathbb{R}^3$ be a closed convex set. We will choose the control $U : [0, T] \to U$ as $U = (l_t, r_t, k_t) \in \mathcal{M}_{ad}(0, T; U)$.

Let us now indicate how the Navier-Stokes system (31)-(35) can be written as the evolution system (22) in a Hilbert Space (details of this transformation can be found in the author's previous papers quoted above.) Construct smooth vectorfields $w^T(\cdot) : \Omega \to \mathbb{R}^3$, $w^R(\cdot) : \Omega \to \mathbb{R}^3$ and $w^B(\cdot) : \Omega \to \mathbb{R}^3$ such that,

$$\nabla \cdot w^T = 0, \quad w^T|_{\partial \Omega} = 0, \quad w^T \to \vec{e}_1 \text{ as } |x| \to \infty;$$

$$\nabla \cdot w^R = 0, \quad w^R|_{\partial \Omega} = \phi, \quad w^R \to 0 \text{ as } |x| \to \infty;$$

and

$$\nabla \cdot w^B = 0, \quad w^B|_{\partial \Omega} = \psi, \quad w^B \to 0 \text{ as } |x| \to \infty.$$ 

Existence and regularity properties of such vectorfields are classical in Navier-Stokes theory and they have their origins in the works of Leray, E. Hopf and Ladyzhenskaya (see for example [12, 15]). In fact it is possible to construct these vectorfields so that they take their respective farfield values identically outside a large sphere. We now set

$$u(x, t) = v(x, t) + l(t)w^T(x) + k(t)w^B(x) + r(t)w^R(x),$$

and note that $v(x, t)$ satisfies homogeneous conditions at the body and at infinity. Using this change of variables in the Navier-Stokes system (31)-(35), we can derive an evolution equation of the form (22) for the state variable $y(t) = (v(\cdot, t), l(t), k(t), r(t))$ using Hodge projection. The resulting operators $\mathcal{A}, \mathcal{N}$ and $\mathcal{B}$ will have the properties posed at the beginning of this section.

A target condition can be posed as

$$y(T) = (v(\cdot, T), l(T), k(T), r(T)) \in Y \subset X = H(\Omega) \times \mathbb{R}^3,$$

where $H(\Omega) = \{ v(\cdot) : \Omega \to \mathbb{R}^3; \ v(\cdot) \in L^2(\Omega) ; \ \nabla \cdot v = 0; v \cdot n|_{\partial \Omega} = 0 \}$. This kind of condition on the state introduces a constraint on the physical properties of the final state of the flowfield. The cost functional of the problem can be written as

$$\frac{1}{2} \int_\Omega |u(x, T) - l(T)\vec{e}_1|^2 dx$$
\[ +2\nu \int_{\tau}^{T} \int_{\Omega} \text{def} u : \text{def} u \, dx \, dt + \int_{\tau}^{T} \mathcal{Y}(U(t)) \, dt, \]

for any lower semicontinuous convex function \( \mathcal{Y}(\cdot) : U \rightarrow \mathbb{R} \) (satisfying (30)

and ) representing the control cost. The control problem is to find the accelerations (rotational, translational and suction rate) such that we can maneuver the obstacle from rest to a given rotational and translational speed (or speed range) with the final flowfield satisfying certain physical constraint such as low vorticity distribution (enstrophy) in the wake and such that the total energy expenditure plus the control cost achieves a global minimum.

Let us now return to the general control problem (22)-(30) and state two of the main results obtained in the earlier papers.

**Theorem 3** Existence of optimal control Let \( A, N, B, \Phi_0, \Theta \) and \( \mathcal{Y} \) satisfy the hypotheses stated in this section and let \( \zeta \in X \). Then \( \exists \) an optimal control \( U(\cdot) \in \mathcal{M}_{ad}(0,T;U) \) such that the corresponding trajectory \( y(\cdot) \in L^2(\tau,T;S_{1/2}) \cap C([\tau,T];X) \) satisfies (22), (26) and

\[
\mathcal{C}(\tau,T;\zeta;U) = \min \{ \mathcal{C}(\tau,T;\zeta;V) ; \ V(\cdot) \in \mathcal{M}_{ad}(0,T;U) \}.
\]

Proof of this theorem for the two dimensional exterior hydrodynamics problem described above can be found in [16]. In [9] this theorem has been proven for a large class of flow control problems.

The above theorem only provides us the knowledge of the existence of an optimal pair \((y(\cdot),U(\cdot))\). The necessary conditions for optimal control is provided by the next theorem [18, 17, 19, 9] (sufficient conditions will be discussed in the next section). Let us first define the pseudo-Hamiltonian (or Pontryagin's controlled Hamiltonian) as

\[
\tilde{H}(t,y,p,U) = \langle \mathcal{F}(y,U),p \rangle - \mathcal{L}(t,y,U), \tag{37}
\]

where \( \mathcal{F}(y,U) = -Ay - N(y) + BU \) and \( \mathcal{L}(t,y,U) = \Theta(t,y) + \mathcal{Y}(U) \) is the Lagrangian defined earlier. Note that if \( y(\cdot) \in L^2(\tau,T;S), U(\cdot) \in L^2(\tau,T;U) \) and \( p \in L^2(\tau,T;X) \) (as in the theorem (4) below) then using the properties of \( \mathcal{L} \) and \( \mathcal{F} \) defined earlier we conclude that the pseudo-Hamiltonian is well defined and \( \tilde{H}(\cdot,\cdot,\cdot) \in L^1(\tau,T) \).

**Theorem 4** Pontryagin maximum principle Let \( A, N, B, \Phi_0, \Theta \) and \( \mathcal{Y} \) satisfy the hypotheses stated in this section and let \( \zeta \in S_{1/2} \). Suppose
that \((y(\cdot), U(\cdot))\) be an optimal pair. If the (Bouligand’s) contingent cone \(K_Y(y(T))\) of the target set \(Y\) at the terminal point \(y(T)\) contains an interior point then \(\exists p \in C([\tau, T]; X) \cap L^2(\tau, T; S_{1/2})\) such that

\[
y_t = \nabla_p \hat{H}(t, y, p, U), \quad t \text{ a.e in } (\tau, T), \tag{38}
\]

\[
-p_t = \nabla_y \hat{H}(t, y, p, U), \quad t \text{ a.e in } (\tau, T), \tag{39}
\]

\[y(\tau) = \zeta \in S_{1/2} \quad \text{and} \quad -p(T) - \nabla \Phi_0(y(T)) \in N_Y(y(T)).\]

Here \(N_Y(y(T))\) is the Clarke normal cone of the target set at the terminal point. Finally the pseudo-Hamiltonian takes its maximum at the optimal control

\[
\hat{H}(t, y, p, U) = \max \{\hat{H}(t, y, p, V); \ V \in U\}, \quad t \text{ a.e in } (\tau, T). \tag{40}
\]

Note that \((38)\) is the same as \((22)\) and \((39)\) can be written as

\[-p_t + Ap + [DN(y)]^* p = -\nabla \Theta(t, y(t)).\]

Let us now obtain a “nonsmooth” version of the maximum principle. Similar result for finite dimensional problems were obtained by Clarke [4, 5]. For this we define the true Hamiltonian in the sense of L. C. Young [22]:

\[
\mathcal{H}(t, y, p) = \max \{\hat{H}(t, y, p, V); \ V \in U\}. \tag{41}
\]

Using the definitions of \(F\) and the Lagrangian \(L\) we get

\[
\mathcal{H}(t, y, p) = \max \{<p, \mathcal{F}(y, V) > - L(t, y, V); \ V \in U\}
\]

\[= - <p, Ay + N(y) > - \Theta(t, y) + \max \{<B^* p, V > - \gamma(V); \ V \in U\}.\]

Hence denoting the conjugate function of \(\gamma(\cdot)\) as \(\gamma^*(\cdot)\), we get

\[
\mathcal{H}(t, y, p) = \gamma^*(B^* p) - <p, Ay + N(y) > - \Theta(t, y). \tag{42}
\]

**Theorem 5** Let \(A, N, B, \Phi_0, \Theta\) and \(\gamma\) satisfy the hypotheses stated in this section and let \(\zeta \in S_{1/2}\). If the (Bouligand’s) contingent cone \(K_Y(y(T))\) of
the target set $Y$ at the terminal point $y(T)$ contains an interior point then the optimal trajectory $y(\cdot)$ and the adjoint state $p(\cdot)$ satisfy

$$y_t = \partial_\nu \mathcal{H}(t, y, p), \ t \ a.e \ in \ (\tau, T),$$

(43)

where $\partial_\nu \mathcal{H}$ is the Clarke generalized gradient,

$$-p_t = \nabla_y \mathcal{H}(t, y, p), \ t \ a.e \ in \ (\tau, T),$$

(44)

$$y(\tau) = \zeta \in S_{1/2} \ and \ \ p(T) - \nabla \Phi_0(y(T)) \in N_Y(y(T)).$$

Proof: We will deduce this result from the theorem (4). We only need to verify (43) since (44) follows immediately from (39). Recall that the conjugate function $\mathcal{Y}^*(\cdot)$ is always convex and lowersemicontinuous [2]. Moreover, we have

$$\mathcal{Y}^*(q) = \sup \{ <q, V> - \mathcal{Y}(V); \ V \in U \}$$

$$= \sup \{ <q, V> - \mathcal{Y}(V) - \delta(V|U); \ V \in F \},$$

where $\delta(\cdot|U) : F \rightarrow \mathbb{R} \cup \{+\infty\}$ is the indicator function defined as

$$\delta(V|U) = \begin{cases} 0 & \text{if } V \in U \\ +\infty & \text{otherwise.} \end{cases}$$

Thus the growth condition (30) $\mathcal{Y}(V) \geq \beta_1 \|V\|_F^2$ implies that

$$\mathcal{Y}^*(q) \leq \sup \left\{ \|q\|_{F^*} \|V\|_F - \beta_1 \|V\|_F^2; \ V \in F \right\}.$$

That is

$$\mathcal{Y}^*(q) \leq \frac{1}{4\beta_1} \|q\|_{F^*}^2, \forall q \in F^*.$$

Now if a convex function is bounded above in a neighborhood then it is locally lipschitz [14]. Thus $\mathcal{Y}^*(\cdot)$ is locally Lipschitz. We noted in section (1) that if a function is convex and locally Lipschitz then the Clarke generalized gradient coincides with the subdifferentials. Thus

$$\partial^{cv} \mathcal{Y}^*(q) = \partial^{-} \mathcal{Y}^*(q) = \partial \mathcal{Y}^*(q).$$
It follows from standard results in convex analysis [2] that, when $\mathbf{Y}(\cdot)$ satisfying (30) is convex and lowersemicontinuous $\exists U \in U$ (depending on $q$) such that

$$\mathbf{Y}^*(q) = < q, U > - \mathbf{Y}(U),$$

with

$$U \in \partial^{CV} \mathbf{Y}^*(q) = \partial \mathbf{Y}^*(q).$$

We now return to the Pontryagin maximum principle and note that

$$\partial \mathbf{H}(t, y, p) = -A y - N(y) + \partial \mathbf{Y}^*(B^*p). \quad (45)$$

But as noted above, we have

$$BU \in \partial \mathbf{Y}^*(B^*p). \quad (46)$$

From (45), (46) and (38) we obtain (43).

4 The nonsmooth verification technique

In this section, we will clarify the “nonsmooth verification technique” which provides the sufficiency condition for optimality. Moreover, we will also explain the roles of value function and (smooth and nonsmooth) verification function in dynamic programming. We will restrict ourselves to the case of $Y = X$ (constraint free end state).

Definition 6 The value function for the optimal control problem formulated in section (3) is defined as

$$V(t, \zeta) = \min \left\{ \Phi_0(y(T)) + \int_t^T \mathcal{L}(r, y(r), V(r))dr, \ V \in \mathcal{M}_{ad}(\tau, T; U) \right\}. \quad (47)$$

Here $(y(\cdot), V(\cdot))$ denotes an admissible pair.

Note that the minimum is indeed achieved due to the existence theorem (3) for the optimal control.
Theorem 6 The value function \( V(\cdot, \cdot) \in C([0, T] \times X) \). For each \( t \in [\tau, T] \), \( V(t, \cdot) \) is locally Lipschitz in \( X \) and for each \( \zeta \in S \), \( V(\cdot, \zeta) \) is absolutely continuous in \( t \in (\tau, T) \). Moreover, \( V(\cdot, \cdot) : [\tau, T] \times X \rightarrow R \) is a viscosity solution of the Hamilton-Jacobi-Bellman equation: (formal expression)

\[
-\partial_t V + \mathcal{H}(t, y, -\partial_y V) = 0.
\]

This theorem has been proven in [18, 17, 9]. Thus combining this result with theorem (1) of section (2) we conclude that \( V \) is also a Clarke-generalized solution and satisfies

\[
-\zeta + \mathcal{H}(t, y, -\xi) = 0, \quad \forall (t, y) \in (0, T) \times S, \quad \forall (\zeta, \xi) \in \partial V(t, y).
\]

(48)

Remark: We note here that the true Hamiltonian defined by (42) does satisfy the required properties for the above conclusion because it is an admissible Hamiltonian in the sense of Definition (1). To see this we consider sequences \( t^n \rightarrow t, y^n \rightarrow y \) strongly in \( S \) and \( p^n \rightarrow p \) weakly in \( X \). Then the term \( \Theta(t^n, y^n) \rightarrow \Theta(t, y) \) due to its continuity. Since the conjugate function \( \mathcal{X}^*(\cdot) \) is convex and lowersemicontinuous it is weakly lowersemicontinuous:

\[
\liminf_{n \to \infty} \mathcal{X}^*(B^* p^n) \geq \mathcal{X}^*(B^* p).
\]

Moreover, if \( y^n \rightarrow y \) strongly in \( S \), then by definition \( N(y^n) \rightarrow N(y) \) strongly in \( X \). This, in combination with the weak convergence of \( p^n \rightarrow p \) in \( X \) imply that

\[
(N(y^n), p^n)_X \to (N(y), p)_X.
\]

Combining these results we get the required semicontinuity property of \( \mathcal{H}(\cdot, \cdot, \cdot) \).

The convexity of \( \mathcal{H}(t, y, \cdot) : X \rightarrow R \) is of course due to the convexity of \( \mathcal{X}^*(\cdot) \).

Let us now introduce the notion of smooth verification function through a lemma.

Lemma 1 Let \((y(\cdot), U(\cdot))\) be an admissible pair satisfying equations (38), (39) and the end conditions of the Pontryagin maximum principle. Suppose \( \exists \) a \( C^1 \) function \( \mathcal{W}(\cdot, \cdot) : [\tau, T] \times X \rightarrow R \) with \( \mathcal{W}(T, y(T)) = \Phi_0(y(T)) \) such that

\[
-\nabla_t \mathcal{W} + \mathcal{H}(t, y(t), -\nabla_y \mathcal{W}) \leq 0, \quad t \in (\tau, T)
\]

(49)

with equality holds if the pair \((y(\cdot), U(\cdot))\) is optimal. Then, the existence of such \( \mathcal{W} \) verifies optimality.
**Proof:** Condition (49) implies, by maximum principle (equation (40)),

$$-\nabla_t \mathcal{W} + \hat{K}(t, y(t), -\nabla_y \mathcal{W}, U) \leq 0, \quad t \in (\tau, T).$$  \hspace{1cm} (50)

That is

$$-\nabla_t \mathcal{W} - \left< \nabla_y \mathcal{W}, y_t \right> - \mathcal{L}(t, y, U) \leq 0, \quad t \in [\tau, T],$$  \hspace{1cm} (51)

and we get,

$$-\frac{d\mathcal{W}}{dt} - \mathcal{L}(t, y, U) \leq 0.$$

Integrating this inequality from $\tau$ to $T$, we get

$$\Phi_0(y(T)) + \int_{\tau}^{T} \mathcal{L}(r, y(r), U(r))dr \geq \mathcal{W}(\tau, \zeta)$$  \hspace{1cm} (52)

with equality holding if $(y(\cdot), U(\cdot))$ is optimal. The inequality (52) establishes a lower bound for the cost functional and the minimum is achieved when the chosen pair is optimal. This verifies the optimality of the chosen trajectory and hence $\mathcal{W}(\cdot, \cdot)$ is a $C^1$-verification function.

---

*Is it possible to always find such a smooth verification function?*. The answer is negative due to the following reason. From (52), we see that if $(y(\cdot), U(\cdot))$ is optimal then $\mathcal{V}(t, y(t)) = \mathcal{W}(t, y(t)), t \in [\tau, T]$. If $z(\cdot)$ is a general trajectory (not necessarily optimal) then (52) shows that

$$\mathcal{V}(t, z(t)) \geq \mathcal{W}(t, z(t)) \quad t \in [\tau, T].$$  \hspace{1cm} (53)

This implies, using the properties of the subdifferential \cite{8},

$$[D\mathcal{W}](t, y(t)) \in \partial^- \mathcal{V}(t, y(t)).$$  \hspace{1cm} (54)

However, since $\mathcal{V}$ is not known to be smoother than Lipschitz, the subdifferential $\partial^- \mathcal{V}$ may be empty. In fact such an example of a function which has a downward corner was presented in section (2). Thus it is not always reasonable to look for a $C^1$-verification function. We will therefore develop the verification technique using Lipschitz functions.
Theorem 7 Let \((y(\cdot), U(\cdot))\) be an admissible pair satisfying equations (38), (39) and the end conditions of the Pontryagin maximum principle. Let the initial data for the trajectory \(y(t) \in S\). Suppose \(\exists\) a Lipschitz function \(W(\cdot, \cdot) : [\tau, T] \times X \to R\) with \(W(T, y(T)) = \Phi_0(y(T))\) which satisfies the Hamilton-Jacobi equation in the sense of Clarke:

\[
\max \{-\zeta + H(t, y(t), -\xi)\}, \quad (\zeta, \xi) \in \partial W(t, y)\} = 0 \quad t \in [\tau, T], \quad (55)
\]

and satisfies the equality

\[
-\zeta + H(t, y(t), -\xi) = 0, \quad \forall (\zeta, \xi) \in \partial W(t, y), \quad t \in [\tau, T] \quad (56)
\]

if the pair \((y(\cdot), U(\cdot))\) is optimal. Then, the existence of such \(W\) verifies optimality.

**Proof:** Note that (55) implies

\[
-\zeta + H(t, y(t), -\xi) \leq 0, \quad \forall (\zeta, \xi) \in \partial W(t, y), \quad t \in [\tau, T]. \quad (57)
\]

Now, let us set

\[
\alpha(t) = W(t, y(t)) - \int_t^T \mathcal{L}(r, y(r), U(r))dr \quad (58)
\]

with \(\alpha(T) = \Phi_0(y(T))\). In order to verify optimality (see equation (52)) it is sufficient to have

\[
\alpha(t) \leq \Phi_0(y(T)), \quad t \in [\tau, T]
\]

with equality holding for \(y(\cdot)\) optimal. We will need the following

**Lemma 2** \(\alpha(\cdot) : [\tau, T] \to R\) is absolutely continuous and

\[
-\alpha_t \in \{-\zeta + H(t, y(t), -\xi)\}, \quad (\zeta, \xi) \in \partial W(t, y)\}, \quad t \ a.e. \ in \ (\tau, T). \quad (59)
\]

We will postpone the proof of this lemma and proceed with our argument concerning the nonsmooth verification technique.

Note that if \(W\) is a Clarke generalized solution then each element in the set on right hand side of (59) is nonpositive and thus we should have \(\alpha_t \geq 0\), almost everywhere in \([\tau, T]\). Moreover, if each element of this set is zero almost everywhere in \([\tau, T]\) then \(\alpha_t = 0\) almost everywhere in this interval.
Thus \( W \) verifies optimality. Let us now prove the Lemma (2) to complete the proof of Theorem (7).

**Proof of Lemma 2.** For \( \tau < t < T \) and \( \delta > 0 \), we consider,

\[
\alpha(t+\delta) - \alpha(t) = W(t+\delta, y(t+\delta)) - W(t, y(t)) - \int_{t}^{t+\delta} \mathcal{L}(r, y(r), U(r)) \, dr.
\]

We note that \( y(\cdot) \in L^2(\tau, T; S_{1/2}) \) and \( U \in \mathcal{M}_{ad}(\tau, T; U) \) imply \( \mathcal{L}(\cdot, \cdot, \cdot) \in L^1(\tau, T) \) and hence

\[
\frac{1}{\delta} \int_{t}^{t+\delta} \mathcal{L}(r, y(r), U(r)) \, dr \rightarrow \mathcal{L}(t, y(t), U(t)) \quad \text{as} \quad \delta \rightarrow 0
\]

if \( t \) is a Lebesgue point [23] of the Lagrangian. Note now that the function \( \beta(t) := W(t, y(t)) \) is a composition of two Lipschitz functions (these properties are proved in our earlier papers) and hence is also Lipschitz. We thus use the mean value theorem of Lebourg [4] to get

\[
\beta(t+\delta) - \beta(t) \in \partial_y W(t+\lambda, y(t+\delta)) \cdot \lambda + \langle \partial_z W(t, z), y(t+\delta) - y(t) \rangle
\]

for some \( 0 < \lambda < \delta \), \( z = y(t) + \mu(y(t+\delta) - y(t)) \) and \( 0 < \mu < 1 \). Since \( \beta(\cdot) \) is Lipschitz, \( \beta_t \) exists almost everywhere by Rademacher's theorem. Hence, to complete the proof of the Lemma, we need to show that the second set on the right hand side of (61) converges. Note first that as \( \delta \rightarrow 0 \), \( z \rightarrow y(t) \) strongly in \( X \). Recall now that the multifunction \( \partial_y W(t, \cdot) : X \rightarrow X \) is weakly closed [4]: if \( z_k \rightarrow y(t) \) strongly in \( X \) and \( \xi_k \in \partial_y W(t, z_k) \) converges weakly to \( \xi \) in \( X \), then \( \xi \in \partial_y W(t, y(t)) \). Hence to justify the convergence

\[
\langle \partial_y W(t, z), \frac{y(t+\delta) - y(t)}{\delta} \rangle \rightarrow \langle \partial_y W(t, y(t)), y_t(t) \rangle,
\]

with \( y_t = \mathcal{F}(t, y(t), U(t)) \), we need the strong convergence of the differential quotient \( (y(t+\delta) - y(t))/\delta \) in \( X \). This follows from the absolute continuity of \( y(\cdot) \) proved in [18, 9].

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