SIGNED RANDOM MEASURES: STOCHASTIC ORDER AND KOLMOGOROV CONSISTENCY CONDITIONS

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Signed Random Measures:  
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Abstract. We consider random measures as maps from the Borel sets $A$ of a Polish space to the lattice $V$ of random variables on a probability space $(\Omega, \mathcal{F}, P)$; for fixed $\omega$ these maps should be signed, bounded measures. The space of such random measures is a $\sigma$-complete vector lattice under the stochastic order. We use this to demonstrate Kolmogorov consistency conditions for bounded, signed random measures, to prove the existence of conditional expectation operator on this lattice, and to prove a Lebesgue-Radon-Nikodym theorem for random measures for which the Radon-Nikodym derivative is a random process.

Key words. Kolmogorov conditions, lattices, random measures, stochastic order

0. Introduction

Rolski and Szelki [R,S] considered partial ordering on the space of non-negative random measures and related this order, the stochastic order, to the theory of point process thinning. The space of bounded, signed measures on a Polish space (a complete, separable, metrizable space) is partially ordered by the same relation

$$\mu \leq \nu \iff (\forall a \in A) \mu(a) \leq \nu(a),$$

where $\mu$ and $\nu$ are random measures and $A$ is the collection of Borel sets. Moreover, this collection forms a $\sigma$-complete vector lattice in this order, as we shall prove. We will use this lattice property to give necessary and sufficient conditions for a system of distributions to determine an essentially unique signed bounded random measure, to prove that the conditional expectation of a signed random measure determines an essentially unique random measure, and to determine general conditions for one random measure to be represented as an indefinite integral with respect to another. According to Daley and Vere-Jones [D,V-J], the first two of these appeared to be open problems. Beyond these results, we hope that the characterization of spaces of random measures by their order properties will produce other interesting results, and that an understanding the properties of signed random measures will be useful in modelling physical problems involving naturally signed quantities such as electrical charge.

The following theorem, the proof of which is mentioned by Dunford and Schwartz [D,S], is useful for dealing with signed set functions which are bounded and countably additive on an algebra. We mention it because it does not seem widely known and because it is key to our proof of the Kolmogorov consistency theorem for signed random measures.

Theorem 0.1. Let $A$ be an algebra, let $\mu$ be a countably additive set function with values only in an interval $[a, b]$. Then, $\mu$ has an unique extension to $\sigma(A)$ which also assumes values only in $[a, b]$. 
1. Vector Lattices

We will briefly summarize the results we need from the theory of vector lattices. A more complete explanation can be found in Yosida's book [Y]. The proof of the representation theorem for semi-simple unital vector lattices can be found there.

An ordered vector space $V$ equipped with a partial order $\leq$ which is additive, for all $x, y, z \in V$, $x \leq y \Rightarrow x + z \leq y + z$, positive homogeneous, for all non-negative real $c$ and all $x, y \in V$ $x \leq y \Rightarrow cx \leq cy$. If such a space is complete with respect to finite infima and suprema, it is a vector lattice. The infimum of two elements $u$ and $v$ is denoted $u \land v$, and the supremum $u \lor v$. The absolute value of $u$ is $|u| = u \lor 0 - u \land 0$, the positive variation $u^+ = u \lor 0$, and the negative variation $u^- = -(u \land 0)$. The collection of elements of $V$ which are non-negative is the positive cone, $V^+$, of $V$.

A linear subspace $U$ is an ideal if whenever a $v$ in $V$ satisfies $|v| \leq |u|$ for some $u$ in $U$, then $v$ is also in $U$. A vector lattice containing no ideals other than $\{0\}$ and $V$ is called simple. The principal ideal associated with $u$ is subspace $V_u = \{v||v| \leq c|u|, c \in \mathbb{R}, c \geq 0\}$. If for some $e \geq 0 V_e = V$, then $e$ is an unit of $V$. For any ideal $U$, the quotient vector space $V/U$ is a vector lattice with the structure determined by $(f + U) \land (g + U) = (f \land g) + U$. The map $u \rightarrow u + V$ which carries an element of $U$ to its equivalence class is a vector lattice homomorphism, a linear map $T$ satisfying $T(|u|) = |T(u)|$.

An ideal is maximal if it is strictly contained in the entire space but in no smaller ideal. If $N$ is a maximal ideal, $V/N$ is a simple vector lattice. The collection of maximal ideals of $V$ is called the maximal ideal space, denoted $\mathcal{M}(V)$. The intersection of all maximal ideals of $V$ is called the radical of $V$, denoted $\mathcal{R}(V)$; $V$ is semi-simple if the radical is equal to $\{0\}$. If $V$ has the unit $e$, the collection of nilpotent elements $\{v \mid n|v| \leq e, n = 1, 2, \ldots\}$ is equal to the radical of $V$ (Yosida [Y]).

If every countable set in a vector lattice $V$ with an upperbound in $V$ has a least upper bound in $V$, then $V$ is $\sigma$-complete; moreover, this implies that any countable collection with a lower bound has a greatest lower bound. A vector lattice which is $\sigma$-complete is called a $K_\sigma$ space. A sequence $\{f_n\}$ in $V$ is order convergent to $f$, $\lim_{n \rightarrow \infty} f_n = f$, if there exists a sequence $\{r_n\}$ which decreases monotonically and has greatest lower bound 0, $r_n \downarrow 0$, and $|f_n - f| \leq r_n$. If $V$ is a $K_\sigma$ space, then so are all of its ideals and quotient lattices. Since it is easy to show that a $K_\sigma$ space cannot have a non-zero nilpotent element, every $K_\sigma$ space is semi-simple.

**Theorem 1.1.** (Yosida [Y]) Let $V$ be a $K_\sigma$ space with unit $e$. Consider the function $\psi : V \times \mathcal{M}(V) \rightarrow \mathbb{R}$ defined by

$$\psi(v)(N) = \inf\{c|v + N \leq ce + N, c \in \mathbb{R}\} = \sup\{c|ce + N \leq v + N, c \in \mathbb{R}\}.$$

Let $Q$ be $\mathcal{M}(V)$ with the coarsest topology which makes $\psi(v)(N)$ continuous in $N$ for every fixed value of $v$. $Q$ is a (quasi-extremally disconnected) compact Hausdorff space in
this topology, the collection of continuous functions on \(Q, C(Q)\), is a \(K_\sigma\) space, and the partial evaluation \(\psi : V \rightarrow C(Q)\) is the unique lattice isomorphism such that \(\psi(e) = 1\).

Familiar examples of \(K_\sigma\) spaces are the collection of measurable functions and the collection of signed measures of bounded total variation on a given measurable space \((X, \mathcal{X})\). Such set functions are maps from a \(\sigma\)-algebra of sets to a \(K_\sigma\) space. We would like to replace the range of the set function by any other \(K_\sigma\) space, \(V\), (e.g. the collection of real random variables over a probability space) and still be able to prove that the collection of maps is a \(K_\sigma\) space. In general, this will depend on \((X, \mathcal{X})\) because \(V\) is \(\sigma\)-complete rather than complete (i.e. conditionally closed under arbitrary infima and suprema). We will see that it is enough that \((X, \mathcal{X})\) be a countable algebra or a countably-generated \(\sigma\)-algebra.

2. Lattice Valued Measures.

We now consider maps from algebras of sets to lattices. We shall see in the final section on integration that these maps generalize to regular operators from \(U\), the lattice of bounded measurable functions on \((X, \mathcal{X})\), to \(V\), some \(K_\sigma\) space. Regular operators are linear operators between \(U\) and \(V\), two vector lattices, and have bounded variation or, equivalently, are the difference of two non-negative operators. The theory of such operators is usually developed assuming that the range vector lattice is complete under bounded infima and suprema of arbitrary cardinality ([\([Ka]\), [\([V]\)]). This is not the case with random measures, for which the range space is only \(\sigma\)-complete. However, since the domain space is countably generated, we can decompose any bounded signed random measure as a difference of non-negative random measures.

By convention we will identify elements of an algebra with their indicator functions denoted by lower case letters, except for the entire space which we will refer to as \(1\) and the null set which we will refer to as \(0\). We will refer to collections of sets by upper case letters.

For an algebra \(A\) and a \(\sigma\)-space \(L\), the collection of bounded countably additive maps from \(A\) to \(L\) is denoted by \(\Lambda(A, L)\) and defined by \(\mu \in \Lambda(A, L)\) if

Boundedness: \((\exists f \geq 0)(\forall a \in A)|\mu(a)| \leq f\)

Finite Additivity: \(a \cap b = 0 \Rightarrow \mu(a \cup b) = \mu(a) + \mu(b)\)

Continuity: \(a_n \downarrow 0 \Rightarrow \lim_{n \to \infty} \mu(a_n) = 0\)

These maps are lattice-valued measures. The collection of all elements of \(\Lambda(A, L)\) with non-negative range is the denoted \(\Lambda(A, L^+)\). We are particularly interested in the cases that \(A\) is a countable algebra or a countably-generated \(\sigma\)-algebra.
THEOREM 2.1. Let $A$ be a countable algebra or a countably generated $\sigma$-algebra, and let $L$ be a $K_\sigma$ space. The space $\Lambda(A, L)$ is a $K_\sigma$ space under the order

$$\mu \leq \nu \iff (\forall a \in A)\mu(a) \leq \nu(a).$$

For our purposes, the importance of this theorem is that if $A$ is the collection of Borel sets on a Polish space $(X, \mathcal{X})$ and $L$ is the collection of random variables on a probability space $(\Omega, \mathcal{F}, P)$, then $\Lambda(A, L)$ is a $K_\sigma$ space. The space $\Lambda(A, L)$ is the space of random measures.

We will prove this by adding the assumption that $L$ has unit $e$ and then dispensing with this assumption. For the next four lemmas, let $A$ be a countable algebra and $L_e$ be a $K_\sigma$ space with unit $e$.

LEMMA 2.1. Every element of $\Lambda(A, L_e)$ extends uniquely to $\Lambda(\sigma(A), L_e)$.

Let $\mu$ be an element of $\Lambda(A, L_e)$. The range of $\mu$ is $L_e$, which by Theorem 1.1 can be represented as $C(S)$, a lattice of continuous functions. For each fixed $s \in S$, the extension holds by Theorem 0.1. We need only verify that the pointwise extension is continuous. Let $A_\sigma$ be the completion of $A$ under countable unions; let $A_{\sigma_\delta}$ be the completion of $A_\sigma$ under countable intersections. Let $\{a_n\} \subseteq A$ and $a = \bigcup_{i=1}^{\infty} a_i$. Since

$$\lim_{n \to \infty} \mu(\bigcup_{i=1}^{n} a_i)(s) = \mu(a)(s)$$

is continuous in $s$ by the $\sigma$-completeness of $C(S)$, $\mu$ extends continuously to $A_\sigma$. Let $\{b_n\} \subseteq A_\sigma$ and $b = \bigcap_{i=1}^{\infty} b_i$. Since

$$\lim_{n \to \infty} \mu(\bigcap_{i=1}^{n} b_i)(s) = \mu(b)(s)$$

is continuous in $s$, $\mu$ extends continuously to $A_{\sigma_\delta}$. Because $A_{\sigma_\delta}$ is $\sigma(A)$, any element of $\Lambda(A, L_e)$ extends uniquely to $\Lambda(\sigma(A), L_e)$. \square

LEMMA 2.2. Any element $\mu \in \Lambda(\sigma(A), L_e)$ can be represented as the difference of two elements of $\Lambda(\sigma(A), L_e^+)$.

For $\mu \in \Lambda(\sigma(A), L_e)$, define $|\mu|$ such that for all $a \in \sigma(A)$

$$|\mu|(a)(s) = \sup_{b \in A} |\mu(a \cap b)(s)| + |\mu(a \cap b^c)(s)| = \sup_{b \in \sigma(A)} |\mu(a \cap b)(s)| + |\mu(a \cap b^c)(s)|.$$

We obtain the second equality by successively substituting $A_\sigma$ and $A_{\sigma_\delta}$ for $A$ in the second supremum, noting that the value at each $s$ would be unchanged. Then $\mu = \mu^+ - \mu^-$, where $\mu^+ = 1/2(|\mu| + \mu)$ and $\mu^- = 1/2(|\mu| - \mu)$. The set functions $|\mu|$, $\mu^+$, and $\mu^-$ are all non-negative, bounded, countably additive for each $s$, and continuous. Therefore, they are in $\Lambda(\sigma(A), L_e^+)$. \square

A similar decomposition holds for elements in $\Lambda(A, L_e)$. Furthermore, the spaces $\Lambda(A, L_e)$ and $\Lambda(\sigma(A), L_e)$ are isomorphic as ordered vector spaces.
Lemma 2.3. The spaces $\Lambda(A, L_e)$ and $\Lambda(\sigma(A), L_e)$ are vector isomorphic lattices.

These spaces are lattices under the formulae, $\mu \lor \nu = 1/2((\mu + \nu) + |\mu - \nu|)$ and $\mu \land \nu = 1/2((\mu + \nu) - |\mu - \nu|)$, which describe set functions which are countably additive for each fixed $s$ and continuous in $s$.

Lemma 2.4. The spaces $\Lambda(A, L_e)$ and $\Lambda(\sigma(A), L_e)$ are $K_\sigma$ spaces.

We need only show that any non-negative non-decreasing sequence $\{\mu_n\}$ with range bounded above by $e$ has a least upper bound $\mu$. We can represent the range of these measures by a lattice of continuous functions $C(S)$ on a compactum $S$ and $e$ by 1 so that $0 \leq \mu_1(a, s) \leq \mu_2(a, s) \leq \cdots \leq 1$ for any $a$ in $A$ or $\sigma(A)$. We have for any $a \in \sigma(A)$,

$$\mu(a, s) = \lim_{n \to \infty} \mu_n(a, s)$$

which is countably additive at each $s \in S$. \(\square\)

Let $A$ be a countable algebra or a countably generated $\sigma$-algebra. Theorem 2.1 is a corollary to this lemma since for any $\mu, \nu \in \Lambda(A, L)$ have their range in the principal ideal $L_e$, where $e = |\mu|(1) \lor |\nu|(1)$ can serve as a unit. Moreover, any non-negative, non-decreasing sequence $\{\mu_n\}$ which is bounded above has range in $L_f$, where $f = \lim_{n \to \infty} \mu_n(1)$.


Let us consider the ordered spaces $\Lambda(A, R)$, $\Lambda(A, N)$, and $\Lambda(A, R/N)$, with $A$ a the Borel sets on a Polish space, a complete, separable, metrizable space, $R$ a $K_\sigma$ space of random variables on a probability space, and $N$ the ideal of almost surely 0 random variables. Because such a topological space has a countable open base, $A$ is countably generated. Therefore, $\Lambda(A, R)$, $\Lambda(A, N)$, and $\Lambda(A, R/N)$ are $K_\sigma$ spaces. We will call $\Lambda(A, R)$ the (strong) random measures, $\Lambda(A, N)$ the null random measures, and $\Lambda(A, R/N)$ the weak random measures.

Theorem 3.1. Every weak random measure has a strong version; this version is unique modulo $\Lambda(A, N)$. Thus, $\Lambda(A, R/N) \simeq \Lambda(A, R)/\Lambda(A, N)$.

This theorem is an extension of the following theorem of Daley and Vere-Jones [D,V-J].

Theorem 3.2. Let $\{\xi_a(\omega)|a \in A\}$ be a collection of non-negative random variables indexed by the Borel sets of a Polish space $(X, \mathcal{X})$. Let this collection satisfy the conditions

Finite Additivity: $a \cap b = 0 \Rightarrow \xi_{a \cup b} = \xi_a + \xi_b$

Continuity: $a_n \downarrow 0 \Rightarrow \lim_{n \to \infty} \xi_{a_n} = 0$
almost surely. Then there is another collection of random variables \( \{\xi_a^*(\omega)|a \in A\} \) that satisfy the conditions pointwise and an event \( \Omega^* \subseteq \Omega \) such that \( P(\Omega^*) = 1 \) and for all \( a \in A \), \( \xi_a(\omega) = \xi_a^*(\omega) \).

To prove Theorem 3.1, suppose that we are given \( \mu \in \Lambda(A, R/N) \), and, for now, we will assume that \( A \) is a countably-generated \( \sigma \)-algebra. Decompose this weak random measure as \( \mu = \mu^+ - \mu^- \). Pick two collections of pointwise non-negative random variables indexed by \( A \) such that \( \mu_a^+(\omega) \in \mu^+(a) \) and \( \mu_a^- (\omega) \in \mu^-(a) \). Use Theorem 3.2 to find versions of these collections which satisfy the conditions pointwise in \( \omega \) and are equal to the original collections off of some set of measure 0. The new versions determine elements of \( \Lambda(A, R^+) \) and their difference determines an element \( \mu^* \) of \( \Lambda(A, R) \) such that for all \( a \in A \), \( \mu^*(a) \in \mu(a) \). Of course the difference of any two such strong versions must be a null random measure. \( \square \)


We will now demonstrate Kolmogorov consistency conditions for bounded, signed random measures (i.e. elements of \( \Lambda(A, R) \)). Daley and Vere-Jones [D,V-J] give such conditions for locally bounded, non-negative random measures, and claim that determining the conditions for signed random measures appeared to be an open problem.

**Definition Consistent System.** Let \( A \) be the Borel sets of a Polish space. A collection, \( \{F(s; \cdot)|s \in A^n, n = 1, 2, \ldots, \} \), of cumulative distribution functions indexed by measurable sets is consistent. For all \( n \) and all \( (a_1, \ldots, a_n) \) in \( A^n \), for all permutations \( \{i_1, \ldots, i_n\} \), for all disjoint \( a, b \in A \), and for all sequences \( \{a_n\} \subseteq A \), let the distribution functions have the following properties:

\[
F(a_1, \ldots, a_n; x_1, \ldots, x_n) = F(a_{i_1}, \ldots, a_{i_n}; x_{i_1}, \ldots, x_{i_n}) \quad (1)
\]
\[
F(a_1, \ldots, a_{n-1}, a_n; x_1, \ldots, x_{n-1}; \infty) = F(a_1, \ldots, a_{n-1}); x_1, \ldots, x_{n-1}) \quad (2)
\]
\[
F(a, b, a \cup b; x, y, z) = 1_{\{x+y \leq z\}} F(a, b; x, y) \quad (3)
\]
\[
a_n \downarrow 0 \Rightarrow (\forall \epsilon, c > 0)(\exists n) \epsilon > 1 - F(a_n; c) + F(a_n; -c) \quad (4)
\]
\[
(\forall \epsilon > 0)(\exists c > 0)(\forall a \in A) \epsilon > 1 - F(a; c) + F(a; -c) \quad (5)
\]

Conditions 1 and 2 guarantee the existence of a probability measure \( P \) on \( (\Omega, \mathcal{F}) \) and a family of random variables indexed by \( A \), \( \{\xi_a|a \in A\} \), satisfying the joint cumulative distributions given by \( F \). This is just an application of a well-known extension theorem due to Kolmogorov [Ko].

Conditions 3 and 4 were given by Daley and Vere-Jones [D,V-J] along with the requirements that the distribution function for the measure of each Borel set be supported on a finite interval non-negative real line, which we have modified to produce condition 5.
Theorem 4.1. Let \( \{F(s; \cdot)|s \in A^n, n = 1, 2, \ldots\} \) be a consistent system of cumulative distribution functions as defined above. Let \( \Omega = \mathbb{R}^A \), and let \( \mathcal{F} \) be the Borel sets of the Tychonoff topology associated with this space. There is a probability measure \( P \) on \((\Omega, \mathcal{F})\) and a random measure \( \xi : A \times \Omega \to \mathbb{R} \) satisfying

\[
P(\cap_{i=1}^n \{\xi(a_i) \leq x_i\}) = F(a_1, \ldots, a_n; x_1, \ldots, x_n).
\]

Moreover, every random measure is described by such a consistent system, and any two random measures described by such a system differ by a null random measure.

Condition 3 translates to finite additivity, \( P(\{|\xi_{a \cup b} - \xi_a + \xi_b| > 0\}) = 0 \), or \( \xi_{a \cup b} = \xi_a + \xi_b \) almost surely. Condition 4 translates to \( a_n \downarrow 0 \Rightarrow \xi_{a_n} \to 0 \) in probability, which is equivalent to \( a_n \downarrow 0 \Rightarrow \xi_{a_n} \to 0 \) almost surely.

Condition 5 translates to \((\forall \varepsilon > 0)(\exists c > 0)(\forall a \in A)P(\{|\xi_a| > c\}) < \varepsilon\). Therefore,

\[
(\forall \varepsilon > 0)(\exists c > 0)(\forall a \in A)P(\{|\xi_a| > c\} \cup \{|\xi_a^c| > c\}) < \varepsilon.
\]

Therefore,

\[
(\forall \varepsilon > 0)(\exists c > 0)(\forall a \in A)P(\{|\xi_a| + |\xi_a^c| > c\}) < \varepsilon.
\]

Let \( A_\infty \) be a countable algebra generating \( A \). \( A_\infty = \bigcup_{n=1}^{\infty} A_n \), where \( \{A_1, A_2, \ldots\} \) is an refining sequence of finite algebras. Let

\[
g_n = \vee_{a \in A_n} (|\xi_{a_n}| + |\xi_{a_n}^c|).
\]

The sequence \( \{g_n\} \) is non-decreasing and bounded on an event of probability 1,

\[
\Omega' = \bigcup_{p=1}^{\infty} \bigcap_{n=1}^{\infty} g_n^{-1}[0, p].
\]

Let \( g_\infty(\omega) = \vee_{n=1}^{\infty} g_n(\omega) \) for \( \omega \in \Omega' \), and let \( g_\infty(\omega) = 0 \) for \( \omega \in \Omega \). Therefore, for all \( a \in A_\infty \), almost surely \( |\xi_a| \leq g_\infty \).

Define \( \xi_a^* = \xi_a + N \) and \( g_\infty^* = g_\infty + N \); the same cumulative distributions apply to \( \{\xi_a^*|a \in A\} \), which is bounded in absolute value by \( g_\infty^* \). Consider the distributions on \( \{\xi_a^*|a \in A_\infty\} \); boundedness, finite additivity, and continuity imply that this collection represents an unique \( \xi^* \in \Lambda(A_\infty, R/N) \), which extends uniquely to a \( \xi^* \in \Lambda(A, R/N) \).

Suppose that there is an \( a \in A \) such that \( \xi_a^* \neq \xi^*(a) \); let \( B_\infty \) be the algebra generated by \( a \cup A_\infty \). Using the above procedure, we construct \( \xi' \in \Lambda(B_\infty, R/N) \) and extend it to \( \xi' \in \Lambda(A, R/N) \). Since \( \xi^* \) agrees with \( \xi' \) on \( A_\infty \) and, therefore, on all of \( A \). This directly contradicts our supposition. Therefore, the distributions specify an unique element of \( \Lambda(A, R/N) \) and, so, an unique element of \( \Lambda(A, R)\Lambda(A, N) \).
5. Regular Conditional Expectation of Random Measures.

Daley and Vere-Jones [D,V-J] prove that the conditional expectation with respect to a
\( \sigma \)-algebra \( \mathcal{F}_0 \) of a (strong) non-negative random measure \( \mu \) is defined by

\[ E[\mu|\mathcal{F}_0](a) = E[\mu(a)|\mathcal{F}_0] \]

and is a weak random measure (in our terminology) which has a strong version. As
they observe, this result is similar to the existence and essential uniqueness of regular
conditional probability [P]. They also caution that this may not be the case for signed
random measures, but, indeed, the situation is the same.

Consider the subspace \( \Lambda^1(A, R/N) \) of \( \Lambda(A, R/N) \) defined by \( \mu \in \Lambda^1(A, R/N) \) if and
only if \( E[|\mu|(X)] < \infty \). This is a vector lattice with a norm defined by

\[ \|\mu\|_1 = E[|\mu|(1)] = \int_\Omega |\mu|(\omega)dP(\omega). \]

Since \( |\mu|(\omega) = |\mu|(1, \omega) \) is the total variation norm at \( \omega \), the norm on \( \Lambda^1(A, R/N) \) is a
vector \( L^1 \) norm, and \( \Lambda^1(A, R/N) \) is, as we shall see, a generalization of \( L^1(\Omega, P) \).

Of course, a norm on a lattice \( V \) is consistent with the lattice structure if for any
\( f, g \in V \)

\[ |f| \leq |g| \Rightarrow \|f\| \leq \|g\|; \]
a vector lattice with a consistent norm is called a normed vector lattice. If the \( V \) is \( \sigma \)-
complete, the norm is order continuous if and only if \( f_n \to f \) implies \( \|f_n\| \to \|f\| \) A norm
has property \( L \) if and only if for \( f, g \in V \),

\[ \| |f| + |g| \| = \|f\| + \|g\|, \]

and property \( N \) if and only if for any non-decreasing, non-negative sequence \( \{f_n\} \),

\[ (\exists c)\|f_n\| \leq c \Rightarrow (\exists f \in V)f_n \uparrow f \]

The norm defined for \( \Lambda^1(A, R/N) \) is consistent; therefore, \( \Lambda^1(A, R/N) \) is \( \sigma \)-complete.
Furthermore, the norm is continuous and has property \( L \). The norm also has property
\( N \) as we shall show; all of these facts together with the following theorem imply that
\( \Lambda^1(A, R/N) \) is a Banach lattice, a normed vector space which is also norm complete.

**Theorem 5.1 (Schaefer [S]).** A \( \sigma \)-complete continuously-normed vector lattice \( V \)
with properties \( L \) and \( N \) is norm-complete.

Let \( \{f_n\} \) be a Cauchy sequence in \( V \). Since we wish only to prove that this sequence has
a limit, we can select any subsequence to construct this limit. Let \( \{f_{n_k}\} \) be a subsequence
satisfying \( \| f_{i_{n+1}} - f_{i_n} \| \leq 2^{-n} \); let \( g_1 = f_{i_1} \) and \( g_{n+1} = f_{i_{n+1}} - f_{i_n} \) for \( n = 1, 2, \ldots \). Let \( s_n = \sum_{i=1}^{n} g_n^+ \) and \( t_n = \sum_{i=1}^{n} g_n^- \). Then \( f_{i_n} = s_n - t_n \), and

\[
\| s_n \| = \| \sum_{i=1}^{n} g_n^+ \| \leq \| \sum_{i=1}^{n} g_n^- \| \leq \sum_{i=1}^{\infty} \| g_n \| \leq \| f_{i_1} \| + 1.
\]

The sequence \( \{ t_n \} \) is bounded in the same way. If \( s_n \uparrow s \) and \( t_n \uparrow t \), then \( f_{i_n} \to s - t \) in order limit; since the norm is continuous, this convergence also occurs in norm. Since the subsequence is convergent, the whole sequence is convergent. \( \Box \)

**Theorem 5.2.** (Monotone convergence [Beppo Levi].) Let \( f_n \) be a non-decreasing, non-negative sequence of random variables such that \( \sup_n E[f_n] < \infty \). There exists a random variable \( f \) such that \( f_n \uparrow f \) almost surely, and \( \sup_n E[f_n] = E[f] \). Obviously, a similar fact applies to equivalence classes of random variables.

Consider a non-decreasing, non-negative sequence \( \{ \mu_n \} \) in \( \Lambda(A, R/N) \) such that

\[
\sup_n E[\mu_n(X)] < \infty.
\]

There is an \( f \) in \( R/N \) such that \( \mu_n(X) \uparrow f \). The ranges of these measures are contained in the principal ideal associated with \( f \); therefore, we can use the same argument as in Lemma 2.4 to show that \( \{ \mu_n \} \) has the least upper bound \( \mu \) defined by

\[
\mu(a) = \sup_n \mu_n(a),
\]

for all \( a \in A \). Therefore, \( \Lambda^1(A, R) \) is a Banach lattice.

Let \( \Lambda^1(A, R/N) \) be the corresponding sublattice in \( \Lambda(A, R/N) \). Of course, \( \Lambda^1(A, R/N) \) is isomorphic as a Banach lattice to \( \Lambda^1(A, R) \). The following theorem is self evident.

**Theorem 5.3.** Given \( \mathcal{F}_0 \), a sub-\( \sigma \)-algebra of \( \mathcal{F} \), the conditional expectation of \( \mu \in \Lambda^1(A, R/N) \) defined by

\[
E[\mu|\mathcal{F}_0](a) = E[\mu(a)|\mathcal{F}_0]
\]

is a well-defined order preserving operator on \( \Lambda^1(A, R/N) \). Because \( \Lambda^1(A, R/N) \) is lattice isomorphic to \( \Lambda^1(A, R) \), the conditional expectation is a well-defined order preserving operator on this space, also.
6. Indefinite Integrals, Hahn Decomposition and Radon-Nikodym Derivatives

For the final result, we consider random measures defined pointwise in $\omega$ (i.e. $\Lambda(A, R)$). We consider a probability space $(\Omega, \mathcal{F}, P)$ and a measurable space $(X, \mathcal{X})$ generated by the a Polish topology on $X$. Now that we have random measures in this scheme, we wish to define indefinite integrals with respect to random measures and to prove necessary and sufficient conditions so that one random measure $\nu$ is the integral of another random measure $\mu$ with respect to some process $F$.

We begin by considering some of the collections of functions, actually processes, we wish to integrate. A step process is an indicator function of a set in the $\sigma$-algebra $\sigma(\{a \times b|a \in \mathcal{X}, b \in \mathcal{F}\})$. A simple process is an finite linear combination of step processes. A process is a function of $x$ and $\omega$ which has as it’s level sets elements of the above $\sigma$-algebra. A bounded process is a process which is less than some constant in absolute value.

If if $\nu$ is a non-negative random measure and $H$ is a step process, $H(x, \omega) = h_1(x)h_2(\omega)$, where $h_1$ and $h_2$ are indicator functions and, by definition,

$$(H : \nu)(a) = \int_a H(x, \omega)d\nu(x; \omega) = h_2(\omega)\nu(a \cap h_1).$$

We extend the integral to non-negative simple functions by linearity. We extend the integral to non-negative bounded processes by taking the supremum

$$(H : \nu) = \vee_{n=1}^\infty(\vee_{q \in Q_n} q1_{H^{-1}[q, +\infty)} : \nu),$$

where $Q_n$ is the first $n$ non-negative rational numbers under some enumeration scheme. (Of course, the sequence of simple processes occurring above converges up to $H$.) We extend the integral to any bounded process by decomposing the process into positive and negative parts.

$$(H : \nu) = (H^+ : \nu) - (H^- : \nu).$$

Finally, we extend the integral to any signed random measure by decomposing the random measure into positive and negative parts,

$$(H : \nu) = (H : \nu^+) - (H : \nu^-).$$

We note that since $\nu$ is a bounded random measure, the integrals defined here are also bounded, signed measures.

**Theorem 6.1 (Hahn Decomposition).** For any signed random measure $\nu$, there is a step process $1_{\{\nu > 0\}}(x, \omega)$ such that

$$(1_{\{\nu > 0\}}(x, \omega) : \nu) = \nu^+.$$
Let \( \{c_n\} \) be a countable open base for the topology of \( X \), for \( n = 1, 2, \ldots \) let \( A_n \) and \( B_n \) be the algebra and the partition generated by \( \{c_1, \ldots, c_n\} \). Let

\[
H_n(x, \omega) = \sum_{a \in B_n} 1_a(x)1_{\{\nu(a, \omega) > 0\}}(\omega).
\]

It is easily shown that

\[
(H_n : \nu)(1, \omega) = \vee_{a \in A_n} \nu(a, \omega).
\]

From which it follows that

\[
(H_n : \nu) \leq (H_{n+1} : \nu),
\]

and

\[
\vee_{n=1}^{\infty} (H_n : \nu)(1, \omega) = \nu^+(1, \omega).
\]

Suppose that \( E(x, \omega) \) is an \( A_k \)-measurable step process in the sense that it can be written as

\[
E(x, \omega) = \sum_{a \in B_k} 1_a(x)1_{e_a}(\omega),
\]

where \( e_a \) is a collection of events indexed by \( a \in B_k \), and that \( 0 \leq (E : \nu)(1, \omega) \). Then

\[
E \land H_k = \sum_{a \in B_k} 1_a(x)(1_{e_a}(\omega) \land 1_{\{\nu(a, \omega) > 0\}}(\omega)),
\]

and

\[
0 \leq (E : \nu)(1, \omega) \leq (E \land H_k : \nu)(1, \omega) \leq (H_k : \nu)(1, \omega)
\]

Since \( E \lor H_k = E + H_k - E \land H_k \),

\[
0 \leq (E : \nu)(1, \omega) \leq (E \lor H_k : \nu)(1, \omega).
\]

By application of the above principle we see that

\[
(H_n \lor \cdots \lor H_{n+p} : \nu)(1, \omega) \leq ((H_n \lor \cdots \lor H_{n+p}) \lor H_{n+p+1} : \nu)(1, \omega);
\]

therefore,

\[
(H_n : \nu)(1, \omega) \leq (H_n \lor H_{n+1} : \nu)(1, \omega) \leq (H_n \lor H_{n+1} \lor H_{n+2} : \nu)(1, \omega) \leq \ldots.
\]

All of these terms are less than or equal to the limit

\[
\lim_{p \to \infty} (\lor_{k=n}^{p} H_k : \nu)(1, \omega) = (\lor_{p=n}^{\infty} H_p : \nu)(1, \omega),
\]
which we calculate using the monotone convergence theorem at each omega and which is bounded above by \( \nu^+(1, \omega) \).

\[
(\limsup_{n \to \infty} H_n : \nu)(1, \omega) = \nu^+(1, \omega).
\]

We can conditionally extend the notion of the indefinite integral of a random measure with respect to a possibly unbounded process by a limiting procedure. Let us assume that \( F \) and \( \nu \) are non-negative and that there is a non-negative random variable \( f \) such that for \( n = 1, 2, \ldots \)

\[
(n \wedge F : \nu)(1, \omega) \leq f(\omega).
\]

Then

\[
(F : \nu)(a, \omega) = \lim_{n \to \infty} (n \wedge F : \nu)(a, \omega)
\]

for all \( a \in A \) defines a random measure. We can extend this result to signed measures and processes by applying it to the positive and negative parts of the measures and processes.

For two non-negative random measures \( \mu \) and \( \nu \), the band projection ([Ka],[V],[S]) of \( \nu \) by \( \mu \) is defined by

\[
[\mu] \nu = \bigvee_{n=1}^{\infty} n \mu \wedge \nu.
\]

Of course, \([\mu] \nu \leq \nu\); when \([\mu] \nu = \nu\), \( \nu \) is said to be absolutely continuous with respect to \( \mu \), written \( \nu \ll \mu \). When \([\mu] \nu = 0\), \( \nu \) is said to be singular with respect to \( \mu \), written \( \nu \perp \mu \); this relation is symmetric. Given any two non-negative random measures \( \mu \) and \( \nu \), there is a Lebesgue decomposition [Y],

\[
\nu = [\mu] \nu + (\nu - [\mu] \nu),
\]

of \( \nu \) with respect to \( \mu \) into non-negative absolutely continuous and singular components.

The band projection for the case that \( \nu \) and \( \mu \) are signed is given by the extension

\[
[|\mu|] \nu = [|\mu|] \nu^+ - [|\mu|] \nu^-.
\]

By the definition of \((F : \nu)\)

\[
[\nu](F : \nu) = (F : \nu).
\]

**Theorem 6.2 (Radon-Nikodym).** Let \( \nu \) and \( \mu \) be non-negative random measures such that \( \nu \ll \mu \). There is a process \( F \) such that \( \nu = (F : \mu) \).

Let

\[
F(x, \omega) = \bigvee_{q \in \mathcal{Q}} 1\{x-q \mu\},
\]

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where $\mathcal{Q}$ is the non-negative rational numbers. Since

$$(\forall q \in \mathcal{Q} \cap \mathbb{Q}^+ \{ \nu - q\mu \} : \mu) \leq \nu,$$

where $\mathcal{Q} \cap \mathbb{Q}$ represents the first $n$ terms in some enumeration of $\mathcal{Q}$, the limit of this sequence of random measures is bounded in the same way and equal to $(F; \mu)$. For any rational $\epsilon > 0$, let

$$F_\epsilon = \bigvee_{n=1}^\infty n \epsilon 1_{\{\nu - n\epsilon \mu\}};$$

$$0 \leq \nu - (F_\epsilon : \mu) \leq \epsilon \mu,$$

so that $(F_\epsilon : \mu) \uparrow \nu$ as $\epsilon \downarrow 0$. Moreover, since

$$(F_\epsilon : \mu) \leq (F : \mu) \leq \nu,$$

we have $(F : \mu) = \nu$. []

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