A VANISHING VISCOSITY APPROACH
ON THE DYNAMICS OF PHASE TRANSITIONS
IN VAN DER WAALS FLUIDS

By

Haitao Fan

IMA Preprint Series # 757
December 1990
A VANISHING VISCOSITY APPROACH
ON THE DYNAMICS OF PHASE TRANSITIONS
IN VAN DER WAALS FLUIDS

HAITAO FAN*

Abstract. The existence of of solutions for the Riemann problem of a mixed type
system is established by a vanishing similarity viscosity approach. The solutions we con-
structed are also admissible by the travelling wave criterion derived from the most common
form of viscosity. The assumption needed includes the constitutive function of van der
Waals fluids.

Table of Contents.

1. Introduction
2. The existence of (1.4)
3. a – priori estimates
4. The Existence of solutions of the Riemann problem (1.1)
5. Solutions of (1.1) admissible according to the travelling wave criterion also exist

§1. Introduction. The isothermal evolution of one dimensionl, compressible media
is generally governed in Eulerian coordinates by the quasilinear system of conservation
laws

\begin{equation}
\rho_t + (\rho u)_x = 0
\end{equation}

\begin{equation}
(\rho u)_t + (\rho u^2 + p(\rho))_x = 0 \quad x \in \mathbb{R}, t > 0.
\end{equation}

Here \(\rho = \rho(x,t), u = u(x,t)\) are the density and the velocity of the media respectively.
The pressure \(p(\rho)\) generally varies for different materials. Typically, for instance the ideal
gas, \(p(\rho)\) is strictly increasing, i.e. \(p'(\rho) > 0\), so that the system (1.1a,b) is hyperbolic. For
some material models, however, \(p'(\rho)\) may not be monotone. A typical example is the van
der Waals fluid, whose constitutive function \(p(\rho)\) is

\[ p(\rho) = \frac{RT}{w - b} - \frac{a}{w^2} \]

*IMA, University of Minnesota, Minneapolis, MN 55455
where $w = 1/\rho$. Let us then consider $p(\rho)$ which satisfies the following assumption: $p(\rho) \in C^1(\mathbb{R})$ and

$$(1.1c) \quad p'(\rho) > 0 \quad \text{for } \rho \notin [\alpha, \beta], \quad p'(\rho) < 0 \quad \text{for } \rho \in (\alpha, \beta).$$

The graph of such a function is shown in Fig.1. The region $\rho < \alpha$ and $\rho > \beta$ are called $\alpha$-phase and $\beta$-phase respectively. With this kind of function $p(\rho)$, the system (1.1a,b) is of hyperbolic-elliptic mixed type.

Fig.1

The analysis of the system (1.1a, b) usually starts with the initial value problem with the following initial data

$$(1.1d) \quad (u(x, 0), w(x, 0)) = \begin{cases} (u_-, w_-), & \text{for } x < 0, \\ (u_+, w_+), & \text{for } x > 0, \end{cases}$$

which is called the Riemann problem for (1.1a, b). To study the dynamics of phase transition which the system (1.1a,b,c) models, we further assume

$$(1.1e) \quad \rho_- < \alpha < \beta < \rho_+.$$

The system (1.1a, b) together with its corresponding system in Lagrangian coordinates

$$(1.2a) \quad u_t + p(w)_x = 0,$$

$$(1.2b) \quad w_t - u_x = 0, \quad x \in \mathbb{R}, t > 0,$$

has been instrumental in the development of the theory of system of conservation laws. For the system (1.1a, b) of hyperbolic type, there are a large literature on its Riemann problems (e.g. [10, 14, 15, 24, 27]). Most of them adopt the approach of the construction of rarefaction and shock curves which are admissible by some admissibility criterion. Epitomizing the experience gained through studies on several concrete systems, Liu [13, 16] proposed a comprehensive admissibility criterion which yields a satisfactory solution for the Riemann problem for strictly hyperbolic systems when the shock waves are of moderate strength. In their celebrated papers, Diperna [5], Ding, Cheng and Luo [4] constructed solutions for the Cauchy problems, where the states at infinity are the same, as limits of viscous regularizations or finite difference schemes.

The wave and shock curve constructing approach has also been applied to the Riemann problems of the hyperbolic-elliptic mixed type system (1.1a, b, c) or its companion systems in Lagrangian coordinate (1.2) (e.g. [8, 11, 12, 18-21]). For example, Shearer [21] proved
that the Riemann problem for (1.2) has solutions, admissible according to the viscosity-capillarity travelling wave criterion proposed by Slemrod [22], if initial datum are close to the Maxwell line. Fan [9] proved the uniqueness of the self similar solution of problem with Riemann initial datum separated by the elliptic region.

An alternative approach, called similarity viscosity approach (cf below) for convenience, was taken by Dafermos in his elegant papers [1, 2] to solve Riemann problems of a broad class of hyperbolic 2×2 system of conservation laws. The ideas and techniques in [1, 2] shades lights on latter developments. However, the assumptions needed in [1, 2] do not cover (1.1a, b) even in hyperbolic case. Recently, Slemrod & Tzavaras [24] employed this approach to solve the Riemann problem (1.1) of hyperbolic type. As pursued by Dafermos [1, 2], Tupcier [25, 26] and applied by Dafermos & Diperna [3] and Slemrod [23], they constructed solutions of (1.1), as $\epsilon \to 0+$ limits of solutions of the system

\begin{align}
(1.3a) & \quad \rho_t + (\rho u)_x = \epsilon t \rho_{xx}, \quad t > 0, x \in \mathbb{R}, \\
(1.3b) & \quad (\rho u)_t + (\rho u^2 + p(\rho))_x = \epsilon t (\rho u)_{xx}
\end{align}

with the initial value (1.1d). For convenience, we shall call solutions constructed in this way to be solutions admissible by similarity viscosity admissibility criterion. To take the advantage of the invariance of (1.2) and (1.1d) under the dilatation of the coordinates, they make the transformation of coordinates $\xi = x/t$ in (1.2). This reduce (1.2) and (1.1d) to a boundary value problem of a system of ordinary differential equations

\begin{align}
(1.4a) & \quad \epsilon \rho'' = -\xi \rho' + m', \\
(1.4b) & \quad \epsilon m'' = -\xi m' + \left(\frac{m^2}{\rho} + p(\rho)\right)', \\
(1.4c) & \quad (m(-\infty), \rho(-\infty)) = (m_-, \rho_-), \quad (m(+\infty), \rho(+\infty)) = (m_+, \rho_+)
\end{align}

where $m = u \rho, m_\pm = u \pm \rho_\pm$ and $''$ is $\frac{d^2}{d\xi^2}$. After difficult estimates involving using entropy pairs, they proved that solutions of Riemann problem for the hyperbolic system (1.1a, b) exist if $\int_{-1}^{\infty} \frac{\rho'}{\rho} d\rho = \infty$ or if $\frac{d}{d\rho} (\rho^2 p'(\rho)) > 0$ for $\rho > 0$ and $\int_{-1}^{\infty} \sqrt{\frac{p'(\rho)}{\rho}} d\rho = \infty$.

For mixed systems, the similarity viscosity approach was applied to its Riemann problem, with Riemann datum lying in different phase regions, by Slemrod [23] ans Fan [7, 9]. After carefully selecting an open subset in $C^1([-L, L]; \mathbb{R})$ and using Leray-Schauder type theory of fixed point, Slemrod [23] proved the system in Lagrangian coordinates corresponding to (1.4) has a solution. Later, Fan [7] proved the existence of weak solutions of (1.1) in Lagrangian coordinates under the assumption that

\begin{align}
(1.5a) & \quad p(1/w) \to \infty \quad \text{as } w \to -\infty,
\end{align}
(1.5b) \[ p(1/w) \to -\infty \text{ as } w \to \infty, \]

which is inphysical. The condition (1.5) is designed to prevent the formation of vacuum state in the solutions. Since the Lagrangian coordinate is not a "good" coordinate to handle vacuum state, the condition (1.5) seems to be a inevitable compromise we have to make in order to take the advantage of the simplicity of (1.2).

In this paper, I shall establish the existence for the Riemann problem (1.1) under the following assumption on \( p(\rho) \):

Assumption 1. Besides (1.1c), \( p(\rho) \) further satisfies

(1.6) \[ p(\rho) \to \infty \text{ as } \rho \to \infty, \text{ and } p(\rho) > p(0). \]

We define \( \delta > 0 \) by

(1.7) \[ p(\delta) = p(\beta), \quad 0 < \delta < \alpha. \]

Our program for this paper is the following: In §2, we prove the boundary value problem (1.4) has a solution if some \( a - priori \) estimates hold. In §3, we provide the estimates needed in §2. In this section, we present techniques which can give us bound of \( u_{\epsilon, \mu, L}(\xi) \) which are independent in \( \epsilon, \mu u > 0 \) and \( L > 1 \) rather than proving that \( u_{\epsilon, \mu, L}(\xi) \) is bounded uniformly in \( \mu > 0, L > 1 \) and then in \( \epsilon, \mu = 1, L = \infty \) as before. In §4, we further prove that solutions of (1.4) have total variations bounded uniformly in \( \epsilon \) and hence we can extract a convergent a.e. subsequence \( (u_{\epsilon_n}(\xi), w_{\epsilon_n}(\xi)) \) which converges to a weak solution of (1.1). This completes our construction of weak solutions of (1.1) by the similarity viscosity approach. In §5, we shall prove that the solutions we constructed in previous sections are also admissible according to the travelling wave criterion derived from the most common form of viscosity

(1.8a) \[ \rho_t + (\rho u)_x = \epsilon \rho_{xx}, \]

(1.8b) \[ (\rho u)_t + (\rho u^2 + p(\rho))_x = \epsilon (\rho u)_{xx}, \quad x \in \mathbb{R}, t > 0. \]

In Corollary 5.5, we imposed, for simplicity, an inessential assumptions on \( p(\rho) \), which is equivalent to, in Lagrangian coordinates, that any straight line in \( w, p \)-plane, \( w = 1/\rho \), intersects the graph of \( p(\rho) \) at at most finitely many points. As a consequence, solutions of (1.1) admissible according to the travelling wave criterion based on (1.8) exist if Assumption 1 holds.
§2. The existence of (1.4). To prove the existence of (1.4), we start with the following altered system

\[(2.1a)\quad \varepsilon \rho'' = -\xi \rho' + \mu m',\]

\[(2.1b)\quad \varepsilon m'' = -\xi m' + \mu \left( \frac{m^2}{\rho} + p(\rho) \right)',\]

\[(2.1c)\quad (\rho(-L), m(-L)) = (\rho_-, \rho_+ u_-), \quad (\rho(+L), m(+L)) = (\rho_+, \rho_+ u_+),\]

where \(\mu \in [0,1], L > 1\). The system (2.1) can be rewritten as

\[(2.2a)\quad \varepsilon \rho'' = -\xi \rho' + \mu (\rho u_\varepsilon)',\]

\[(2.2b)\quad \varepsilon (\rho u'' + 2 \rho' u') = -\xi \rho u' + \mu (\rho u u' + p(\rho)'),\]

\[(2.2c)\quad (\rho(\pm L), u(\pm L)) = (\rho_\pm, u_\pm).\]

The following lemma is borrowed from [24]:

**Lemma 2.1.** Let \((u(\xi), \rho(\xi))\) be a solution of (2.2) with \(\rho(\xi) > 0\). Then on any interval \((l_1, l_2) \subset [-L, L]\) for which \(p'(\rho(\xi)) > 0\), one of the following holds

(i) \(\rho(\xi)\) is strictly increasing (decreasing) with no critical point in \((l_1, l_2)\) while \(u(\xi)\) has at most one critical point in \((l_1, l_2)\) which must be a minimum (maximum).

(ii) \(u(\xi)\) is strictly increasing (decreasing) with no critical point in \((l_1, l_2)\) while \(\rho(\xi)\) has at most one critical point in \((l_1, l_2)\) which must be a minimum (maximum).

**Lemma 2.2.** Let \((u(\xi), \rho(\xi))\) be a solution of (2.2) with \(\mu > 0\) satisfying

\[(2.3)\quad \rho(\xi) > 0 \quad \text{for} \quad \xi \in [-L, L] \quad \text{and} \quad \rho'(\xi) > 0 \quad \text{when} \quad \rho(\xi) \in [\alpha, \beta].\]

Let \(\tau\) be a local maximum (minimum) point of \(u(\xi)\) in \((-L, L)\). Then

\[\rho(\tau) \in [\alpha, \beta] \quad (\rho(\tau) \notin (\alpha, \beta)).\]

**Proof.** Suppose, for contradiction, that \(\tau\) is a local maximum point of \(u(\xi)\) in \((-L, L)\) and \(\rho(\tau) \notin [\alpha, \beta]\). Without loss of generality we assume \(\rho(\tau) < \alpha\). By Lemma 2.1 \(\rho(\xi)\), in this case, is strictly decreasing when \(\rho(\xi) < \alpha\) and hence \(\rho(L) < \alpha\) which violates (2.2c).

Thus \(\rho(\tau) \in [\alpha, \beta]\).

Let \(\tau\) be a local minimum of \(u(\xi)\) in \((-L, L)\). Then (2.2b) reads

\[\varepsilon \rho(\tau) u''(\tau) = \mu p'(\rho(\tau)) \rho'(\tau) \geq 0.\]

By (2.3), \(\rho(\tau) \notin (\alpha, \beta)\). \(\square\)
Theorem 2.3. Suppose, for each solution \((u(\xi), \rho(\xi))\) of (2.2) satisfying (2.3), the following \(a-priori\) estimates

\[
(2.4a) \quad \sup_{-L \leq \xi \leq +L} (|m(\xi)| + |m'(\xi)| + |\rho(\xi)| + |\rho'(\xi)|) \leq M,
\]

\[
(2.4b) \quad \inf_{\xi \in [-L,L]} \rho(\xi) > \nu > 0
\]

hold, where \(\nu, M\) are constants independent of \(\mu \in [0,1]\) and \(L > 1\). Then (1.4) has a solution \((u(\xi), \rho(\xi))\) satisfying

\[
(2.5) \quad \rho(\xi) > 0, \quad \rho'(\xi) > 0 \quad \text{when} \quad \rho(\xi) \in [\alpha, \beta].
\]

Proof. We rewrite (2.1) as

\[
(2.6) \quad \varepsilon y''(\xi) = \mu f(y)' - \xi y'(\xi)
\]

where

\[
y(\xi) = \begin{pmatrix} \rho(\xi) \\ m(\xi) \end{pmatrix}, \quad f(y(\xi)) = \begin{pmatrix} m(\xi) \\ m^2(\xi) \rho(\xi) + p(\rho(\xi)) \end{pmatrix}
\]

A straightforward calculation shows

\[
y(\xi) = y(-L) + z(y) \int_{-L}^{\xi} \exp \left( \frac{-\zeta^2}{2\varepsilon} \right) d\tau + \frac{\mu}{\varepsilon} \int_{-L}^{\xi} f(y(\tau)) d\tau
\]

\[
- \frac{\mu^2}{\varepsilon} \int_{-L}^{\xi} \int_{-L}^{\zeta} \tau f(y(\tau)) \exp \left( \frac{\tau^2 - \zeta^2}{2\varepsilon} \right) d\tau d\zeta
\]

where

\[
z(x) = \frac{1}{\int_{-L}^{L} \exp \frac{-\xi^2}{2\varepsilon} d\xi} \left[ y(+L) - y(-L) - \frac{\mu}{\varepsilon} \int_{-L}^{L} f(x(\tau)) d\tau \right]
\]

\[
+ \frac{\mu}{\varepsilon^2} \int_{-L}^{L} \int_{-L}^{\zeta} \tau f((x(\tau)) \exp \left( \frac{\tau^2 - \zeta^2}{2\varepsilon} \right) d\tau d\zeta
\]

\[
= z_1(x) + \mu z_2(x)
\]
Consider the following bounded open subset of $C^1([-L, L]; \mathbb{R}^2)$:

\begin{equation}
\Omega := \{ (\rho, m) \in C^1([-L, L]; \mathbb{R}^2) \mid \| (\rho, m) \|_{C^1([-L, L]; \mathbb{R}^2)} < M + 1, \\
\inf_{\xi \in [-L, L]} \rho(\xi) > \nu/2 > 0, \text{ and } \rho'(\xi) > 0 \text{ when } \rho(\xi) \in [\alpha, \beta] \} .
\end{equation}

We define an integral operator

\[ T : \Omega \times [0, 1] \to C^1([-L, L]; \mathbb{R}^2) \]

by

\begin{equation}
T(x, \mu)(\xi) = y(-L) + z(x) \int_{-L}^{\xi} \exp\left(\frac{-\zeta^2}{2\varepsilon}\right) d\zeta + \frac{\mu}{\varepsilon} \int_{-L}^{\xi} f(x(\zeta)) d\zeta - \frac{\mu}{\varepsilon^2} \int_{-L}^{\xi} \int_{-L}^{\zeta} \tau f(x(\tau)) \exp\left(\frac{\tau^2 - \zeta^2}{2\varepsilon}\right) d\tau d\zeta .
\end{equation}

To prove our theorem, it suffices to show that $T(x, 1)$ has a fixed point in $\Omega$. It is a matter of routine analysis to show that $T$ maps $\Omega \times [0, 1]$ continuously into $C^1([-L, L]; \mathbb{R}^2)$. Furthermore, we can verify, by taking $\frac{d}{d\xi}$ twice on (2.10), that

\[ \varepsilon (T(x, \mu)(\xi))'' = \mu f(x(\xi))' - \xi (T(x, \mu)(\xi))' . \]

This implies that $T$ maps $\Omega \times [0, 1]$ into a bounded, with bound independent of $\mu$, subset of $C^2([-L, L]; \mathbb{R}^2)$. Thus $T$ is a compact operator from $C^1([-L, L]; \mathbb{R}^2) \times [0, 1]$ into $C^1([-L, L]; \mathbb{R}^2)$.

At this moment we would like to recall the following fixed point theorem (J. Mawhin [17], Thm IV.1).

**Proposition 2.4.** Let $X$ be a real normed vector space and $\Omega$ a bold open subset of $X$. Let $T : \Omega \times [0, 1] \to X$ be a compact operator. If

(i) $T(x, \mu) \neq x$ for $x \in \partial \Omega$ , $\mu \in [0, 1]$ , and (ii) $T(x, 0) = x$ for some $x \in \Omega$ ,

then $T(x, 1) = x$ has at least one solution in $\Omega$.

Here, we take $X = C^1([-l, +l]; \mathbb{R}^2)$. We can see that (ii) is satisfied by

\begin{equation}
x(\xi) = \frac{y(L) - y(-L)}{\int_{-L}^{\xi} \exp\left(\frac{-\zeta^2}{2\varepsilon}\right) d\zeta - \int_{-L}^{\xi} \exp\left(\frac{-\zeta^2}{2\varepsilon}\right) d\zeta + y(-L) \in \Omega .
\end{equation}
To verify (i) in Proposition 2.4, we assume \( x = (\rho, m) \in \partial \Omega \) and \( T(x, \mu) = x \) for some \( \mu \in [0,1] \). Then the (2.4) excludes the possibility of \( \|(\rho, m)\|_{C^1([-L,L]:\mathbb{R}^2)} = M + 1 \) or 
\[
\inf_{\xi \in [-L,L]} \rho(\xi) = \nu/2. \]
Thus

\[
(2.12) \quad \rho(\xi) \geq 0 \text{ when } \rho(\xi) \in [\alpha, \beta], \\
\rho'(\xi_0) = 0 \text{ and } \rho(\xi_0) \in [\alpha, \beta] \text{ for some } \xi_0 \in \mathbb{R}.
\]

We have the following three cases:

Case 1. \( \rho''(\xi_0) = 0 \).

In this case, (2.1b) implies \( m'(\xi_0) = 0 \) and hence \( \rho(\xi), m(\xi) \equiv (\rho(\xi_0), m(\xi_0)) \) by the uniqueness of initial value problem of (2.1a,b). This violates (2.1c). Thus this case cannot happen.

Case 2. \( \rho''(\xi_0) < 0 \).

Clearly, \( \xi_0 \) is a local maximum point of \( \rho(\xi) \). We claim that \( \rho(\xi_0) \not\in (\alpha, \beta] \). Indeed, if otherwise, there would be a \( \eta > 0 \) such that

\[
\rho'(\xi) < 0 \text{ and } \rho(\xi) \in (\alpha, \beta) \text{ for } \xi \in (\xi_0, \xi_0 + \eta).
\]

This violates (2.12) and hence impossible.

We further claim that \( \rho(\xi_0) \neq \alpha \) and thus Case 2 cannot happen. If otherwise,

\[
(2.13) \quad \rho'(\xi) < 0 \quad \text{and} \quad \rho(\xi) < \alpha \quad \text{for} \quad \xi \in (\xi_0; \xi_0 + \eta)
\]

for some \( \eta > 0 \). To satisfy the boundary condition \( \rho(+L) = \rho_+ > \beta > \alpha \), it is necessary for \( \rho(\xi) \) to have a local minimum point \( \xi_1 > \xi_0 \). Let \( \xi_1 \) be the infimum of such \( \xi_1 \). Then by Lemma 2.1,

\[
(2.14) \quad u'(\xi) > 0 \quad \text{for} \quad \xi \in (\xi_0, \xi_1]
\]

On the other hand, by (2.1a)

\[
0 > \rho''(\xi_0) = \mu \rho(\xi_0) u'(\xi_0)
\]

and hence \( u'(\xi_0) < 0 \) which violates (2.14).

Case 3. \( \rho''(\xi_0) > 0 \).

By arguments similar to what we used for Case 2, this case is also impossible.

From our discussion of above three cases, we conclude that

\[
T(x, \mu) \neq x \quad \text{for} \quad x \in \partial \Omega, \quad \mu \in [0,1].
\]
Therefore, by Proposition 2.4, we have proved the existence of (2.1) with \( \mu = 1 \).

To prove the existence of (1.4), we need to pass to the limit \( L \to \infty \). We follow Dafermos [1] and extend \((\rho(\xi), m(\xi))\) as follows

\[
(\rho(\xi; L), m(\rho; L)) = \begin{cases} 
(\rho_+, m_+) & \xi > L, \\
(\rho_-, m_-) & \xi < -L. 
\end{cases}
\]

By the hypothesis (2.4a), we see that \( \{ (\rho(\cdot L), m(\cdot L)) \} \) is precompact in \( C((-\infty, \infty); \mathbb{R}^2) \). So, there is a sequence \( L_n \to \infty \) as \( n \to \infty \) such that \( (\rho(\xi; L_n), m(\rho; L_n)) \to (\rho(\xi), m(\xi)) \) uniformly as \( n \to \infty \). By integrating (1.4a,b) from \( \xi \) to \( \xi_0 \), we can prove \((\rho(\xi), m(\xi))\) satisfies (1.4a,b). It remains to prove that \((\rho(\pm \infty), m(\pm \infty)) = (\rho_{\pm}, m_{\pm})\). To this end, we manipulate (1.4a,b) to obtain

\[
\frac{d}{d\xi} \left( \exp(\xi^2/2\varepsilon)y'(\xi) \right) = \frac{1}{\varepsilon} \left[ f(y(\xi))^y' \exp \left( \frac{\xi^2}{2\varepsilon} \right) \right]
\]
or

\[
\exp(\xi^2/2\varepsilon)y'(\xi) = y'(0) + \frac{1}{\varepsilon} \int_0^\xi \nabla f(y) y'(\zeta) \exp \left( \frac{\zeta^2}{2\varepsilon} \right) d\zeta.
\]

(2.15)

Applying (2.4) and Gronwall's inequality on (2.15), we obtain

\[
|y'(\xi)| \leq |y'(0)| \exp \left( \frac{2R|\xi| - \xi^2}{2\varepsilon} \right) 
\]

(2.16)

\[
\leq M \exp \left( \frac{2R|\xi| - \xi^2}{2\varepsilon} \right)
\]

where \( R > 0 \) depend at most on \( M, \nu \) and \( \varepsilon > 0 \). Inequality (2.16) holds for \( y(\xi; L) \) also. Then

\[
(\rho(\pm \infty), m(\pm \infty)) = (\rho_{\pm}, m_{\pm})
\]

follows from (2.16) easily.

From (2.4b), it is clear that \( \rho(\xi) > 0 \). By its construction, \( \rho'(\xi) \geq 0 \) when \( \rho(\xi) \in [\alpha, \beta] \). The same reasoning used in Case 2.3 yields that \( \rho'(\xi) > 0 \) when \( \rho(\xi) \in [\alpha, \beta] \)

**Theorem 2.5.** The conclusion of Theorem 2.3 remains valid if (2.4a) is replaced by

\[
\sup_{-L \leq \xi \leq L} (|\rho(\xi)| + |m(\xi)|) \leq M_1
\]

where \( M_1 \) is independent of \( \mu \in [0, 1] \) and \( L > 1 \).

**Proof.** The proof is the same as that of Theorem 1.3 in [23].
§3. a-priori estimates. In this section, we shall prove that each possible solution \((u_\varepsilon(\xi), \rho_\varepsilon(\xi))\) (2.1) satisfying

\[
(3.1) \quad \rho_\varepsilon(\xi) > 0, \quad \rho'_\varepsilon(\xi) > 0 \text{ when } \rho_\varepsilon(\xi) \in [\alpha, \beta].
\]

must satisfy the following inequalities

\[
\sup_{\xi \in [-L, L]} |u_\varepsilon(\xi)| \leq C
\]
\[
\sup_{\xi \in (-L, L)} \rho_\varepsilon(\xi) \leq C
\]
\[
\inf_{\xi \in (-L, L)} \rho_\varepsilon(\xi) \geq \nu > 0
\]

where \(C\) is a constant independent of \(\mu \in [0,1]\), \(L > 1\), \(\varepsilon > 0\) and \(\nu\) is a constant independent of \(\mu \in [0,1]\), \(L > 1\).

**Theorem 3.1.** Let \((u_\varepsilon(\xi), \rho_\varepsilon(\xi))\) be a solution of (2.1) satisfying (3.1). Then

\[
(3.2) \quad u_\varepsilon(\xi) \geq \min\left(u_-, u_+, \frac{\rho_+ u_+}{\rho_-}, -\left(\frac{\rho_+}{\rho_-} \max_{\rho \in [\rho_-, \rho_+]} |p'(\rho)|\right)^{1/2}\right).
\]

**Proof.** When \(\mu = 0\), the solution of (2.1) is given by (2.11) where \(u_\varepsilon(\xi)\) is clearly a monotone function and hence

\[
u_\varepsilon(\xi) \geq \min(u_+, u_-).
\]

Now we consider the case \(\mu > 0\). Without loss of generality, we can assume \(u_\varepsilon(\xi)\) has a local minimum point \(\tau_\varepsilon \in [-L, L]\) with

\[
(3.3) \quad u_\varepsilon(\tau_\varepsilon) < \min(u_+, u_-, 0), \quad u'_\varepsilon(\tau_\varepsilon) = 0.
\]

Then, by Lemma 2.1,

\[
(3.4a) \quad \rho'_\varepsilon(\tau_\varepsilon) > 0 \quad \text{and}
\]
\[
(3.4b) \quad \rho_- \leq \rho_\varepsilon(\tau_\varepsilon) \leq \rho_+.
\]

**Fig.2**

Consider the set

\[
(3.5) \quad A := \left\{ L \geq \zeta > \tau_\varepsilon \mid m_\varepsilon(\xi) < m_\varepsilon(\tau_\varepsilon) \text{ for any } \xi \in (\tau_\varepsilon, \zeta) \right\}.
\]

10
Since, by (3.3),

(3.6) \[ \frac{d m_\varepsilon(\xi)}{d \xi} \bigg|_{\xi=\varepsilon} = u_\varepsilon(\varepsilon) \rho_\varepsilon'(\varepsilon) < 0, \]

\( \varepsilon \) is nonempty and

(3.7) \[ \eta := \sup A > \varepsilon. \]

By Lemma 2.1 and (3.1), we can see that (cf. Fig.2)

(3.8) \[ \rho_\varepsilon(\varepsilon) < \rho_\varepsilon(\eta). \]

**Case 1.** \( \eta = L. \)

In this case

\[ m_\varepsilon(\varepsilon) \geq m_\varepsilon(L) = \rho_+ u_+. \]

Thus,

(3.9) \[ u_\varepsilon(\varepsilon) \geq \frac{\rho_+ u_+}{\rho_\varepsilon(\varepsilon)} \geq \min \left( 0, \frac{\rho_+}{\rho_-} u_+ \right). \]

**Case 2.** \( \eta < L \)

For this case, we infer from the definitions (3.5) and (3.7) that

(3.10) \[ m_\varepsilon(\eta) = m_\varepsilon(\varepsilon) \quad \text{and} \quad m_\varepsilon'(\eta) \geq 0. \]

Integrating (2.1b) over \((\varepsilon, \eta)\), we obtain

\[ 0 < \varepsilon m_\varepsilon'(\eta) - \varepsilon m_\varepsilon'(\varepsilon) = \int_{\varepsilon}^{\eta} -\zeta m_\varepsilon'(\xi)d\xi + \]

\[ + \mu \frac{m_\varepsilon^2(\eta)}{\rho_\varepsilon(\eta)} - \mu \frac{m_\varepsilon^2(\varepsilon)}{\rho_\varepsilon(\varepsilon)} + \mu p(\rho_\varepsilon(\eta)) - \mu p(\rho_\varepsilon(\varepsilon)) \]

\[ = \int_{\varepsilon}^{\eta} (m_\varepsilon(\xi) - m_\varepsilon(\varepsilon))d\xi + \mu m_\varepsilon^2(\varepsilon) \left( \frac{1}{\rho_\varepsilon(\eta)} - \frac{1}{\rho_\varepsilon(\varepsilon)} \right) \]

\[ + \mu (p(\rho_\varepsilon(\eta)) - p(\rho_\varepsilon(\varepsilon))). \]

This and (3.7), (3.8) and (3.10) imply that

\[ 0 < m_\varepsilon^2(\varepsilon) \left( \frac{1}{\rho_\varepsilon(\eta)} - \frac{1}{\rho_\varepsilon(\varepsilon)} \right) + p(\rho_\varepsilon(\eta)) - p(\rho_\varepsilon(\varepsilon)) \]

11
or

\[ m_e^2(\tau_e) \leq \rho_e(\tau_e) \rho_e(\eta) \frac{p(\rho_e(\eta)) - p(\rho_e(\tau_e))}{\rho_e(\eta) - \rho_e(\tau_e)}. \]  

If \( \rho_e(\eta) \leq \rho_+ \), then (3.11) and (3.4) yields

\[ u_e^2(\tau_e) \leq \frac{\rho_+}{\rho_-} \max_{\rho \in [\rho_-, \rho_+]} |p'(\rho)|, \]

or

\[ u_e(\tau_e) \geq \left\{ \frac{\rho_+}{\rho_-} \max_{\rho \in [\rho_-, \rho_+]} |p'(\rho)| \right\}^{1/2}. \]  

We finish our proof by verifying the claim that \( \rho_e(\eta) \leq \rho_+ \). Indeed, if \( \rho_e(\eta) > \rho_+ \), then \( \rho_e(\xi) \) has a maxima and hence, by Lemma 2.1,

\[ u_e'(\xi) < 0 \quad \text{when} \quad \rho_e(\xi) > \beta, \quad \text{and} \quad \rho_e(\tau_e) \leq \alpha. \]

Then there is a \( \tau_e \leq \theta < \eta \) (cf. Fig.2) such that \( \rho_e(\theta) = \rho_+ \). Recalling (3.3) and (3.10), we conclude

\[ m_e(\theta) = \rho_e(\theta) u_e(\theta) > \rho_e(\eta) u_e(\eta) = m_e(\eta) \]

which is in contradiction with (3.7). The inequality (3.2) is a combination of (3.12), (3.9) and (3.3). \( \Box \)

**Theorem 3.2.** Let \((u_e(\xi), \rho_e(\xi))\) be a solution of (2.1) satisfying (3.1). Then

\[ u_e(\xi) \leq C \]

where \( C \) is a constant independent of \( \mu \in [0, 1] \), \( L > 1 \) and \( \varepsilon > 0 \).

**Proof.** Assume the contrary, then there is a sequence \( \{(\varepsilon_n, \mu_n, L_n)\} \) such that each \( u_{\varepsilon_n}(\xi) \) has a local maximum point \( \tau_n \in [-L_n, L_n] \) and

\[ u_{\varepsilon_n}(\tau_n) \to +\infty \quad \text{as} \quad n \to \infty. \]  

By Lemma 2.1,

\[ \rho_{\varepsilon_n}(\tau_n) \in (\alpha, \beta). \]
For simplicity, we shall write $\epsilon_n, \mu_n, L_n$ as $\epsilon, \mu, L$ in this proof. Integrating (2.2b) from $\tau_n$ to $\xi$, we obtain

$$
\varepsilon u'_\epsilon(\xi)\rho_\epsilon(\xi) = \int_{\tau_n}^{\xi} (-\zeta \rho_\epsilon(\zeta) + \mu \rho_\epsilon(\zeta) u_\epsilon(\zeta) - \varepsilon \rho'_\epsilon(\zeta)) u'_\epsilon(\zeta) d\zeta + \mu p(\rho_\epsilon(\xi)) - \mu p(\rho_\epsilon(\tau_n)).
$$

(3.16)

We assume that $f_\epsilon(\tau_n) \geq 0$ where

$$f_\epsilon(\xi) = -\xi \rho_\epsilon(\xi) + m_\epsilon(\xi) - \varepsilon \rho'_\epsilon(\xi).$$

(3.17)

A simple calculation based on (2.2a) shows that

$$\frac{df_\epsilon}{d\xi} = -\rho_\epsilon(\xi) < 0.$$ 

(3.18)

Thus $f_\epsilon(\xi) > 0$ for any $\xi < \tau_n$.

By Lemma 2.2, the subset of $\mathbb{R}$

$$B := \{ \eta \in [-L, L] \mid \eta < \tau_\epsilon, u'_\epsilon(\xi) > 0, \xi \in [\eta, \tau_\epsilon) \}$$

(3.19)

is nonempty. Letting $\xi \in B$ in (3.16) and using (3.15), we obtain

$$0 < \varepsilon u'_\epsilon(\xi)\rho_\epsilon(\xi) = \int_{\tau_\epsilon}^{\xi} f_\epsilon(\zeta) u'_\epsilon(\zeta) d\zeta + \mu(p(\rho_\epsilon(\xi)) - p(\rho_\epsilon(\tau_\epsilon)))$$

$$< \mu[p(\rho_\epsilon(\xi)) - p(\rho_\epsilon(\tau_\epsilon))] \leq \mu[p(\rho_\epsilon(\xi)) - p(\beta)].$$

Since $\rho'_\epsilon(\theta) > 0$ when $\rho_\epsilon(\theta) \in (\alpha, \beta)$,

(3.21)

$$\rho_\epsilon(\xi) < \rho_\epsilon(\tau_\epsilon) < \beta.$$

An inspection on the graph of $p(\rho)$ and (3.20) tell us that

(3.22)

$$\rho_\epsilon(\xi) > \delta > 0$$

for any $\xi \in B$.

(3.20) and (3.22) give us another useful inequality:

$$0 < \varepsilon u'_\epsilon(\xi) < \frac{\mu}{\rho_\epsilon(\xi)}(p(\rho_\epsilon(\xi)) - p(\beta))$$

$$\leq \frac{\mu}{\delta}(p(\alpha) - p(\beta))$$

(3.23)
for $\xi \in B$.

Claim: There is an $\eta_{\epsilon} \in (\inf B, \tau_{\epsilon})$ such that

\[
(3.24a) \quad u_{\epsilon}(\eta_{\epsilon}) \leq \max(u_+, u_-) + \frac{(\gamma - \delta)}{\delta} \max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right) + \frac{4[p(\alpha) - p(\beta)]}{\delta \max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)},
\]

\[
(3.24b) \quad \frac{d\rho_{\epsilon}(\xi)}{du_{\epsilon}(\xi)} \bigg|_{\xi = \eta_{\epsilon}} = \frac{\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)}.
\]

By Lemmas 2.1, 2.2,

\[
(3.25) \quad u_{\epsilon}(\inf B) \leq u_-.
\]

By virtual of (3.14), we can take $\xi_{\epsilon} \in B$ such that

\[
(3.26) \quad u_{\epsilon}(\xi_{\epsilon}) = u_{\epsilon}(\inf B) + \frac{(\gamma - \delta)}{\delta} \max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)
\]

\[
\leq u_- + \frac{(\gamma - \delta)}{\delta} \max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right).
\]

There is a $\theta \in (\inf B, \xi_{\epsilon})$ such that

\[
(3.27) \quad \frac{d\rho_{\epsilon}(\xi)}{du_{\epsilon}(\xi)} \bigg|_{\xi = \theta} = \frac{\rho_{\epsilon}(\xi_{\epsilon}) - \rho_{\epsilon}(\inf B)}{u_{\epsilon}(\xi_{\epsilon}) - u_{\epsilon}(\inf B)}.
\]

Substituting the denominator of the above by equation (3.26) and noticing that

\[
|\rho_{\epsilon}(\xi_{\epsilon}) - \rho_{\epsilon}(\inf B)| \leq \gamma - \delta
\]

we obtain

\[
\left| \frac{d\rho_{\epsilon}(\xi)}{du_{\epsilon}(\xi)} \bigg|_{\xi = \theta} \right| \leq \frac{\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)}.
\]

The subset $B_1$ of $B$ defined by

\[
(3.28) \quad B_1 := \left\{ \eta \in B \mid \eta > \theta, \left| \frac{d\rho_{\epsilon}(\xi)}{du_{\epsilon}(\xi)} \right| \leq \frac{\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)} \text{ for } \xi \in [\theta, \eta] \right\}
\]

is nonempty because $\theta \in B_1$. 

14
We deduce from (2.2) that

\[
\frac{d^2 \rho_\varepsilon(\xi)}{du_\varepsilon^2(\xi)} = \frac{\mu \rho_\varepsilon(\xi)}{\varepsilon^2 u_\varepsilon'(\xi)} \left( 1 - \frac{1}{\rho_\varepsilon^2(\xi)} \left( p'(\rho_\varepsilon(\xi)) - 2\varepsilon u_\varepsilon'(\xi) / \mu \right) \left( \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right)^2 \right).
\]

For \( \xi \in B_1 \), (3.29), (3.23) and (3.19), imply

\[
\frac{d^2 \rho_\varepsilon(\xi)}{du_\varepsilon^2(\xi)} \geq \frac{\mu \rho_\varepsilon(\xi)}{\varepsilon^2 u_\varepsilon'(\xi)} \left( 1 - \frac{p'(\rho_\varepsilon(\xi))}{\rho_\varepsilon^2(\xi)} \left( \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right)^2 \right) \geq \frac{\delta^2}{2(p(\alpha) - p(\beta))}.
\]

Now we show that (3.24) hold at \( \eta_\varepsilon = \text{sup} B_1 \). Indeed, by definition (3.28)

\[
\left| \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right|_{\xi = \eta_\varepsilon} = \frac{\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)}.
\]

If

\[
\left| \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right|_{\xi = \eta_\varepsilon} = \frac{-\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)},
\]

then (3.30) implies that

\[
\frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \left|_{\xi = \eta_\varepsilon} < \frac{d \rho_\varepsilon(\xi)}{d \theta_\varepsilon(\xi)} < 0 \right.
\]

for \( \xi \in (\eta_\varepsilon, \eta_\varepsilon + \nu) \) for some \( \nu > 0 \). This, however, contradicts the definition \( \eta_\varepsilon := \text{sup} B_1 \). Thus

\[
\left(3.32\right) \quad \left| \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right|_{\xi = \eta_\varepsilon} = \frac{\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)}.
\]

Combining (3.32), (3.28), (3.30) and (3.19), we obtain

\[
\frac{2\delta}{\max_{\rho \in [\delta, \gamma]} \left( \sqrt{2|p'(\rho)|} \right)} \geq \left| \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right|_{\xi = \theta} - \left| \frac{d \rho_\varepsilon(\xi)}{d u_\varepsilon(\xi)} \right|_{\xi = \eta_\varepsilon} = \left| \int_{\eta_\varepsilon}^{\theta} \frac{d^2 \rho_\varepsilon(\zeta)}{d u_\varepsilon^2(\zeta)} d(u_\varepsilon(\zeta)) \right| \geq \frac{\delta^2}{2(p(\alpha) - p(\beta))} |u_\varepsilon(\theta) - u_\varepsilon(\eta_\varepsilon)|.
\]
Noticing (3.26) and that $\xi_\epsilon \leq \theta \leq \text{sup } B_1 < \tau_\epsilon$, we conclude that

$$u(\eta_\epsilon) \leq u_\epsilon(\theta) + \frac{4[p(\alpha) - p(\beta)]}{\delta \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)}$$

$$\leq u_\epsilon(\xi_\delta) + \frac{4[p(\alpha) - p(\beta)]}{\delta \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)}$$

(3.33)

$$\leq \max(u_-, u_+) + \frac{(\gamma - \delta)}{\delta} \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right) + \frac{4[p(\alpha) - p(\beta)]}{\delta \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)}.$$

Thus (3.24) holds at $\eta_\epsilon = \text{sup } B_1$. Similar analysis on (3.29) shows that

$$\frac{d\rho_\epsilon(\xi)}{du_\epsilon(\xi)} \geq \frac{\delta}{\max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)}$$

or

(3.34)

$$\frac{du_\epsilon(\xi)}{d\rho_\epsilon(\xi)} \leq \frac{1}{\delta} \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)$$

for $\xi \in [\eta_\epsilon, \tau_\epsilon]$. Then, by applying (3.34), (3.15) and (3.22) on

$$u_\epsilon(\tau_\epsilon) - u_\epsilon(\eta_\epsilon) = \int_{\eta_\epsilon}^{\tau_\epsilon} \frac{du_\epsilon(\xi)}{d\rho_\epsilon(\xi)} d(\rho_\epsilon(\xi)),$$

we conclude that

(3.35)

$$u(\tau_\epsilon) \leq u_\epsilon(\eta_\epsilon) + \frac{1}{\delta} (\beta - \delta) \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)$$

$$\leq \max(u_-, u_+) + \frac{2(\gamma - \delta)}{\delta} \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)$$

$$+ \frac{4[p(\alpha) - p(\beta)]}{\delta \max_{\rho \in [\delta, \gamma]} \left(\sqrt{2|p'(\rho)|}\right)}.$$

The contradiction between (3.35) and (3.14) completes our proof. □

**Theorem 3.3.** Let $(u_\epsilon(\xi), \rho_\epsilon(\xi))$ be a solution of (2.1) satisfying (3.1). Then

(i) $\rho_\epsilon(\xi) \leq C(\epsilon)$ where $C(\epsilon)$ is independent of $\mu \in [0, 1]$, $L > 1$. 

16
(ii) if \( p(\rho) \) satisfies Assumption 1 and if \( \mu = 1 \), then \( \rho_\varepsilon(\xi) \leq C \) where \( C \) is independent of \( L > 1 \) and \( \varepsilon > 0 \).

Proof. Without loss of generality, we assume \( \rho_\varepsilon(\xi) \) has a local maximum point \( \tau_\varepsilon \). After some manipulations on (2.2a), we obtain

\[
\frac{\varepsilon \rho''_\varepsilon(\xi)}{\rho_\varepsilon(\xi)} - \frac{\varepsilon \rho'^2_\varepsilon(\xi)}{\rho^2_\varepsilon(\xi)} = \frac{1}{\rho^2_\varepsilon(\xi)} (-\xi \rho_\varepsilon(\xi) + \mu \rho_\varepsilon(\xi) u_\varepsilon(\xi) - \varepsilon \rho'_\varepsilon(\xi)) \rho'_\varepsilon(\xi) + \mu u'_\varepsilon(\xi). \tag{3.36}
\]

We assume that

\[
f_\varepsilon(\tau_\varepsilon) \geq 0 \tag{3.37}
\]

where

\[
f_\varepsilon(\xi) = -\xi \rho_\varepsilon(\xi) + \mu \rho_\varepsilon(\xi) u_\varepsilon(\xi) - \varepsilon \rho'_\varepsilon(\xi). \tag{3.38}
\]

The proof for the other case is similar.

Since

\[
\frac{df_\varepsilon(\xi)}{d\xi} = -\rho_\varepsilon(\xi) < 0, \tag{3.39}
\]

\( f_\varepsilon(\xi) \) is a strictly decreasing function. By Lemma 2.1, there is an \( \eta < \tau_\varepsilon \) such that

\[
\rho_\varepsilon(\eta) = \beta, \quad \text{and} \quad \rho'_\varepsilon(\xi) > 0, \quad u'_\varepsilon(\xi) < 0 \quad \text{for any} \quad \xi \in [\eta, \tau_\varepsilon). \tag{3.40a,b}
\]

Integrating (3.36) over \((\xi, \tau_\varepsilon)\) for \( \xi \in [\eta, \tau_\varepsilon) \), we obtain

\[
-\frac{\varepsilon \rho'_\varepsilon(\xi)}{\rho_\varepsilon(\xi)} = \int_{\xi}^{\tau_\varepsilon} \frac{f_\varepsilon(\zeta)}{\rho^2_\varepsilon(\zeta)} \rho'_\varepsilon(\zeta) d\zeta + \mu (u_\varepsilon(\tau_\varepsilon) - u_\varepsilon(\zeta)). \tag{3.41}
\]

If

\[
\eta \leq \xi \leq \tau_\varepsilon \quad \text{and} \quad f_\varepsilon(\xi) \geq \rho_\varepsilon(\xi), \tag{3.42}
\]

then, by the monotonicity of \( f_\varepsilon(\xi) \) and (3.40b),

\[
f_\varepsilon(\theta) > \rho_\varepsilon(\theta) \quad \text{for} \quad \theta \in [\eta, \xi). \tag{3.43}
\]
Recalling (3.40a), (3.37), (3.41), we derive that

\[
\ln \rho_\epsilon(\xi) - \ln \beta = \int_{\eta}^{\xi} \frac{\rho'_\epsilon(\zeta)}{\rho_\epsilon(\zeta)} \, d\zeta \leq \int_{\eta}^{\xi} \frac{f_\epsilon(\zeta)}{\rho_\epsilon^2(\zeta)} \rho'_\epsilon(\zeta) \, d\zeta
\]

(3.44)

\[
\leq \int_{\eta}^{\tau_\epsilon} \frac{f_\epsilon(\zeta)}{\rho_\epsilon^2(\zeta)} \rho'_\epsilon(\zeta) \, d\zeta = -\frac{\epsilon \rho'_\epsilon(\eta)}{\rho_\epsilon(\eta)} + \mu(u_\epsilon(\eta) - u_\epsilon(\tau_\epsilon))
\]

\[
\leq \mu(u_\epsilon(\eta) - u_\epsilon(\tau_\epsilon)) \leq u^* - u_\star.
\]

where

\[
u^* := \sup \left\{ u_\epsilon(\xi) \mid \epsilon > 0, \mu \in [0,1], L > 1, \xi \in [-L,L] \right\},
\]

\[
u_\star := \inf \left\{ u_\epsilon(\xi) \mid \epsilon > 0, \mu u \in [0,1], L > 1, \xi \in [-L,L] \right\}
\]

whose finiteness are guaranteed by Theorems 3.1 and 3.2. Thus \(\rho_\epsilon(\xi)\) is bounded from above uniformly in \(\mu \in [0,1], L > 1\) and \(\epsilon > 0\) if (3.42) is satisfied.

Suppose \(0 \leq f_\epsilon(\tau_\epsilon) < \rho_\epsilon(\tau_\epsilon)\). We define

\[
(3.45) \quad \xi_0 = \begin{cases} 
\xi, & \text{if there is } \xi \text{ satisfying (3.42)}, \\
\eta, & \text{else}.
\end{cases}
\]

Then

\[
(3.46) \quad \xi_0 \in [\eta, \tau_\epsilon], \quad f_\epsilon(\xi_0) \leq \rho_\epsilon(\xi_0) < C,
\]

and \(\rho_\epsilon(\xi) \leq C\) for \(\xi \leq \xi_0\),

where \(C\) is a constant independent of \(\mu \in [0,1], L > 1, \epsilon > 0\). Recalling (3.39), and (3.40b), we have

\[
\rho(\xi_0) \geq f_\epsilon(\xi_\epsilon) \geq f_\epsilon(\xi_0) - f_\epsilon(\tau_\epsilon)
\]

\[
= \int_{\xi_0}^{\tau_\epsilon} \rho_\epsilon(\zeta) \, d\zeta \geq \rho_\epsilon(\xi_0)(\tau_\epsilon - \xi_0) > 0
\]

and hence

\[
(3.47) \quad 1 \geq \tau_\epsilon - \xi_0 > 0.
\]

18
Now, (3.41) and (3.47) imply that
\[
\varepsilon (\ln \rho_\varepsilon (\tau_\varepsilon) - \ln \rho_\varepsilon (\xi_0)) = \int_{\xi_0}^{\tau_\varepsilon} \frac{\varepsilon \rho_\varepsilon' (\zeta)}{\rho_\varepsilon (\zeta)} \, d\zeta
\]
\[
= \int_{\xi_0}^{\tau_\varepsilon} \left[ - \int_{\zeta}^{\tau_\varepsilon} \frac{f_\varepsilon (\theta)}{\rho_\varepsilon^2 (\theta)} \rho_\varepsilon' (\theta) \, d\theta + \mu (u(\tau_\varepsilon) - u(\zeta)) \right] \, d\zeta
\]
\[
\leq \mu (u^* - u_*) (\tau_\varepsilon - \xi_0) \leq u^* - u_*
\]

and hence \(\rho_\varepsilon (\xi)\) is bounded from above uniformly in \(\mu \in [0, 1]\) and \(L > 1\).

It remains to prove that \(\rho_\varepsilon (\tau_\varepsilon)\) is bounded from above uniformly in \(\varepsilon > 0\) when \(\mu = 1\) and \(L = \infty\) in the case
\[
0 \leq f(\tau_\varepsilon) < \rho_\varepsilon (\tau_\varepsilon).
\]

To this end, we choose \(\theta \in (\xi_0 - 1, \xi_0)\), where \(\xi_0\) is defined by (3.45), such that
\[
(3.48) \quad |u'(\theta)| \leq u^* - u_*.
\]

We rewrite (2.1b) with \(\mu = 1\) as follows:
\[
\varepsilon u''_\varepsilon (\xi) \rho_\varepsilon (\xi) + \varepsilon u'_\varepsilon (\xi) \rho_\varepsilon' (\xi) = f_\varepsilon (\xi) u'_\varepsilon (\xi) + p(\rho_\varepsilon (\xi))'.
\]

Integrating above equation over \((\theta, \tau_\varepsilon)\), we obtain
\[
\varepsilon \int_{\theta}^{\tau_\varepsilon} (u''_\varepsilon (\xi) \rho_\varepsilon (\xi) + u'_\varepsilon (\xi) \rho_\varepsilon' (\xi)) \, d\xi = \varepsilon u'_\varepsilon (\tau_\varepsilon) \rho_\varepsilon (\tau_\varepsilon) - \varepsilon u'_\varepsilon (\theta) \rho_\varepsilon (\theta)
\]
\[
= \int_{\theta}^{\tau_\varepsilon} f_\varepsilon (\xi) u'_\varepsilon (\xi) \, d\xi + p(\rho_\varepsilon (\tau_\varepsilon)) - p(\rho_\varepsilon (\theta)).
\]

From this, we deduce
\[
(3.49) \quad p(\rho_\varepsilon (\tau_\varepsilon)) = p(\rho_\varepsilon (\theta)) - \int_{\theta}^{\tau_\varepsilon} f_\varepsilon (\xi) u'_\varepsilon (\xi) \, d\xi + \varepsilon u'_\varepsilon (\tau_\varepsilon) \rho_\varepsilon (\tau_\varepsilon) - \varepsilon u'_\varepsilon (\theta) \rho_\varepsilon (\theta)
\]
\[
< p(\rho_\varepsilon (\tau_\varepsilon)) = p(\rho_\varepsilon (\theta)) - \int_{\theta}^{\tau_\varepsilon} f_\varepsilon (\xi) u'_\varepsilon (\xi) \, d\xi - \varepsilon u'_\varepsilon (\theta) \rho_\varepsilon (\theta)
\]
\[
\leq p(\rho_\varepsilon (\theta)) - \varepsilon (u^* - u_*) \rho_\varepsilon (\theta) - \int_{\theta}^{\tau_\varepsilon} f_\varepsilon (\xi) u'_\varepsilon (\xi) \, d\xi.
\]

By (3.46), the first two terms of the right hand side of (3.49) are bounded uniformly in \(\varepsilon\). Thus, if the last term \(- \int_{\theta}^{\tau_\varepsilon} f_\varepsilon (\xi) u'_\varepsilon (\xi) \, d\xi\) is bounded uniformly in \(\varepsilon\), then Assumption
1 will imply that \( \rho_\varepsilon(\xi) \) is bounded uniformly in \( \varepsilon \) from the above. Indeed, by (3.39), for \( \xi \in [\theta, \xi_0] \),

\[
\begin{align*}
f_\varepsilon(\xi) &= f_\varepsilon(\xi_0) - \int_\xi^{\xi_0} \frac{df_\varepsilon(\zeta)}{d\zeta} \\
&= f_\varepsilon(\xi_0) + \int_\xi^{\xi_0} \rho_\varepsilon(\zeta)d\zeta \\
&\leq \rho(\xi_0) + \rho_\varepsilon(\xi_0)(\xi_0 - \theta) < 2\rho_\varepsilon(\xi_0).
\end{align*}
\]

Thus, because \( f_\varepsilon(\xi) \) is decreasing,

\[
(3.50) \quad \left| \int_\theta^{\xi_0} f_\varepsilon(\xi)u_\varepsilon'(\xi)d\xi \right| \leq 2\rho_\varepsilon(\xi_0) \int_\theta^{\xi_0} |u_\varepsilon'(\xi)|d\xi \leq 2\rho_\varepsilon(\xi_0)TV(u_\varepsilon).
\]

Since \( u_\varepsilon(\xi) \) consists finitely many monotone pieces and is bounded uniformly in \( \varepsilon \), \( TV(u_\varepsilon) \) is also bounded uniformly in \( \varepsilon \). This completes our proof. \( \Box \)

After a slight modification of the proof of Lemma 3.5 of [24], we obtain the following theorem.

**THEOREM 3.4.** Let \( (u_\varepsilon(\xi), \rho_\varepsilon(\xi)) \) be a solution of (2.1) with \( \rho_\varepsilon(\xi) > 0 \). Then

\[
\rho(\xi) \geq \delta > 0
\]

for some constant \( \delta > 0 \) independent of \( \mu \in [0, 1] \) and \( L > 1 \).

**Proof.** Without loss of generality, we assume \( \rho_\varepsilon(\xi) \) has a local minimum point \( \tau_\varepsilon \). We define \( \eta \geq -L \) by

\[
\rho_\varepsilon(\eta) = \rho_- \ , \ \eta > \tau.
\]

Then for any \( \xi \in [-L, \tau] \), there is a \( \xi' \in (\tau, \eta] \) such that \( \rho_\varepsilon(\xi) = \rho_\varepsilon(\xi') \) By Lemma 2.1, \( \rho'_\varepsilon(\xi') < 0 \) and \( \rho'_\varepsilon(\xi) > 0 \). Integrating (2.2a) over \([\xi, \xi']\), we find that

\[
0 < \varepsilon \rho'_\varepsilon(\xi') - \varepsilon \rho'(\xi)
\]

\[
(3.51) \quad = \int_\xi^{\xi'} (\rho_\varepsilon(\zeta) - \rho_\varepsilon(\xi))d\zeta + \mu \rho_\varepsilon(\xi)[u_\varepsilon(\xi') - u_\varepsilon(\xi)].
\]

Since \( \rho_\varepsilon(\zeta) \leq \rho_\varepsilon(\xi) \) for \( \zeta \in [\xi, \xi'] \), (3.51) implies that

\[
(3.52) \quad 0 < \varepsilon \rho'_\varepsilon(\xi) + A\rho_\varepsilon(\xi) \quad \text{for} \quad -L \leq \xi < \tau
\]

where

\[
A = u^* - u_*.\n\]
Integrating (3.52), we obtain

\begin{equation}
\rho_{\varepsilon}(\tau) \geq \rho(\zeta) \exp \left[ -\frac{A}{\varepsilon} (\tau - \xi) \right].
\end{equation}

On the other hand, (3.51) shows that

\[ \int_{-\eta}^{\eta} (\rho_{-} - \rho(\zeta)) d\zeta < A\rho_{-}. \]

Let \(\xi \in [-L, \tau)\), then

\[ (\tau - \xi)(\rho_{-} - \rho(\xi)) \leq \int_{-\xi}^{\eta} (\rho_{-} - \rho(\zeta)) d\zeta \]

\[ \leq \int_{-\bar{L}}^{\eta} (\rho_{-} - \rho(\zeta)) d\zeta < A\rho_{-}. \]

Thus,

\begin{equation}
\rho(\xi) > \rho_{-} - \frac{A\rho_{-}}{\tau - \xi}.
\end{equation}

**Case 1:** If \(\tau + L \leq 2A\), then (3.53) and (3.54) with \(\xi = -L\) gives us

\[ \rho_{\varepsilon}(\tau) \geq \rho_{-} \exp \left( \frac{-2A^2}{\varepsilon} \right). \]

**Case 2:** If \(\tau + L > 2A\). Then we take \(\xi = \tau - 2A\) in (3.53) and (3.54). This gives us

\[ \rho_{\varepsilon}(\tau_{\varepsilon}) \geq \frac{1}{2} \rho_{-} \exp \left( -\frac{2A^2}{\varepsilon} \right). \]

\[ \square \]

**§4. The existence of solutions of the Riemann problem (1.1).** With the a-priori estimates established in last section for solutions of \((u, \rho)\) of (2.1) satisfying \(\rho_{\varepsilon}(\xi) > 0\) and

\[ \rho'_{\varepsilon}(\xi) > 0 \quad \text{when} \quad \rho_{\varepsilon}(\xi) \in [\alpha, \beta], \]

we are ready to state the following existence theorem for (1.1):
THEOREM 4.1. (i) There is a solution of (1.4) satisfying $\rho_\varepsilon(\xi) > 0$ and
\[ \rho_\varepsilon'(\xi) > 0 \quad \text{when} \quad \rho_\varepsilon(\xi) \in [\alpha, \beta]. \]

(ii) There is a subsequence \( \{\varepsilon_n\} \), \( \varepsilon_n \to 0^+ \) as \( n \to \infty \), such that \( u_{\varepsilon_n}(\xi) \cdot \rho_{\varepsilon_n}(\xi) \) given in (i) converges a.e. to a weak solution \((u(\xi), \rho(\xi))\) of the Riemann problem (1.1). Furthermore \( \rho(\xi) \geq 0. \)

Proof. (i) Theorem 3.1 – 3.5 provide a-priori estimates needed by Theorem 2.3. Thus, the existence of (1.1) is established.

(ii) Lemmas 2.1, 2.2 together with Theorems 3.1 – 3.4 show that \( \{u_{\varepsilon}(\xi), \rho_{\varepsilon}(\xi)\} \)-given in (i) has total variation bounded uniformly in \( \varepsilon > 0. \) Now the same arguments used in the proof of Theorem 3.2 of [1] or Theorem 4.1 of [24] prove our assertion. "

§5 Solutions of (1.1) admissible according to the travelling wave criterion also exist. In this section, we shall prove that solutions of (1.1) constructed in §2-§4, which are admissible by the similarity viscosity criterion, are also admissible by the travelling wave criterion. In this section, sequences \( \{\varepsilon_n\} \) and \( \{u_{\varepsilon_n}(\xi), \rho_{\varepsilon_n}(\xi)\} \) are given in (ii) of Theorem 4.1. We know that \((u_{\varepsilon_n}(\xi), \rho_{\varepsilon_n}(\xi)) \) converges to a weak solution of (1.1) as \( n \to \infty. \)

We use \( \xi_{\varepsilon_n}(\rho) \) to denote the inverse function of \( u_{\varepsilon_n}(\xi). \) By lemma 2.1 and (3.1), \( \xi_{\varepsilon_n}(\rho) \) may have at most two continuous branches. We define
\[ \bar{u}_{\varepsilon_n}(\rho) =: u_{\varepsilon_n}(\xi_{\varepsilon_n}(\rho)) \]
on the range of \( \rho_{\varepsilon_n}(\xi). \)
\[ \rho_* =: \inf \{\rho_{\varepsilon_n}(\xi) | \xi \in \mathbb{R}, n = 1, 2, \ldots\}, \]
\[ \rho^* =: \sup \{\rho_{\varepsilon_n}(\xi) | \xi \in \mathbb{R}, n = 1, 2, \ldots\}. \]

Without loss of generality, we can assume that \( \min_{\xi \in \mathbb{R}} \rho_{\varepsilon_n}(\xi) \to \rho_* \) as \( n \to \infty, \) and \( \max_{\xi \in \mathbb{R}} \rho_{\varepsilon_n}(\xi) \to \rho^* \) as \( n \to \infty. \) Then each \( \rho \in (\rho_* , \rho^*) \) is in the domain of \( \bar{u}_{\varepsilon_n}(\rho) \) for sufficiently large \( n. \) After a slight modification of the proof for Lemma 2.3 of [6], we obtain

LEMMA 5.1. There is a subsequence of \( \{\varepsilon_n\} \), denoted by \( \{\varepsilon_n\} \) again for simplicity, such that either
\[
\lim_{n \to \infty} \max \left\{ \left| \frac{d u_{\varepsilon_n}(\xi)}{d \rho_{\varepsilon_n}(\xi)} \right| \right\} \mathbb{R}, \rho_{\varepsilon_n}(\xi) \geq r < \infty
\]
or
\[
\lim_{n \to \infty} \max \left\{ \left| \frac{d \rho_{\varepsilon_n}(\xi)}{d u_{\varepsilon_n}(\xi)} \right| \right\} \mathbb{R}, \rho_{\varepsilon_n}(\xi) \geq r < \infty
\]
for any \( r > 0. \)

Based on this lemma, we can prove the following theorem ( cf. Theorem 2.5 of [6], or Lemma 4.2 of [7]):

22
Theorem 5.2. There is a subsequence of \( \{ \epsilon_n \} \), denoted as \( \{ \epsilon_n \} \), such that,

\[
(5.2) \quad \tilde{u}_{\epsilon_n}(\rho) \to \tilde{u}(\rho) \quad \text{as} \ n \to \infty
\]

where \( \tilde{u}(\rho) \) is Lipschitz on any compact subset of \((0, \rho^*)\). Further, \((u(\xi), \rho(\xi))\) lies on the curve \( \tilde{u}(\rho) \) for a.a. \( \xi \in \mathbb{R} \).

Let \( \xi_0 \) be a point of discontinuity of \((u(\xi), \rho(\xi))\). By checking the Rankine-Hugoniot conditions for (1.1), we can easily prove that either \( \rho(\xi_0 \pm) = 0 \) or \( \rho(\xi_0 \pm) \neq 0 \) holds (cf [16]). We assume that \( \rho(\xi_0 \pm) \neq 0 \). We use \( C_{\xi_0} \) to denote the portion of the curve \( \tilde{u}(\rho) \) connecting points \((u(\xi_0 -), \rho(\xi_0 -))\) and \((u(\xi_0 +), \rho(\xi_0 +))\). Let \((\tilde{u}, \tilde{\rho}) \in C_{\xi_0} \) with \( \tilde{\rho} > 0 \) and \( \tilde{\rho} \neq \rho_*, \rho^* \). Then we can define \( \xi_{\epsilon_n}(\rho; \tilde{u}, \tilde{\rho}) \), the branch of the inverse function of \( \rho = \rho_{\epsilon_n}(\xi) \) for which

\[
(5.3) \quad u_{\epsilon_n}(\xi_{\epsilon_n}(\tilde{\rho}; \tilde{u}, \tilde{\rho})) \to \tilde{u}
\]
as \( n \to \infty \). We define, for \( n \) large, that

\[
(5.4) \quad \xi_{\epsilon_n} = \xi_{\epsilon_n}(\tilde{\rho}) + \epsilon \zeta,
\]

\[
(5.5) \quad \hat{u}_{\epsilon_n}(\zeta) = u_{\epsilon_n}(\xi_{\epsilon_n}),
\]

\[
(5.6) \quad \hat{\rho}_{\epsilon_n}(\zeta) = \rho_{\epsilon_n}(\xi_{\epsilon_n}).
\]

Lemma 5.3. [7]. Let \( \xi_0 \) be a point of discontinuity of \((u(\xi), \rho(\xi))\) with \( \rho(\xi_0 \pm) > 0 \). For \((\hat{u}_{\epsilon_n}(\zeta), \hat{\rho}_{\epsilon_n}(\zeta))\) defined above with \( \tilde{\rho} > 0 \), there is a subsequence of \( \{ \epsilon_n \} \), denoted by \( \{ \epsilon_n \} \) for simplicity, such that

\[
(5.7) \quad (\hat{u}_{\epsilon_n}(\zeta), \hat{\rho}_{\epsilon_n}(\zeta)) \to (\hat{u}(\zeta), \hat{\rho}(\zeta)) \in C^1(\mathbb{R}; \mathbb{R}^2) \quad \text{as} \ n \to \infty,
\]

uniformly for \( \zeta \) in a compact subset of \( \mathbb{R} \). \((\hat{u}(\zeta), \hat{\rho}(\zeta))\) satisfies the following initial value problem:

\[
(5.8a) \quad \frac{d\hat{\rho}(\zeta)}{d\zeta} = \hat{m}(\zeta) - m(\xi_0 -) - \xi_0 (\hat{\rho}(\zeta) - \rho(\xi_0 -)),
\]

\[
(5.8b) \quad \frac{d\hat{m}(\zeta)}{d\zeta} = \hat{u}(\zeta)\hat{m}(\zeta) - m(\xi_0 -)u(\xi_0 -)
- \xi_0 (\hat{m}(\zeta) - m(\xi_0 -)) + p(\hat{\rho}(\zeta)) - p(\rho(\xi_0 -)).
\]
\[(5.9) \quad \dot{u}(0) = \bar{u}, \quad \dot{\rho}(0) = \bar{\rho}\]

where \(\dot{m}(\zeta) = \dot{u}(\zeta)\dot{\rho}(\zeta)\). Furthermore, \((\dot{u}(\zeta), \dot{\rho}(\zeta)) \in C_{\xi_0}\), if \(\dot{\rho}(\zeta) > 0\).

The most common form of viscosity for system (1.1a,b) is

\[(5.10a) \quad \rho_t + (\rho u)_x = \epsilon \rho_{xx},\]

\[(5.10b) \quad (\rho u)_t + (\rho u^2 + p(\rho))_x = \epsilon (u\rho)_{xx}.\]

The corresponding travelling wave equations with speed \(s\) are

\[(5.11a) \quad \frac{d\dot{\rho}(\zeta)}{d\zeta} = \dot{m}(\zeta) - m_1 - s(\dot{\rho}(\zeta) - \rho_1),\]

\[(5.11b) \quad \frac{d\dot{m}(\zeta)}{d\zeta} = \dot{u}(\zeta)\dot{m}(\zeta) - m_1 u_1 - s(\dot{m}(\zeta) - m_1) + p(\dot{\rho}(\zeta)) - p(\rho_1).\]

We can see that (5.8) is the same as (5.11) when \(s = \xi_0\), \((u_1, \rho_1) = (u(\xi_0^-), \rho(\xi_0^-))\).

We consider the boundary value problem of (5.11a, b) with

\[(5.11c) \quad (\dot{u}(-\infty), \dot{\rho}(-\infty)) = (u_1, \rho_1), \quad (\dot{u}(+\infty), \dot{\rho}(+\infty)) = (u_2, \rho_2).\]

where \((u_1, \rho_1)\) and \((u_2, \rho_2)\) satisfy the Rankine-Hugoniot conditions:

\[(5.12a) \quad -s(\rho_2 - \rho_1) + u_2 \rho_2 - u_1 \rho_1 = 0,\]

\[(5.12b) \quad -s(u_2 \rho_2 - u_1 \rho_1) + u_2^2 \rho_2 - u_1^2 \rho_1 + p(\rho_2) - p(\rho_1) = 0.\]

The travelling wave criterion for system (1.1a, b), based on (5.11a, b) can be stated in a general setting:

**Definition 5.4.** (i) We say \((u_1, \rho_1)\) and \((u_2, \rho_2)\) can be connected by a shock with speed \(s\), where \(s\) is determined by (5.12), if (5.11) has a solution.

(ii) A shock of speed \(s\), with \((u_1, \rho_1)\) and \((u_2, \rho_2)\) on its sides, where \((u_1, \rho_1), (u_2, \rho_2)\) satisfy the Rankine-Hugoniot conditions (5.12), is admissible by traveling wave criterion if there are \((v_k, \phi_k), k = 1, 2, ..., n \in \mathbb{N}\), and \((u_1, \rho_1) = (v_1, \phi_1), (u_2, \rho_2) = (v_n, \phi_n)\) such that \((v_k, \phi_k)\) can be connected by a shock with speed \(s\) to \((v_{k+1}, \phi_{k+1}), k=1,2,\ldots,n-1\).

(iii) We say a solution \((u(\xi), \rho(\xi))\) of (1.1) is admissible by the traveling wave criterion if every discontinuity of \((u(\xi), \rho(\xi))\) is a jump discontinuity which is admissible in the sense of (ii).

By arguments similar to that used in the proof of Corollary 5.7 in [9], we can show the following:
Corollary 5.5. Let \( p(w) \) satisfies Assumption 1 and the property that (5.12) can have at most finitely many solutions \((u_2, \rho_2)\) for fixed \(s\) and \((u_1, \rho_1)\). Then the solutions of (1.1) given by Theorem 4.1, which are admissible by the similarity viscosity criterion, are also admissible according to the traveling wave criterion. Hence, solutions of (1.1) admissible by the traveling wave criterion exist.

Proof. Let \( \xi_0 \) be a point of discontinuity of \((u(\xi), \rho(\xi))\) given in Theorem 4.1. By the property of \( p(w) \) assumed in this theorem, we know that there are only finitely many points \((u_2, \rho_2)\) satisfy the Rankine-Hugoniot conditions (5.12) with \(s = \xi_0\). Since \( C_{\xi_0} \), the portion of the curve \((U(s), P(s))\) connecting \((u(\xi_0^-), \rho(\xi_0^-))\) and \((u(\xi_0^+), \rho(\xi_0^+))\), can be oriented in the direction from \((u(\xi_0^-), \rho(\xi_0^-))\) to \((u(\xi_0^+), \rho(\xi_0^+))\), we can assume that the points on \( C_{\xi_0} \) satisfying (5.12) are \((v_1, \phi_1) = (u_1, \rho_1), (v_2, \phi_2), ..., (v_{n-1}, \phi_{n-1}) \) and \((v_n, \phi_n) = (u_2, \rho_2)\), which are ordered in the direction of \( C_{\xi_0} \). Let \((\bar{u}_1, \bar{\rho}_1) \in C_{\xi_0} \) be a point on the portion of \( C_{\xi_0} \) between \((v_1, \phi_1) = (u_1, \rho_1)\) and \((v_2, \phi_2)\) and \((\hat{u}_1(\xi), \hat{\rho}_1(\xi))\) to be the corresponding solution of (5.8) with the initial condition (5.9) being \((u, \rho) = (\bar{u}_1, \bar{\rho}_1)\). By Lemma 5.3, we can see that \((v_1, \phi_1)\) is connected to some \((\hat{u}_1(\sup), \hat{\rho}_1(\sup)) = (v_{j_1}, \phi_{j_1})\), \(1 < j_1 \leq n\), by a solution of (5.11) with \(s = \xi_0\). If \(j_1 = n\) then our theorem is proved. If otherwise, we repeat above procedure to see that \((v_{j_1}, \phi_{j_1})\) is connected to \((v_{j_2}, \phi_{j_2})\), \(j_1 < j_2 \leq n\), by a travelling wave solution. Repeating this process finite times, we can prove that \(w_{j_4}\) can be connected to \(w_{j_4+1}\), \(k = 0, 1, ..., m \leq n\), where \(w_{j_0} = w(\xi_0^-)\) and \(w_{j_m} = w(\xi_0^+)\). By Definition 5.4 (ii), the jump discontinuity of \((u(\xi), \rho(\xi))\) at \(\xi = \xi_0\) is admissible by the travelling wave criterion. Thus, the first statement of our assertion is proved. The last statement is a consequence of Theorem 4.1 and the first statement of this theorem.

Acknowledgement

I would like to thank Prof. M. Slemrod for his valuable suggestions and comments. I also thank Prof. M.-C Shen for inspiring discussions.

References


Recent IMA Preprints

Title

1. Andrew Majda and Kevin Lamb, Simplified equations for low Mach number combustion with strong heat release
2. Ju. S. Il'’yashenko, Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation
3. James F. Reineck, Continuation to gradient flows
4. Mohamed Sami Elbialy, Simultaneous binary collisions in the collinear N-body problem
5. John A. Jacquez and Carl P. Simon, AIDS: The epidemiological significance of two different mean rates of partner-change
7. Matthew Stafford, Markov partitions for expanding maps of the circle
8. Ciprian Foias and Edriss S. Titi, Determining nodes, finite difference schemes and inertial manifolds
9. M.W. Smiley, Global attractors and approximate inertial manifolds for abstract dissipative equations
10. M.W. Smiley, On the existence of smooth breathers for nonlinear wave equations
11. Hitay Özbay and Janos Turi, Robust stabilization of systems governed by singular integro-differential equations
12. Mary Silber and Edgar Knobloch, Hopf bifurcation on a square lattice
13. Christophe Golé, Ghost circles for twist maps
14. Christophe Golé, Ghost tori for monotone maps
15. Christophe Golé, Monotone maps of $T^n \times R^n$ and their periodic orbits
16. E.G. Kalnins and W. Miller, Jr., Ilypo-geometric expansions of Heun polynomials
17. Victor A. Pliss and George R. Sell, Perturbations of attractors of differential equations
18. Avner Friedman and Peter Knabner, A transport model with micro- and macro-structure
19. E.G. Kalnins and W. Miller, Jr., A note on group contractions and radar ambiguity functions
20. George R. Sell, References on dynamical systems
21. Shui-Nee Chow, Kening Lu and George R. Sell, Smoothness of inertial manifolds
22. Shui-Nee Chow, Xiao-Biao Lin and Kening Lu, Smooth invariant foliations in infinite dimensional spaces
24. Christophe Golé and Glen R. Hall, Poincaré’s proof of Poincaré’s last geometric theorem
25. Mario Taboada, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations
26. Peter Rejto and Mario Taboada, Weighted resolvent estimates for Volterra operators on unbounded intervals
27. Joel D. Avrin, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion
28. Susan Friedlander and Misha M. Vishik, Lax pair formulation for the Euler equation
29. H. Scott Dumas, Ergodization rates for linear flow on the torus
30. A. Eden, A.J. Milani and B. Nicolaenko, Finite dimensional exponential attractors for semilinear wave equations with damping
32. A. Eden, C. Foias, B. Nicolaenko & R. Temam, Hölder continuity for the inverse of Mañé’s projection
33. Michel Chipot and Charles Collins, Numerical approximations in variational problems with potential wells
34. Huanan Yang, Nonlinear wave analysis and convergence of MUSCL schemes
35. László Gerencsér and Zsuzsanna Vágó, A strong approximation theorem for estimator processes in continuous time
36. László Gerencsér, Multiple integrals with respect to $L$-mixing processes
37. David Kinderlehrer and Pablo Pedregal, Weak convergence of integrands and the Young measure representation
38. Bo Deng, Symbolic dynamics for chaotic systems
40. Charles Collins and Mitchell Luskin, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
41. Peter Gritzmann and Victor Klee, Computational complexity of inner and outer $j$-radii of polytopes in finite-dimensional normed spaces
42. A. Ronald Gallant and George Tauchen, A nonparametric approach to nonlinear time series analysis: estimation and simulation
43. H.S. Dumas, J.A. Ellison and A.W. Sáenz, Axial channeling in perfect crystals, the continuum model and the method of averaging
44. M.A. Kaashoek and S.M. Verduyn Lunel, Characteristic matrices and spectral properties of evolutionary systems