RECURSIVE SOLUTION FOR DIFFUSE TOMOGRAPHIC SYSTEMS
OF ARBITRARY SIZE

By

S.K. Patch

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
RECURSIVE SOLUTION FOR DIFFUSE TOMOGRAPHIC SYSTEMS OF ARBITRARY SIZE

S. K. PATCH*

Abstract. The first presentation of recursive scheme for recovering Markov transition probabilities from boundary-value data generated by two dimensional diffuse tomographic systems was of prohibitively high computational complexity. Matrix factorizations, the Laplace expansion, and the Graßmann-Plücker identities are combined in an inductive argument to present a efficient algorithm for splitting a $n \times n$ system into four $n/2 \times n/2$ subsystems.

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1. Introduction. Applications in optical imaging motivated Grünbaum to study anisotropic random walks on two dimensional lattices of finite size [1], [2]. One possible application is imaging the brains of neonates. (Cranial bleeds are a leading cause of death among premature infants. Repeated CT scans would overexpose the infant’s brain to harmful X-rays; MRI scans are prohibitively expensive for periodic monitoring.) Imaging requires solving the inverse problem. Later work on Grünbaum’s “diffuse tomographic” model resulted in a cumbersome recovery algorithm. In its initial form, the complexity of this recursive algorithm skyrocketed with the number of recursive levels. Naturally, as system size increases the complexity of the recovery scheme presented here grows faster than the number of unknowns. The author is optimistic, however, that

* Institute for Mathematics and its Applications, 514 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455. email: patch@ima.umn.edu. The author was supported by a NSF postdoctoral fellowship held at the IMA.
several recursive levels can be carried out in the generality presented here. Use of a priori information is required to make inversion practical for seven or eight recursions, enough to generate an image of $256 \times 256$ pixels.

At each recursive level a $n \times n$ system is broken into four $n/2 \times n/2$ subsystems and families of solutions for the subsystems’ data sets are computed. Too many parameters are introduced in all but the lowest level. Most of the work required by this algorithm goes into eliminating the unnecessary parameters. This is done by enforcing range conditions upon the newly-found subsystems’ data. “Hidden-outgoing” transition probabilities representing travel from the middle of the system outward are much more helpful in efficiently eliminating parameters than their “incoming-hidden” counterparts.

1.1. Description of the model. Consider a $n \times n$ array of pixels in the plane. On each outer face there are two devices. One device shoots photons across the outside edge into the neighboring pixel; the other device detects photons as they leave the system. For each of the $4n$ outside edges $4n$ pieces of data are collected. These data are stored as a $4n \times 4n$ transition matrix, $Q$. Within the system, photons travel either horizontally or vertically. Label the horizontal and vertical axes $x_1$ and $x_2$, respectively. A photon traveling parallel to the $x_i$ axis in the positive direction moves with velocity $x_i^+$. Photons traveling in the opposite direction travel with velocity $x_i^-$. They do not interact and may be absorbed within a pixel. Photons move according to a Markov process. The probabilities with which a photon moves to a neighboring pixel depend upon its previous, as well as present, location. In this two step formulation the state space consists of locations. The state space may be redefined so that photons move according to a one step Markov process. In the new state space a single state consists of the photon’s location and direction of travel.

There are three different types of these Markov states: incoming, outgoing, and hidden. The probabilities with which photons move from one state to another are referred to as transition probabilities. For example, a photon which travels with velocity
$x_i^\dagger$ into pixel $O$ and travels straight through pixel $O$ does so with some probability, denoted by $x_i^\dagger O x_i^\dagger$. For each pixel $O$ and incident direction the sum of the absorption probability and the four possible transition probabilities must be identically one. The absorption probability is therefore be neglected in the rest of this paper. Each pixel corresponds to 16 transition probabilities. The same photon travels to $O$’s neighbor in the $x_j^\dagger$ direction with probability $x_i^\dagger O x_j^\dagger$. These probabilities are the nonzero entries of the Markov transition matrix $M$. $M$ is sparse and may be written as a block matrix with nontrivial subblocks $P_{io}$, $P_{ih}$, $P_{ho}$, and $P_{hh}$. $P_{io}$, for example, contains the probabilities with which photons in incoming states move directly to outgoing states. $P_{ih}$ contains the probabilities with which photons in incoming states move to hidden states. $P_{ho}$ and $P_{hh}$ are the transition matrices for photons starting in hidden states traveling to outgoing and hidden states, respectively. $P_{io}$ and $P_{hh}$ are always square matrices.

**1.2. Forward Problem.** The forward map takes $16n^2$ transition probabilities to the $4n \times 4n$ data matrix $Q$. The domain of the forward map lies in the unit cube in $\mathbb{R}^{16n^2}$ and is defined by

\begin{equation}
(x_i^\dagger O x_1^\dagger + x_i^\dagger O x_1^-) + (x_i^\dagger O x_2^\dagger + x_i^\dagger O x_2^-) \leq 1 \quad \pm = +, - \quad i = 1, 2
\end{equation}
for each pixel \( O \). Furthermore, none of these transition probabilities is permitted to be zero. \( Q^j_i \) represents the probability that a photon which enters the system at source \( i \) exits the system at detector \( j \). \( Q \) provides no time-of-flight information. Because \( Q \) is a transition matrix acceptable solutions lie in the unit cube in \( \mathbb{R}^{16n^2} \) and satisfy

\[
0 \leq \sum_{\lambda=1}^{4n} Q^\lambda_i \leq 1, \quad i = 1, 2, \ldots, 4n
\]

The forward map is given by the following matrix expression:

\[
Q = P_{io} + P_{ih} \sum_{n=0}^{\infty} P_{hh}^n P_{ho} = P_{io} + P_{ih} (I - P_{hh})^{-1} P_{ho}
\]

1.3. Consistency Conditions. Range conditions appear as rank deficient submatrices of \( Q \). Each of these rank deficient submatrices represents travel from one “side” of the system to the other “side”. Let \( b \) be a (not necessarily straight) barrier of \#\( b \) hidden states separating the “sides”. The Markovian nature of the system can be used to show that the corresponding submatrix is generically of rank \#\( b \) [3]. For the purposes of this paper it is only necessary to consider straight barriers.

Notation: Let \( Q^c_r \) denote the submatrix of \( Q \) taken from rows \( r \) and columns \( c \). Let \( dQ^c_r \) denote the determinant of this submatrix. Furthermore, let \( a - b \) denote \( a, a+1, \ldots, b \) where \( a, b \in \mathbb{N}^+ \) and \( a < b \)

For example, the data matrix for a \( 2 \times 2 \) system has many rank deficient submatrices of rank two. See figure 1. The submatrix representing travel from left to right, \( Q_{1,2,3,4}^{5,6,7,8} \), is generically rank two, as is \( Q_{5,6,7,8}^{1,2,3,4} \). Similarly, the submatrices \( Q_{3,4,5,6}^{1,2,7,8} \) and \( Q_{1,2,7,8}^{3,4,5,6} \) are generically of rank two as well.

2. The Modified Problem. Although the final goal is to recover the microscopic transition probabilities for each pixel from BV data, the purpose of this section is more modest. The original \( n \times n \) system is broken into four \( n/2 \times n/2 \) subsystems; \( 2n^2 \) parameter families of the subsystems’ data sets is computed from the data set for the
$n \times n$ array. (Here $n = 2^k$, $k \in \mathbb{N}$.) Once the data sets for each of the $n/2 \times n/2$ subsystems are found, the recursion can be implemented in parallel until $k = 1$.

2.1. Matrix Solutions. A “trick” is to solve the inverse problem for a $2 \times 2$ system; a recursive scheme takes advantage of the same “trick” many times over [4]. In the standard formulation of the forward problem $Pio$ is $4n \times 4n$; $Pih$ is $4n \times 4n(n-1)$; $Phh$ is $4n(n-1) \times 4n(n-1)$; and $Pho$ is $4n(n-1) \times 4n$. The “trick” requires invertibility of $Pho$, a non-square matrix for $n > 2$. By considering only the $4n$ hidden states dividing the $n \times n$ system into four $n/2 \times n/2$ subsystems, the “modified” transition matrices are square. In fact, they have the same block structure as their $2 \times 2$ counterparts. See figure 2. The only difference is that each entry of a $2 \times 2$ transition matrix is now a $n/2 \times n/2$ block of the modified transition matrices for the $n \times n$ system. Note that the entries of these modified transition matrices are the data for the $n/2 \times n/2$ subsystems.

and define $A = Pho^{-1}$. Then 1.3 expresses the data generated by the $n \times n$ system in terms the $n/2 \times n/2$ subsystems’ data. Note that the only “bad” term in 1.3 is $(I - Phh)^{-1}$ and define $A \equiv Pho^{-1}$. The governing equations may be written as a matrix equation of degree three in $A$, $Pio$, $Phh$, and $Pih$:

$$(2.4) \quad Q \ A - Pio \ A - Q \ A \ Phh + (Pio \ A \ Phh - Pih) = 0$$

The following notation is useful in manipulating 2.4, a matrix equation whose summands are functions of sparse block matrices.

Notation: $[M : N]$ denotes the concatenation of matrices $M$ and $N$ (where $M$ and $N$ have the same number of rows). left, right, top, and bottom denote any choice of one half of the states on the left, right, top, and bottom of the system. There are $\binom{n}{n/2}$ possibilities for each. Finally, let $i = (i - 1)n/2 + 1, \ldots, in/2$ for $i = 1, \ldots, 8$.

Just as was done in [5], it is possible to solve for each of the nontrivial subblocks of $Pio$, $Pih$, $Phh$, and $Pho$ as functions of $A$ and $Q$. The following blocks will be used later to eliminate some of the parameters $A_i^j$ from the solutions:
Fig. 2. The leftmost array represents the block structure of \( \Pi_0 \) and \( \Phi_0 \) (and of course \( A \)) for a modified \( n \times n \) problem where \( n = 2^k \), \( k \in \mathbb{N}^+ \). Each * is an \( n/2 \times n/2 \) block. The array on the right gives the off diagonal block structure for modified transition matrices \( \Pi_1 \) and \( \Phi_1 \).

\[
\Pi_{1,8}^{7,8} = \Phi_{0,7,8}^{7,8} (Q_{\text{bot}}^{7,8})^{-1} Q_{\text{bot}}^{1,2} A_{1,2}^{1,2}
\]

\[
(2.6) \quad \Pi_{1,2}^{7,8} = Q_{1,2}^{7,8} - Q_{1,2}^{7,8} (Q_{\text{bot}}^{7,8})^{-1} Q_{\text{bot}}^{1,2} A_{1,2}^{1,2} : Q_{1,2}^{3,4} (Q_{\text{right}}^{3,4})^{-1} Q_{\text{right}}^{1,2} A_{1,2}^{1,2} \Phi_{1,2}^{7,8}
\]

\[
(2.7) \quad \Pi_{1,8}^{7,8} = \left[ Q_{1,2}^{7,8} - (Q_{7,8}^{7,8} - \Pi_{0,7,8}^{7,8}) (Q_{\text{bot}}^{7,8})^{-1} Q_{1,2}^{7,8} \right] A_{1,2}^{1,2}
\]

The solution for \( \Pi_0^{7,8} \) is not used above as it will be considerably simplified in section 2.3. Expressions for other nontrivial blocks of \( \Pi_{1,8} \), \( \Pi_0 \), and \( \Pi_{1,8} \) take the same forms as 2.5, next two sections parallel work done in [5], they are included here for completeness.

2.2. Matrix Identities. Since the inverse problem involves linear systems, it is not surprising that Graßmannians and the Graßmann-Plücker embedding come into play. The identities which embed Graßmannians \( G(k, n) \) in \( \mathbb{P}^{(2)} \) are derived below. A cursory explanation of the embedding can be found in [4, 6]. For a more thorough exposition see [7, 8]. Let \( \Lambda \) be any rectangular matrix with \( k \) rows and \( n \) columns where
Fig. 3. A $4 \times 4$ system. The incoming and outgoing states are labeled; all unlabeled states are hidden states. There are 16 incoming and 16 outgoing states, but 48 hidden states.

Fig. 4. Decomposition of a $4 \times 4$ system into four $2 \times 2$ subsystems. The thick lines separate the subsystems. The “modified” $4 \times 4$ system disregards individual pixels. Only the subsystems are relevant at the first level of this recursive procedure.
$k < n - 1$ and $\Lambda = (a)_{ij}$. Let $I = (i_1, i_2, i_3, \ldots, i_{k-1})$ index $(k - 1)$ distinct columns of $\Lambda$. Let $J = (j_1, j_2, j_3, \ldots, j_{k+1})$ index $(k + 1)$ distinct columns of $\Lambda$. Then,

$$
\sum_{\lambda=1}^{k+1} \pi(i_1, i_2, \ldots, i_{k-1}, j_{\lambda}) \pi(j_1, j_2, \ldots, j_{\lambda-1}, j_{\lambda+1}, \ldots, j_{k+1}) = 0
$$

Equation 2.8 defines the Grassmann relations. Let $\alpha, \beta, \gamma, \eta, \kappa \in \mathbb{N}^k$. For any matrix $Q$ basic matrix properties imply

$$
(Q^\gamma)^{-1}Q^\eta = \left(\frac{dQ^\gamma_\alpha \gamma_\eta \gamma_{k-1} \gamma_{k+1} \ldots \gamma_n}{dQ^\gamma_\alpha}\right)_{i,j}
$$

$$
Q^\kappa (Q^\gamma)^{-1}Q^\eta = Q^\kappa - (1/dQ^\gamma_\alpha)(dQ^\eta_\kappa_\alpha)_{i,j}
$$

The identity 2.9 combines with Grassmann-Plücker relations to imply

$$
(dQ^\gamma_\kappa \beta)_i (Q^\gamma)^{-1}Q^\eta_{i,k} = (dQ^\gamma_\kappa \beta)_i, k \left(\frac{dQ^\gamma_\alpha \gamma_\eta \gamma_{k-1} \gamma_{k+1} \ldots \gamma_n}{dQ^\gamma_\alpha}\right)_{i,j}
$$

$$
= \frac{-1}{dQ^\gamma_\alpha} \sum_{k=1}^{n} (-1)^k \left(\frac{dQ^\gamma_\kappa \beta}{dQ^\gamma_\alpha} dQ^\eta_\kappa \gamma_\alpha \gamma_{k-1} \gamma_{k+1} \ldots \gamma_n\right)_{i,j}
$$

$$
= \left(\frac{dQ^\eta_\kappa \beta}{dQ^\gamma_\alpha} - \frac{dQ^\gamma_\beta}{dQ^\gamma_\kappa_\alpha} \right)_{i,j}
$$

These matrix identities will be used to simplify 2.6 and 2.7, as will

$$
I = A_{2\alpha-1,2\alpha} P h_{2\alpha-1,2\alpha} = A_{2\alpha-1,2\alpha} P h_{2\alpha-1,2\alpha} + A_{2\alpha-1,2\alpha} P h_{2\alpha-1,2\alpha}
$$

where $\alpha = 1, 2, 3, 4$.

2.3. Matrix Solutions Revisited. Identity 2.10 permits the solution for $Pio_{1,2}^{1,2}$ to be written quite succinctly:

$$
Pio_{1,2}^{1,2} = M^1 + \left(\frac{dQ^\gamma_3 \alpha}{dQ^\gamma_3 \alpha}\right)_{i,j}
$$
where

\begin{equation}
M_1^1 \equiv \frac{\left( \frac{dQ_{i,bot}^{j,7,8}}{dQ_{bot}^{7,8}} - \frac{dQ_{i,right}^{j,3,4}}{dQ_{right}^{3,4}} \right)}{i,j} A_{1,2}^{1,2} Pho_{1,2}^{1,2}
\end{equation}

and

\begin{equation}
M_4^4 \equiv \left( \frac{dQ_{i+3n,bot}^{j,5,6}}{dQ_{bot}^{5,6}} - \frac{dQ_{i+3n,bot}^{j,1,2}}{dQ_{bot}^{1,2}} \right)_{i,j} A_{7,8}^{7,8} Pho_{7,8}^{7,8}
\end{equation}

Furthermore,

\begin{equation}
Pio_{7,8}^{7,8} = M_4^4 + \left( \frac{dQ_{i+3n,bot}^{j,1,2}}{dQ_{bot}^{1,2}} \right)_{i,j}
\end{equation}

All of the nonzero blocks of \( Pio \) can be written in such simple form. These simpler solutions for \( Pio \) can be used to express \( Pih \) more succinctly. For instance, substituting 2.16 and the following identities

\begin{equation}
Q_{7,8}^{7,8} \left( Q_{bot}^{7,8} \right)^{-1} Q_{bot}^{1,2} = Q_{7,8}^{1,2} - \left( \frac{dQ_{i+3n,bot}^{j,7,8}}{dQ_{bot}^{7,8}} \right)_{i,j}
\end{equation}

\begin{equation}
\left( \frac{dQ_{i+3n,bot}^{j,1,2}}{dQ_{bot}^{1,2}} \right)_{i,j} \left( Q_{bot}^{7,8} \right)^{-1} Q_{bot}^{1,2} = \left( \frac{dQ_{i+3n,bot}^{j,1,2}}{dQ_{bot}^{1,2}} - \frac{dQ_{i+3n,bot}^{j,7,8}}{dQ_{bot}^{7,8}} \right)_{i,j} \equiv - \left( \frac{dQ_{i+3n,bot}^{j,7,8}}{dQ_{bot}^{7,8}} \right)_{i,j}
\end{equation}

\begin{equation}
\left( \frac{dQ_{i+3n,bot}^{j,5,6}}{dQ_{bot}^{5,6}} \right)_{i,j} \left( Q_{3,4}^{7,8} \right)^{-1} Q_{3,4}^{1,2} = \left( \frac{dQ_{i+3n,bot}^{j,5,6}}{dQ_{bot}^{5,6}} - \frac{dQ_{i+3n,bot}^{j,7,8}}{dQ_{bot}^{7,8}} \right)_{i,j}
\end{equation}

into equation 2.7 yields

\begin{equation}
Pih_{7,8}^{1,2} = M_4^4 \left( Q_{bot}^{7,8} \right)^{-1} Q_{bot}^{1,2} A_{1,2}^{1,2}
\end{equation}

Similarly,

\begin{equation}
Pih_{1,2}^{3,4} = M_1^1 \left( Q_{right}^{1,2} \right)^{-1} Q_{right}^{3,4} A_{3,4}^{3,4}
\end{equation}

The nonzero blocks of \( Pih, Pio, Pho, \) and \( Phh \) can be written in terms of the data and \( A \), a matrix with four \( n \times n \) blocks along its diagonal.
3. Elimination of Parameters. In this section $4(n^2 - 2n)$ of the $A_i^j$'s will be eliminated from the solutions of $Pho$, $Phh$, $Pih$, and $Pio$ in terms of $Q$ and $A$. There are two classes of identities: those resulting from zero-valued minors of predominantly “hidden-outgoing” submatrices and those from predominantly “incoming-hidden” submatrices. Derivations of the former identities parallel those of their counterparts for $4 \times 4$ systems. The latter are more difficult to derive. Only when $n = 4$ does the derivation found in [5] hold. An inductive argument is used to derive the remaining “incoming-hidden” identities in section 3.3.

The solutions derived above give the data for each of the $n/2 \times n/2$ subsystems. These solutions, however, do not obey the range conditions mentioned in section 1.3. The particular range conditions studied below are zero valued $(n/2+1) \times (n/2+1)$ minors of the subsystems’ data sets. Forcing these minors to be zero results in polynomials in the $A_i^j$ which factor and have a linear relevant term. Although there are many such conditions, only $4(n^2 - 2n)$ of them are independent. $8n$ of the parameters $A_i^j$ in the solutions for $Pho$, $Phh$, $Pih$, and $Pio$ cannot be eliminated by virtue of these range conditions.

Each of the four subsystems has a $2n \times 2n$ data matrix. The data matrix for the 1,1 subsystem is shown below:

\[
Q11 = \begin{bmatrix}
\frac{Pio}{n/4+1}, \ldots, n & \frac{Pih}{n/4+1}, \ldots, n \\
\frac{Pho}{n,n-1}, \ldots, 1 & \frac{Phh}{n,n-1}, \ldots, 1 \\
\frac{Pio}{n/4+1}, \ldots, n/4 & \frac{Pih}{n,1}, \ldots, n/4
\end{bmatrix}
\]

(3.22)

$Q11$ has many rank deficient submatrices. They are $(n/2 + 2k) \times (3n/2 - 2k)$ submatrices of rank $n/2$ (or less). Substituting the solutions for the modified transition probabilities into the zero-valued $(n/2 + 1) \times (n/2 + 1)$ minors forces highly nonlinear polynomials of the $A_i^j$'s to be identically zero. Since the solutions for $Pho$ are much simpler than those for $Pih$, the rank-deficient submatrices with the greatest proportion of hidden-outgoing transitions are the simplest to analyse. Those with a large fraction of
incoming-hidden transitions are more difficult and are left until section 3.3. One quarter of the conditions are identities of the form \( A_i^j = 0 \). The rest reduce (at a generic point) to four term linear equations. In the rest of this section, the \((n/2 + 2k) \times (3n/2 - 2k)\) right-left, left-right, top-bottom, and bottom-top rank deficient submatrices are labeled as \( Q_{ij}^k \), \( Q_{ij}^{k_1} \), \( Q_{ij}^{k_2} \), and \( Q_{ij}^{k_3} \) where \( i, j = 1, 2, k = 1, 2, \ldots, (n/2 - 1) \). For example, for some permutation matrices \( P, P' \),

\[
Q^{1_k}_{rl} = Q^{1_{(5n/4-k)(7n/4+k+1),\ldots,2n}}_{(5n/4-k+1),(7n/4+k)} = P \begin{bmatrix} Pio^{(k+1),\ldots,n}_{1, \ldots, k} & \pih^{(n+1),\ldots,(3n/2-k)}_{1, \ldots, k} \\ Pho^{(k+1),\ldots,n}_{1, \ldots, n/2+k} & Phh^{(n+1),\ldots,(3n/2-k)}_{1, \ldots, n/2+k} \end{bmatrix} P'
\]  

(3.23)

3.1. “Hidden-Outgoing” Transitions. The rank deficient submatrices in

\[
\begin{bmatrix} Pho^{(k+1),\ldots,n}_{1, \ldots, n/2+k} & Phh^{(n+1),\ldots,(3n/2-k)}_{1, \ldots, n/2+k} \end{bmatrix}
\]

have zero-valued determinants. Forcing these determinants to be zero results in identities and linear conditions upon the \( A_i^j \)'s. Not surprisingly, the simplest identities result from zero-valued “hidden-outgoing” minors. The entries of \( A \) furthest from the diagonal are in fact identically zero, as is shown by the following

CLAIM 1.

\[
A_{mn+k}^{mn+\beta} = 0 \equiv A_{mn+k}^{mn+\beta} \quad \text{where} \quad k = 1, 2, \ldots, (n/2 - 1); \quad \beta = (n/2 + k + 1), \ldots, n; \quad \text{and} \quad m = 0, 1, 2, 3.
\]

Proof. Since \( Q^{1_k}_{rl} \) is rank \( n/2 \), rank \( Pho^{(k+1),\ldots,n}_{1, \ldots, n/2+k} = n/2 \). Then

rank \( Pho^{(k+1),\ldots,n}_{1, \ldots, \beta - 1, \beta + 1, \ldots, n} \leq (n - k - 1) \); rank \( Pho^{1, \ldots, k-1, k+1, \ldots, n}_{1, \ldots, \beta - 1, \beta + 1, \ldots, n} \leq (n - 2) \). However,

\[
A_\beta = (-1)^{\beta+k}dPho^{1, \ldots, k-1, k+1, \ldots, n}_{1, \ldots, \beta - 1, \beta + 1, \ldots, n} / dPho^{1, \ldots, n}_{1, \ldots, n} = 0.
\]

The same argument holds upon the hidden-outgoing blocks within the other “hidden-outgoing” submatrices to prove the claim. □
These identities account for one quarter of the conditions upon the parameters $A_i^j$s. Another quarter of the conditions can be easily derived, following the method used to derive their counterparts for the $4 \times 4$ problem. They result from forcing the remaining $(n/2 + 1) \times (n/2 + 1)$ minors of 3.24 to be zero. Since

$$P hh_{i_1,\ldots,i_k,(n/2+k)}^{(n+1),\ldots,(3n/2-k)} = P h_{i_1,\ldots,i_k,(n/2+k)}^{1,2} \left(Q_{right}^{1,2}\right)^{-1} Q_{right}^{3,4} A_{3,4}^{(n+1),\ldots,(3n/2-k)}$$

it helps to define

$$v = \left(Q_{right}^{1,2}\right)^{-1} Q_{right}^{3,4} A_{3,4} = \frac{1}{dQ_{right}^{1,2}} \left(dQ_{right}^{1,2}\right)_{i,j} A_{3,4}^{3,4}$$

Then rank \[ P h_{i_1,\ldots,i_k,(n/2+k)}^{(n+1),\ldots,n} P h_{i_1,\ldots,i_k,(n/2+k)}^{1,2} v_{1,\ldots,(n/2-k)} \] = $n/2$ Since rank $P h_{i_1,\ldots,i_k,(n/2+k)}^{(n+1),\ldots,n}$ = $n/2$, it is sufficient to force for $\alpha = 1, \ldots, k$, $k = 1, 2, \ldots, (n/2 - 1)$

$$0 = \begin{vmatrix} P h_{i_1,\ldots,i_k,(n/2+\alpha)}^{(n/2+1),\ldots,n} P h_{i_1,\ldots,i_k,(n/2+\alpha)}^{1,\ldots,(n/2-k)} \end{vmatrix}$$

$$= \sum_{\eta=1}^{k} v_{\eta}^{(n/2-k)} \begin{vmatrix} P h_{i_1,\ldots,i_k,(n/2+\alpha)}^{(n/2+1),\ldots,n} \end{vmatrix}$$

$$= \sum_{\eta=1}^{\alpha} v_{\eta}^{(n/2-k)} d P h_{i_1,\ldots,i_k,(n/2+\alpha)}^{\eta,(n/2+1),\ldots,n}$$

Equation 3.27 follows from 3.26 because for $\eta > \alpha$, $d P h_{i_1,\ldots,i_k,(n/2+\alpha)}^{\eta,(n/2+1),\ldots,n} = 0$. Since 3.27 holds for $k$ different values of $\alpha$, it is a homogeneous system of $k$ equations for \{v_{\eta}^{(n/2-k)}\}_{\eta=1,\ldots,k}.

The Jacobian for this system is lower triangular and has a generically nonzero determinant, $\Pi_{i=1}^{k} d P h_{i_1,\ldots,i_k,(n/2+\alpha)}^{\eta,(n/2+1),\ldots,n}$. Therefore, $0 = v_{\alpha}^{(n/2-k)}$,

$$0 = v_{\alpha}^{(n/2-k)} = \sum_{j=1}^{n} d Q_{right}^{1,j,\ldots,\alpha-1,j+n,\alpha+1,\ldots,n} A_{j+n}^{(3n/2-k)}$$

for $\alpha = 1, 2, \ldots, k$

This identity forces the submatrix 3.24 to be rank $n/2$, as required by the range conditions upon $Q1_{i_1}^{k}$. Similar calculations hold for other primarily "hidden-outgoing"
rank deficient submatrices of $Q_{11}$, $Q_{12}$, $Q_{21}$, and $Q_{22}$. These conditions are shown below and can be used to eliminate $2\times 4 \times \sum_{k=1}^{n/2-1} k = n(n-2)$ of the $A_i^j$'s from the solutions for $Pio$, $Pih$, $Phh$, and $Pho$ in terms of $Q$ and $A$.

\[
\begin{array}{|c|c|}
\hline
\text{Identity} & \alpha \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{bottom}}^{1,\ldots,\alpha-1,j+3n,\alpha+1,\ldots,n} A_{j+3n}^{(7n/2+k+1)} & (n-k+1), \ldots, n \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{bottom}}^{3n+1,\ldots,3n+\alpha-1,j+3n+\alpha+1,\ldots,4n} A_{j}^{(n/2-k)} & 1, \ldots, k \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{left}}^{3n+1,\ldots,3n+\alpha-1,j+2n,3n+\alpha+1,\ldots,4n} A_{j+2n}^{(5n/2+k+1)} & (n-k+1), \ldots, n \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{right}}^{n+1,\ldots,n+\alpha-1,j+n+\alpha+1,\ldots,2n} A_{j}^{(n/2+k+1)} & (n-k+1), \ldots, n \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{top}}^{n+1,\ldots,n+\alpha-1,j+2n,n+\alpha+1,\ldots,3n} A_{j+2n}^{(5n/2-k)} & 1, \ldots, k \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{top}}^{2n+1,\ldots,2n+\alpha-1,j+n,2n+\alpha+1,\ldots,3n} A_{j+n}^{(3n/2+k+1)} & (n-k+1), \ldots, n \\
\hline
\sum_{j=1}^{n} d \hat{Q}_{\text{left}}^{2n+1,\ldots,2n+\alpha-1,j+3n,2n+\alpha+1,\ldots,3n} A_{j+3n}^{(7n/2-k)} & 1, \ldots, k \\
\hline
\end{array}
\]

These equations may be combined with the identities in claim 1 to eliminate $A_i^j$'s. Afterwards $4n^2 - 2n(n-2) = 2n^2 + 4n$ of the $A_i^j$'s remain. Modulo the identities in claim 1 and equations 3.28 and 3.29 the solutions for the subsystems' data can be written in terms of $Q$ and $2n^2 + 4n$ of the $A_i^j$'s. In section 3.3 another $2n(n-2)$ linear conditions upon the remaining $A_i^j$'s are derived.

3.2. More Matrix Identities. In the following section, Graßmann-Plücker relations and the Laplace expansion of a determinant are often used in conjunction with range conditions. For example, the Laplace expansion and Graßmann-Plücker relations can be used to simplify the minor
\[ (3.30) \quad \left| dQ_{\beta, \kappa}^{\gamma, \eta} \right|_{i,j=1, \ldots, m} = dQ_{\beta, \kappa}^{\gamma, \eta} (dQ_{\alpha}^{\gamma})^{m-1} \quad \text{where } \beta, \kappa \in \mathbb{R}^m \text{ and } \alpha, \gamma \in \mathbb{R}^n. \]

Furthermore, range conditions also imply that for \( \gamma = (3n + 1), \ldots, (7n/2 - k), \)

\[
Q_{1, \ldots, (n/2-k):\alpha; (7n/2-k+1), \ldots, 4n}^{(n/2+k+1), \ldots, 2n} \text{ is a } (n+1) \times (3n/2 - k) \text{ matrix of rank } n \text{ and }
\]

\[
Q_{1, \ldots, (n/2-k):\alpha; (7n/2-k+1), \ldots, 4n}^{(n/2+k+1), \ldots, 2n} \text{ is a } (n+1) \times (3n/2 - k + 1), \text{ rank } n \text{ matrix.}
\]

Adding a few more rows to these rank deficient rectangular matrices results in rank deficient square matrices. Therefore,

\[ (3.31) \quad dQ_{1, \ldots, (n/2-k);\alpha; (3n+1), \ldots, \gamma-1, \gamma+1, \ldots, 4n}^{(n/2+k+1), \ldots, 2n} = 0 = dQ_{1, \ldots, (n/2-k);\alpha; (3n+1), \ldots, 4n}^{(n/2+k+1), \ldots, 2n} \]

Finally,

\[ (3.32) \quad dQ_{\alpha; (n+1), \ldots, 2n}^{(3n+1), \ldots, 4n} \equiv 0 \quad \text{for each } j = (2n+1), \ldots, 3n \]

These identities are used to derive another \( 2n^2 - 4n \) coditions upon the remaining \( A_{i}^{j} \)'s in the following section.

### 3.3. “Incoming-Hidden” Conditions.

In this section the remaining range conditions upon the subsystem’s data are enforced. The submatrices of \( Q_{11}, Q_{12}, Q_{21}, \) and \( Q_{22} \) which should be rank deficient and (primarily) represent travel from incoming states to hidden states are used. Consider now the “incoming-hidden” submatrix

\[ (3.33) \quad Q_{11 k_{tb}} = \begin{bmatrix}
Pi_{1, \ldots, (n/2+k)}^{(n/2+k+1), \ldots, n} & Pi_{1, \ldots, (n/2+k)}^{(n+1), \ldots, 3n/2(7n/2+1), \ldots, 4n} \\
Ph_{1, \ldots, k}^{(n/2+k+1), \ldots, n} & Ph_{1, \ldots, k}^{(n+1), \ldots, 3n/2(7n/2+1), \ldots, 4n}
\end{bmatrix} \]

\( Q_{11 k_{tb}} \) should be of rank \( n/2 \). Only for \( k = (n/2 - 1) \) can \( Q_{11 k_{tb}} \) be factored as was done in [5] to derive linear conditions upon the \( A_{i}^{j} \)'s. An inductive argument is used to
derive analogous linear conditions from \( Q_{11b}^k \) for \( k \neq (n/2 - 1) \). Start on the inductive step by defining

\[
(3.34) \quad l_{1,\ldots,\eta} \equiv \left( \frac{dQ_{1bot}^7.8}{dQ_{bot}^1} - \frac{dQ_{1right}^3.4}{dQ_{right}^3.4} \right)_{i=1,\ldots,\eta}
\]

\[
(3.35) \quad L_{1,\ldots,\eta} = \begin{bmatrix} l_{1,\ldots,\eta} & A_{1,2}^{1,1} & \cdots & A_{1,2}^{1,n-\eta} \\ I_{\eta-n/2} & \theta^{n-\eta}_{\eta-n/2} \end{bmatrix}
\]

and recall the definition of \( M^1 \). Rather than forcing all of \( Q_{11b}^k \) to be rank \( n/2 \) we undertake a less ambitious endeavor, forcing

\[
(3.36) \quad \text{rank} \left[ \begin{array}{cc} Pio_{1,\ldots,(n/2+k)}^{(n/2+k+1),\ldots,n} & Pih_{1,\ldots,(n/2+k)}^{(n+1),\ldots,3n/2} \\ Pho_{1,\ldots,k}^{(n/2+k+1),\ldots,n} & Phh_{1,\ldots,k}^{(n+1),\ldots,3n/2} \end{array} \right] = \frac{n}{2}
\]

The blocks of 3.36 can be written as follows:

\[
(3.37) \quad Pio_{1,\ldots,(n/2+k)}^{(n/2+k+1),\ldots,n} = l_{1,\ldots,(n/2+k)} A_{1,2}^{1,1} Pho_{1}^{n/2+k+1,\ldots,n} + \frac{dQ_{1right}^3.4}{dQ_{right}^3.4}_{1,\ldots,n/2+k}
\]

\[
(3.38) \quad Pih_{1,\ldots,(n/2+k)}^{(n+1),\ldots,3n/2} = l_{1,\ldots,(n/2+k)} A_{1,2}^{1,1} Pho_{1}^{n/2+k+1,\ldots,n} \left( \frac{1}{Q_{right}^3.4} \right)^{-1} Q_{right}^3.4 A_{3,4}^3.4
\]

\[
(3.39) \quad Phh_{1,\ldots,k}^{(n+1),\ldots,3n/2} = Pho_{1,\ldots,k}^{1,2} \left( \frac{1}{Q_{right}^3.4} \right)^{-1} Q_{right}^3.4 A_{3,4}^3.4
\]

For \( k = (n/2 - 1) \) the matrix in 3.36 is the basis of the inductive argument and equals

\[
(3.40) \quad L_{1,\ldots,(n-1)} \text{Pho}_1^{1,2} + \begin{bmatrix} dQ_{1right}^3.4 \\ dQ_{right}^3.4 \end{bmatrix}_{i=1,\ldots,(n-1)}^{j=n} \theta^{n-1}_{(n/2-1)\times 1} \rightarrow L_{1,\ldots,(n-1)} \text{Pho}_1^{1,2} \left( \frac{1}{Q_{right}^3.4} \right)^{-1} Q_{right}^3.4 A_{3,4}^3.4
\]

\( n - 2 = 2k \) conditions are required to force a \( (3n/2 - 2) \times (n/2 + 1) \) matrix to be rank \( n/2 \). This occurs if for each \( \alpha = 2,3,\ldots,(n-1) \), row \( \alpha \) lies in the space spanned by row one and the last \( n/2 - 1 \) rows of 3.40. In that case, the minor taken from rows \( 1, \alpha; (n/2 + 1), \ldots, (n-1) \) of 3.40 is identically zero. This minor can be simplified, resulting in linear conditions upon the \( A_i^{n/2} \)s.
\[
0 = \begin{bmatrix}
    l_{1,\alpha} A_{1,2}^{1,2} \\
    I_{(n/2-1)} & \theta_1^{1,2}_{(n/2-1)}
\end{bmatrix}
\begin{bmatrix}
    \text{Pho}^n_1 : \text{Pho}^{1,2}_1 (Q_{\text{right}}^{1,2})^{-1} Q_{\text{right}}^{3,4} A_{3,4}^{3,4}
\end{bmatrix}
+ \\
\frac{C}{dQ_{\text{right}}^{3,4}}
\begin{bmatrix}
    \left( dQ_{i,\text{right}}^{3,4} \right)_{i=1,\alpha} \\
    \theta_{(n/2-1)\times 1}
\end{bmatrix}
\begin{bmatrix}
l_{1,\alpha} A_{1,2}^{1,2} \\
I_{(n/2-1)} & \theta_1^{1,2}_{(n/2-1)}
\end{bmatrix}
\]

(3.41)

\[
0 \pm \frac{C}{dQ_{\text{right}}^{3,4}}
\begin{bmatrix}
dQ_{i,\text{right}}^{3,4} \\
dQ_{\alpha,\text{right}}^{3,4}
\end{bmatrix}
\begin{bmatrix}
l_{1,\alpha} A_{1,2}^{1,2}
\end{bmatrix}
\]

(3.42)

where \( C = \left| \text{Pho}^{1,2}_1 (Q_{\text{right}}^{1,2})^{-1} Q_{\text{right}}^{3,4} A_{3,4}^{3,4} \right| \neq 0 \), just as in [5]. Equation 3.42 implies that

(3.43)

\[
\left( dQ_{i,\text{right}}^{3,4} \right)_{1,2,\ldots,(n-1)} \parallel (l)_{1,2,\ldots,(n-1)} A^{n/2}_{1,2}
\]

**CLAIM 2.** Forcing \( Q_{11}^{k}_{tb} \) to be rank \( n/2 \) for each \( k \) implies

\[
(l)_{1,2,\ldots,(n/2+k)} A^{k+1}_{1,2} \in \text{colspan} \left( dQ_{i,\text{right}}^{3,4} \right)_{i=1,2,\ldots,(n/2+k)}
\]

**Proof.** (by induction) Thanks to 3.43, the claim holds for \( k = n/2 - 1 \). The fact that rank \( Q_{tb}^{n/2-1} = n/2 \) results in 3.43. Similarly, enforcing rank \( Q_{tb}^{k} = n/2 \) implies the statement of the claim. Assuming the claim holds for \( k \), we shall show that it also holds for \( k - 1 \). By assumption,

(3.44) \[ n/2 = \text{rank } Q_{11}^{k}_{tb} \geq \text{rank } \begin{bmatrix}
P_{io}^{(n/2+k+1),\ldots,n}_{1,\ldots,(n/2+k)} & P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,(n/2+k)} \\
P_{io}^{(n/2+k+1),\ldots,n}_{1,\ldots,k} & P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,k}
\end{bmatrix}
\]

We want to show that

(3.45) \[ n/2 = \text{rank } Q_{11}^{(k-1)}_{tb} \geq \text{rank } \begin{bmatrix}
P_{io}^{(n/2+k),\ldots,n}_{1,\ldots,(n/2+k-1)} & P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,(n/2+k-1)} \\
P_{io}^{(n/2+k),\ldots,n}_{1,\ldots,(k-1)} & P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,(k-1)}
\end{bmatrix}
\]

Generically, rank \( P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,(n/2+k-1)} = n/2 \) so it is sufficient to require that the inequality in 3.45 be an equality. This is equivalent to forcing for each \( \alpha = (n/2 - k + 2), \ldots, (n/2 + k - 1) \)

(3.46) \[ 0 = \begin{bmatrix}
P_{io}^{(n/2+k),\ldots,n}_{1,\ldots,(n/2-k+1)\alpha} & P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,(n/2-k+1)\alpha} \\
P_{io}^{(n/2+k),\ldots,n}_{1,\ldots,(k-1)} & P_{ih}^{(n+1),\ldots,3n/2}_{1,\ldots,(k-1)}
\end{bmatrix}
\]
Plugging equations 3.37, 3.38, and 3.39 into 3.46 allows one to rewrite 3.46 as a sum of two minors. (The first column in 3.46 is the sum of two columns.) Both minors can be factored as is done below.

\[
0 = \begin{bmatrix}
 l_{1,\ldots,(n/2-k+1),\alpha}A_{1,2}^{1,2} \\
 I_{(k-1)}
\end{bmatrix}
\begin{bmatrix}
 P\phi_{1}^{(n/2+k)} \\
 \theta_{(k-1)}^{(n/2-k+1)}
\end{bmatrix}
\begin{bmatrix}
 P\phi_{1}^{1,2} Q_{right}^{1,2} & -Q_{right}^{3,4} A_{3,4}^{3,4}
\end{bmatrix}
\]

\[
(3.47) \quad + \quad \frac{C}{dQ_{right}^{3,4}} \begin{bmatrix}
 (dQ_{i,\text{right}}^{i,(n/2+k),3,4})_{i=1,\ldots,(n/2-k+1),\alpha} \\
 \theta_{(k-1)\times 1}
\end{bmatrix}
\begin{bmatrix}
 l_{1,\ldots,(n/2-k+1),\alpha}A_{1,2}^{1,2} \\
 I_{(k-1)}
\end{bmatrix}
\]

\[
(3.48) = 0 \pm \frac{C}{dQ_{right}^{3,4}} \begin{bmatrix}
 (dQ_{i,\text{right}}^{i,(n/2+k),3,4})_{i=1,\ldots,(n/2-k+1),\alpha} \\
 \theta_{(k-1)\times 1}
\end{bmatrix}
\begin{bmatrix}
 l_{1,\ldots,(n/2-k+1),\alpha}A_{1,2}^{k,\ldots,n/2} \\
 I_{(k-1)}
\end{bmatrix}
\]

Since 3.48 holds for all \( \alpha = (n/2 - k + 2), \ldots, (n/2 + k - 1) \),

\[
l_{1,\ldots,(n/2+k-1)}A_{1,2}^{k} \in \text{colspan} \begin{bmatrix}
 dQ_{i,\text{right}}^{i,(n/2+k),3,4} \\
 l_{i}^{(k+1),\ldots,n/2}
\end{bmatrix}_{i=1,\ldots,(n/2+(k-1))}
\]

\[
(3.49) \quad \in \text{colspan} \begin{bmatrix}
 dQ_{i,\text{right}}^{i,(n/2+k),3,4} \\
 j=(n/2+k),\ldots,n
\end{bmatrix}_{i=1,\ldots,(n/2+(k-1))}, \text{ by the inductive hypothesis.}
\]

\[
\square
\]

The above claim implies linear conditions upon the \( A_{i}^{j} \)s:

\[
(3.50) \quad 0 = \begin{bmatrix}
 (dQ_{i,\text{right}}^{i,(n/2+k+1),\ldots,n})_{i=1,\ldots,(n/2-k)\alpha} \\
 l_{1,\ldots,(n/2-k),\alpha}A_{1,2}^{k+1}
\end{bmatrix}
\]

for each \( \alpha = (n/2 - k + 1), \ldots, (n/2 + k) \). (This holds for \( k = (n/2 - 1), (n/2 - 2), \ldots, 1 \).

To avoid cumbersome notation in the next few lines we absorb the cumbersome-ness in the following notation. Define

\[
r \equiv 1, \ldots, (n/2 - k) ; \alpha
\]

\[
r_{m} \equiv 1, \ldots, (m - 1), (m + 1), \ldots, (n/2 - k) ; \alpha \quad \text{for} \quad m = 1, 2, \ldots, (n/2 - k)
\]

\[
r_{(n/2-k+1)} \equiv 1, \ldots, (n/2 - k)
\]

\[
l_{(n/2-k+1)} \equiv \ l_{\alpha}
\]

Note the dependence of \( r, r_{m}, \text{and } l_{(n/2-k+1)} \) upon \( \alpha \). Furthermore,

\[
(3.52) \quad \begin{bmatrix}
 (dQ_{i,\text{right}}^{i,(n/2+k+1),\ldots,n})_{i=1,\ldots,(n/2-k)\alpha} \\
 dQ_{r_{m},\text{right}}^{(n/2+k+1),\ldots,2n}
\end{bmatrix} = dQ_{r_{m},\text{right}}^{3,4} \left( dQ_{right}^{3,4} \right)^{n/2-k-1}
\]
according to identity 3.30. Expanding 3.50 along the last column and dividing by 
\((dQ_{right})^{n/2-k-1} \) yields a zero-valued vector-vector multiply.

\[
0 = \left( dQ_{right}^{\mathbf{3,4}} \right)^{-(n/2-k-1)} \sum_{m=1}^{n/2-k+1} (-1)^m \left( dQ_j^{\mathbf{3,4}} \right)_{i=\Gamma_m}^{j=(n/2+k+1),...,n} l_m A^{k+1}_{1,2} \\
(3.53) = \left( \sum_{m=1}^{n/2-k+1} (-1)^m dQ_{i=\Gamma_m}^{(n/2+k+1),...,2n} l_m \right) \cdot A^{k+1}_{1,2}
\]

where the following (painful) notation is used:

Graßmann-Plücker identities can be used to simplify the first vector in this vector-vector multiply

\[
\sum_{m=1}^{n/2-k+1} (-1)^m dQ_{i=\Gamma_m}^{(n/2+k+1),...,2n} l_m
(3.54)
\]

Graßmann identities imply

\[
\sum_{m=1}^{n/2-k+1} (-1)^m dQ_{i=\Gamma_m}^{(n/2+k+1),...,2n} dQ_j^{\mathbf{3,4}} = (-1)^{(k+1)} dQ_{i=\Gamma_m}^{(n/2+k+1),...,2n} dQ_{right}^{\mathbf{3,4}}
(3.55)
\]

Recall the definition of \( l \) and plug 3.55 into 3.54. Substituting the whole thing into 3.53 gives

\[
0 = \left( \sum_{m=1}^{n/2-k+1} (-1)^m dQ_{i=\Gamma_m}^{(n/2+k+1),...,2n} dQ_{right}^{\mathbf{7,8}} + (-1)^k dQ_{i=\Gamma_m}^{(n/2+k+1),...,2n} dQ_{bot}^{\mathbf{7,8}} \right)_{j=1,...,n} \cdot A^{k+1}_{1,2}
(3.56)
\]

This holds for all \( \alpha = (n/2 - k + 1), \ldots, (n/2 + k) \), so we have a homogeneous system of \( 2k \) linear conditions upon the \( A^{k+1}_i \). The system is unnecessarily complicated, however.

In claim 3 a simpler system of linear conditions is shown to be equivalent to 3.56.

**CLAIM 3.** The system of conditions in 3.56 is equivalent to

\[
0 = \left( dQ_j^{\beta;\mathbf{3,4}} \right)_{j=1,...,n} \cdot A^{k+1}_{1,2} \text{ for } \beta = (\frac{7n}{2} - k + 1), \ldots, (\frac{7n}{2} + k)
(3.57)
\]

**Proof.** The first step in this proof is to show that 3.57 constitutes a system of \( 2k \) independent conditions. The second step is to show that for any fixed
\( \alpha \in \{(n/2 - k + 1), \ldots, (n/2 + k)\} \), the condition from 3.56 corresponding to \( \alpha \) is redundant amongst those in 3.57. Note that 3.30 implies

\[
\begin{align*}
&\begin{vmatrix}
\frac{\partial dQ^{j(2n+1), \ldots, (7n/2-k)}}{\partial \beta(7n/2-k+1), \ldots, 4n; 3, 4}
\end{vmatrix}_{\beta=(7n/2-k+1), \ldots, (7n/2+k)} \\
&\beta=(7n/2-k+1), \ldots, (7n/2+k) \\
&\beta=(7n/2-k+1), \ldots, (7n/2+k)
\end{align*}
\]

\[= dQ^{j(2n+1), \ldots, (7n/2-k)}(7n/2-k+1), \ldots, 4n; 3, 4 \left( dQ^{(2n+1), \ldots, (7n/2-k)}(7n/2-k+1), \ldots, 4n; 3, 4 \right)^{2k-1} \]

\[\neq 0\]

in the generic case. Therefore, the Jacobian of 3.57 is generically of full rank, so all of the conditions are independent. It remains to show that the Jacobian of the conditions 3.57 and the condition from 3.56 corresponding to \( \alpha \) is a \((2k + 1) \times n\) matrix of rank \(2k\). We prove here that when \( \text{bot} = (n+1), \ldots, 2n \equiv 3, 4 \) and \( \text{right} = (3n+1), \ldots, 4n \equiv 7, 8 \) the first \(2k + 1\) columns are rank deficient; the same proof holds for other choices of columns. The determinant of the first \(2k + 1\) columns can be written as

\[\begin{vmatrix}
\sum_{m=1}^{n/2-k+1} (-1)^m dQ^{(n/2-k+1), \ldots, 2n; 7, 8} \frac{\partial dQ^{j(2n+1), \ldots, (7n/2-k)}}{\partial \beta(7n/2-k+1), \ldots, 4n; 3, 4} + (-1)^k dQ^{j(2n+1), \ldots, 2n; 7, 8} \frac{\partial dQ}{\partial \beta(7n/2-k+1), \ldots, (7n/2+k)}
\end{vmatrix}_{\beta=(7n/2-k+1), \ldots, (7n/2+k)} \]

(3.59)

To show that 3.59 is identically zero, it is broken into a sum of minors. The summands are minors which are easily expanded along the top row. For instance,

\[\frac{1}{C} \begin{vmatrix}
\frac{\partial dQ^{j, 7, 8}}{\partial \beta(7n/2-k+1), \ldots, 4n; 3, 4}
\end{vmatrix}_{\beta=(7n/2-k+1), \ldots, (7n/2+k)} =
\]

\[=\sum_{j=1}^{2k+1} (-1)^j dQ^{1, \ldots, j-1, j+1, \ldots, (2k+1); (2n+1), \ldots, (7n/2-k); \beta(7n/2-k+1), \ldots, 4n; 3, 4} \frac{\partial dQ^{j, 7, 8}}{\partial \beta(7n/2-k+1), \ldots, (7n/2+k)}
\]

\[=\sum_{j=2n+1}^{(7n/2-k)} (-1)^j dQ^{1, \ldots, j-1, j+1, \ldots, (2k+1); (2n+1), \ldots, (7n/2-k); \beta(7n/2-k+1), \ldots, 4n; (n+1), \ldots, 2n} \frac{\partial dQ^{j, 7, 8}}{\partial \beta(7n/2-k+1), \ldots, (7n/2+k)}
\]

\[=\sum_{j=2n+1}^{2k+1} (-1)^j dQ^{1, \ldots, j-1, j+1, \ldots, (2k+1); (2n+1), \ldots, (7n/2-k); \beta(7n/2-k+1), \ldots, 4n; (n+1), \ldots, 2n} \frac{\partial dQ^{j, 7, 8}}{\partial \beta(7n/2-k+1), \ldots, (7n/2+k)}
\]

\[= (-1)^k dQ^{1, \ldots, (2k+1); (2n+1), \ldots, (7n/2-k); \alpha(7n/2-k+1), \ldots, 4n; (n+1), \ldots, 2n} \frac{\partial dQ^{7, 8}}{\partial \alpha(7n/2-k+1), \ldots, 4n; 3, 4}
\]

(3.60)
where \( C \equiv \left( dQ^{(2n+1), \ldots,(7n/2-k)}_{(7n/2+k+1), \ldots,4n;(n+1), \ldots,2n} \right)^{2k-1} \). The last equality is a direct result of 3.31.

Also,

\[
\frac{1}{C} \sum_{j=1}^{2k+1} (-1)^j \sum_{j=1}^{(7n/2-k)} \sum_{j=1}^{(7n/2-k)} dQ^{j;j'=j-k, \ldots,(7n/2-k)}_{\beta,(7n/2-k+1), \ldots,4n;3,4, \ldots,2n} dQ^{j'=j-k, \ldots,(7n/2-k)}_{\beta,(7n/2-k+1), \ldots,4n;3,4, \ldots,2n} = 
\]

\[
= - \sum_{j=1}^{2k+1} (-1)^j dQ^{1, \ldots,(j-1),(j+1), \ldots,(7n/2-k)}_{(7n/2-k+1), \ldots,4n;(n+1), \ldots,2n} dQ^{j'=j-k, \ldots,(7n/2-k)}_{\beta,(7n/2-k+1), \ldots,4n;3,4, \ldots,2n} 
\]

\[
= - \sum_{j=1}^{2k+1} (-1)^j dQ^{1, \ldots,(j-1),(j+1), \ldots,(7n/2-k)}_{(7n/2-k+1), \ldots,4n;(n+1), \ldots,2n} dQ^{j'=j-k, \ldots,(7n/2-k)}_{\beta,(7n/2-k+1), \ldots,4n;3,4, \ldots,2n} 
\]

\[
= \pm \sum_{m=1}^{(7n/2-k+1)} (-1)^m dQ^{1, \ldots,(7n/2-k)}_{m,(7n/2-k+1), \ldots,4n;(n+1), \ldots,2n} dQ^{(7n/2-k)}_{m,(7n/2-k+1), \ldots,4n;(n+1), \ldots,2n} 
\]

\[
= (-1)^{(k+1)} \sum_{m=1}^{(n/2-k+1)} (-1)^m dQ^{1, \ldots,(7n/2-k)}_{m,(7n/2-k+1), \ldots,4n;3,4, \ldots,2n} dQ^{(7n/2-k)}_{m,(7n/2-k+1), \ldots,4n;3,4, \ldots,2n} 
\]

Expand 3.59 along its top row and substitute in identities 3.59 and 3.62. The result is a lengthy quadratic polynomial in minors of \( Q \) which turns out to be identically zero. The same argument applies to other column choices, so the Jacobian 3.59 is rank deficient. The conditions from 3.56 is redundant amongst those in 3.57. \( \Box \)

Similar arguments apply to other “incoming-hidden” rank deficient submatrices
and yield the following identities:

<table>
<thead>
<tr>
<th>Identity</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (7n/2+k+1), \ldots, 3, 4}^{j; (2n+1), \ldots, (7n/2-k)} A_j^{k+1}$</td>
<td>$(\frac{7n}{2} - k + 1), \ldots, (\frac{7n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (n+1), \ldots, (3n/2-k)}^{j; 7, 8} A_j^{n-k}$</td>
<td>$(\frac{3n}{2} - k + 1), \ldots, (\frac{3n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (n/2+k+1), \ldots, n}^{j+n; 1, \ldots, (n/2-k)} A_j^{n+k+1}$</td>
<td>$(\frac{n}{2} - k + 1), \ldots, (\frac{n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (2n+1), \ldots, (5n/2-k)}^{j+n; 1, 2} A_j^{2n-k}$</td>
<td>$(\frac{5n}{2} - k + 1), \ldots, (\frac{5n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (3n/2+k+1), \ldots, 2n}^{j+2n; 1, \ldots, (3n/2-k)} A_j^{2n+k+1}$</td>
<td>$(\frac{3n}{2} - k + 1), \ldots, (\frac{3n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (3n/2+k+1), \ldots, 3n}^{j+2n; 1, 2} A_j^{3n-k}$</td>
<td>$(\frac{7n}{2} - k + 1), \ldots, (\frac{7n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta (5n/2-k)}^{j+3n; 1, 2} A_j^{3n+k+1}$</td>
<td>$(\frac{5n}{2} - k + 1), \ldots, (\frac{5n}{2} + k)$</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} dQ_{\beta 1, \ldots, (n/2-k)}^{j+3n; 2n} A_j^{n-k}$</td>
<td>$(\frac{n}{2} - k + 1), \ldots, (\frac{n}{2} + k)$</td>
</tr>
</tbody>
</table>

(3.63)

The conditions above can be used to eliminate another $2n(n - 2)$ of the $A_i^j$'s from the solutions for $Q11$, $Q12$, $Q21$, and $Q22$. The end result is $8n$ parameter families of solutions for the $n/2 \times n/2$ subsystems’ data.

4. **Conclusion.** A practical and concise method of computing $8n$ parameter families of data sets for $n/2 \times n/2$ subsystems from the data generated by a $n \times n$ system is given by equations like 2.5, 2.6, and 2.7. This gives a $p$-parameter family of solu-
tions, where $p$ is large. Many of the parameters may be removed by enforcing range conditions upon the subsystems’ data. These conditions are expressed most succinctly in 3.28, 3.29, and 3.63. The conditions presented here are significantly improved over their predecessors. Earlier versions of these conditions were also linear in the $A_i^j$s but their coefficients were prohibitively cumbersome.

Before making even the most preliminary conclusions on the practicality of applying this Markovian model to imaging three major projects must be completed. The problem must be done in three dimensions, the underdetermined system of equations 1.3 must be closed, and stability of the recovery algorithm must be established. Optical imaging requires a three dimensional attack because low energy radiation will diffuse out of an imaging plane. Additional information is required to pick a single solution from the family of solutions found here. The author hopes to combine microscopic reversibility and time-of-flight information into a more practical model, permitting unique solution of the inverse problem. Finally, a careful stability study must be done. It is crucial that the problem be reasonably well-posed. (The isotropic case, for example, is extremely ill-posed [1], [17].) To have any hope of recovering accurate images from boundary value data it must be shown that both the physics of the problem and the inversion scheme choosen are stable.

REFERENCES


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