ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF
COAGULATION-FRAGMENTATION MODELS

By

Avner Friedman
and
Fernando Reitich

IMA Preprint Series # 1479
June 1997

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF COAGULATION-FRAGMENTATION MODELS

AVNER FRIEDMAN* AND FERNANDO REITICH†

Abstract. In this paper we study the evolution of the number density \( n(x, c, t) \) of droplets of volume \( x \) with chemical concentration \( c \) subject to both coalescence (coagulation) and rupture (fragmentation, breakage). It is proved that as \( t \to \infty \) the concentration tends to a limit value \( c_\infty \) and the measure \( xn(x, c, t)dc \) converges to \( \delta(c - c_\infty)dx \) where \( \delta(y) \) is the Dirac measure and \( dL \) is a measure satisfying the appropriate equilibrium equation.

1. The model. Consider a mixture of a large number of incompressible spherical droplets of varying volume size \( x \) in a solution (e.g., water or air). When the mixture is well stirred the distribution of droplet size can be taken to be homogeneous in space. We may then introduce the notion of density number \( n(x) \):

\[
n(x) \, dx = \text{number of droplets in unit volume}
\]

whose volume size lies between \( x \) and \( x + dx \);

\( n(x) \) is independent of the location of the volume element.

Since the droplets are in continuous motion, they may collide and, as a result, coalesce. The motion may also cause rupture. We assume that if two droplets of volumes \( x \) and \( \xi \) coalesce, they will form a droplet of volume \( x + \xi \). The rate of coalescence is given by a function \( K(x, \xi) \), which is called the coalescence or coagulation kernel.

We also introduce the fragmentation or breakage kernel \( B(x, \xi) \) which is the rate that a droplet of volume \( x \) breaks into two droplets of volumes \( \xi \) and \( x - \xi \).

Throughout this paper we assume that

\[
K(x, \xi) \text{ is continuous and } \geq 0 \text{ for } x, \xi \geq 0, \\
K(x, \xi) = K(\xi, x), \\
K(x, \xi) \leq C_0, \\
B(x, \xi) \text{ is continuous and } \geq 0 \text{ for } x \geq \xi \geq 0, \\
B(x, \xi) = B(x, x - \xi), \\
\int_0^x B(x, \xi)d\xi \leq C_1, \\
\int_0^x \xi B(x, \xi)d\xi \leq C_2x.
\]

(1.1)

Remark 1.1. Note that if volume is preserved under rupture, then

\[
\int_0^x B(x, \xi)\xi d\xi = x,
\]

* University of Minnesota, Institute for Mathematics and its Applications, Minneapolis, Minnesota 55455.
† Department of Mathematics and Center for Research in Scientific Computation, North Carolina State University, Box 8205, Raleigh, NC 27695-8205.
so that the last condition in (1.1) is very natural.

Due to coagulation and fragmentation the number density will actually vary with time, and we introduce
\[ n(x, t) = \text{number density at time } t. \]

The following conservation law holds:
\[ \frac{d}{dt} n(x, t) dx = \text{number gained through coalescence} + \text{number gained through rupture} - \text{number lost through coalescence} - \text{number lost through rupture}. \]

This can be expressed by the evolution equation:
\[
\frac{d}{dt} n(x, t) = \frac{1}{2} \int_{0}^{x} K(x - \xi, \xi)n(x - \xi, t)n(\xi, t)d\xi + \int_{0}^{\infty} B(\xi, x)n(\xi, t)d\xi - n(x, t) \int_{0}^{x} K(x, \xi)n(\xi, t)d\xi - n(x, t) \int_{x}^{\infty} \frac{\xi}{x} B(x, \xi)d\xi.
\]
(1.2)

Here the underlying assumption is that, although \( n(x, t) \) varies with \( t \), the mixture is maintained homogeneous in space at any moment (due to continuous fast stirring).

Together with (1.2) we prescribe an initial condition
\[
(1.3) \quad n(x, 0) = n_0(x), \quad n_0(x) \geq 0.
\]

There is substantial literature on coagulation and coagulation-fragmentation models. Melzak [10] proved a general existence, uniqueness and positivity of a solution for all \( t > 0 \). For other results we refer to [2], [4], [9] and the references therein.

We shall actually be interested in a situation where each droplet carries a chemical species which is uniformly distributed within it; the concentration \( c \) of the chemical species is a variable quantity. We then introduce
\[ n(x, c, t)dx dc = \text{number of droplets at time } t \]
with volume size in the interval \( (x, x + dx) \)
and concentration (of the chemical species) in the interval \( (c, c + dc) \).
Analogously to (1.2), the number density \( n(x, c, t) \) evolves as follows:

\[
\frac{d}{dt} n(x, c, t) = \frac{1}{2} \int_0^\infty \int_0^\infty K(x - \xi, \xi) n(x - \xi, \frac{xc - \xi \gamma}{x - \xi}, t) n(\xi, \gamma, t) \frac{x}{x - \xi} d\gamma d\xi \\
+ \int_x^\infty B(\xi, x) n(\xi, c, t) d\xi \\
- n(x, c, t) \int_0^\infty \int_0^\infty K(x, \xi) n(\xi, \gamma, t) d\gamma d\xi \\
- n(x, c, t) \int_0^\infty \frac{\xi}{x} B(x, \xi) d\xi,
\]

(1.4)

where we define

\[ n(x, \xi', t) = 0 \quad \text{if} \quad \xi' < 0. \]

Here we used the fact that rupture does not change the concentration. On the other hand, if particles \( (\xi, \gamma) \) and \( (\eta, \beta) \) coalesce to produce \( (x, c) \), then

\[ x = \xi + \eta, \quad xc = \xi \gamma + \eta \beta \]

so that

\[ \eta = x - \xi, \quad \beta = \frac{xc - \xi \gamma}{x - \xi} \]

and \( d\gamma d\xi d\eta d\beta = d\gamma d\xi \frac{x}{x - \xi} dx dc \), since the Jacobian determinant

\[
\left| \begin{array}{cc}
\frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial c} \\
\frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial c}
\end{array} \right|
\]

is equal to \( \frac{x}{x - \xi} \);

this explains the appearance of \( x/(x - \xi) \) in the first integral on the right-hand side of (1.4).

**Remark 1.2.** The last integral on the right-hand side of (1.2), or (1.4), can also be written in the form

\[
\frac{1}{2} \int_0^x B(x, \xi) d\xi
\]

since

\[
\frac{1}{2} \int_0^x B(x, \xi) d\xi = \frac{1}{2} x \int_0^x \left[ \xi B(x, \xi) + (x - \xi) B(x, x) \right] d\xi \\
= \frac{1}{2} x \left[ \int_0^x \xi B(x, \xi) d\xi + \int_0^x (x - \xi) B(x, x - \xi) d\xi \right] \\
= \frac{1}{x} \int_0^x \xi B(x, \xi) d\xi.
\]
Together with (1.4) we prescribe an initial condition

\begin{equation}
(1.5) \quad n(x, c, 0) = n_0(x, c), \quad n_0(x, c) \geq 0.
\end{equation}

The model (1.4) arises in the process of making a photographic film. The film is made of several emersion layers. One of them consists of oil droplets in aqueous solution. The mixture, prior to being coated on the film substrate, is subjected to coalescence and rupture by means of stirring. A key problem is to determine the rate of coalescence (which depends on parameters, such as the amount of surfactant added at the initial time). One of the methods to determine coalescence is based on adding chemiluminescent species to the mixture:

The optical signal, which is the number of photons emitted per unit time from a droplet, is a function, \( f(x, c) \), of the species concentration \( c \) in the droplet and the droplet volume \( x \). One can measure the total signal

\begin{equation}
(1.6) \quad S(t) = \int_0^\infty \int_0^\infty f(x, c)n(x, c)dxdc,
\end{equation}

and this is used to evaluate the rate of coalescence [8, Chap. 6].

Note that if two drops with parameters \( (x_1, c_1) \) and \( (x_2, c_2) \) coalesce to form a drop with parameters \( (\bar{x}, \bar{c}) \) then

\[ \bar{x} = x_1 + x_2, \quad \bar{c} = \frac{x_1 c_1 + x_2 c_2}{x_1 + x_2}. \]

Typically \( f(x, c) = x c^2 \), and then

\[ f(\bar{x}, \bar{c}) < f(x_1, c_1) + f(x_2, c_2), \]

so that the total signal should decrease.

The model (1.4) was developed by David Ross; see [8, Chap. 6]. He conjectured that as \( t \to \infty \) the concentrations \( c \) will all tend to a constant, say \( c_\infty \). With this in mind, one defines the degree of coalescence

\begin{equation}
(1.7) \quad D(t) = \int_0^\infty \int_0^\infty (c - c_\infty)^2 xn(x, c, t)dxdc
\end{equation}

and one would like to prove rigorously that

\begin{equation}
(1.8) \quad D(t) \to 0 \text{ if } t \to \infty.
\end{equation}

Consider next the measures \( d\mu_t \) defined by

\begin{equation}
(1.9) \quad d\mu_t(x, c) = xn(x, c, t)dxdc
\end{equation}

If (1.8) is true then we may go one step further and try to prove that as \( t \to \infty \)

\begin{equation}
(1.10) \quad d\mu_t(x, c) \to \delta(c - c_\infty)dL(x)
\end{equation}

in the sense of weak convergence of measures, where \( \delta(y) \) is the Dirac measure and

\begin{equation}
(1.11) \quad dL \text{ is a solution of the equilibrium equation (1.2)}.
\end{equation}
In this paper we shall prove, under mild additional conditions on $K$ and $B$ that (1.8) holds. We shall also establish (1.10), (1.11) under the additional condition that

$$K(x, \xi) \leq \eta(x + \xi), \quad \gamma_0 \equiv \inf_{x > 0} \frac{1}{x} \int_0^x \xi B(x, \xi) d\xi > 0,$$

where

$$2\eta \int_0^\infty \int_0^\infty x n_0(x, c) dx dc < \gamma_0. \quad (1.12)$$

The structure of the paper is as follows:

In §2 we establish existence, uniqueness and positivity of the solution of (1.4), (1.5) for all $t > 0$. Assuming that

$$n_0(x) = \int_0^\infty n_0(x, c) dc \quad (1.13)$$

we also prove that the function

$$n(x, t) = \int_0^\infty n(x, c, t) dc \quad (1.14)$$

is the unique solution of (1.2), (1.3). In §3 we prove conservation of mass

$$\int_0^\infty x n(x, t) dx = \text{const.}$$

We also show that if $n_0(x, c) = 0$ whenever $c > c_*$ ($c_*$ constant) then $n(x, c, t) = 0$ for $c > c_*$. Using these results we derive in §4 a formula for $dS/dt$ (see (4.2)) when $S$ is defined in (1.6) and $f(x, c)$ is any function satisfying $|f(x, c)| \leq C(1 + x)$. This formula is the key to the proofs of both (1.8) (in §5) and (1.10), (1.11) (in §6).

We finally remark that a coalescence problem for $n(x, c, t)$ was studied in [1], with $c$ being the concentration of surfactant which inhibits coalescence. The limit behavior of the solution and the mathematical method for deriving it are entirely different from the limit behavior and the methods of the present paper.

2. Existence and uniqueness. We shall need the following assumptions:

$$n_0(x) \text{ is continuous for } x \geq 0,$$

$$0 \leq n_0(x) \leq A_0, \quad \int_0^\infty n_0(x) dx \leq A_0, \quad (2.1)$$

$$n_0(x, c) \text{ is continuous for } x \geq 0, \ c \geq 0,$$

$$0 \leq n_0(x, c) \leq A_1, \quad \int_0^\infty \int_0^\infty n_0(x, c) dx dc \leq A_1 \quad (2.2)$$

where $A_0, A_1$ are positive constants.
THEOREM 2.1. If \((1.1), (2.1)\) hold then there exists a unique solution of \((1.2), (1.3)\) with \(n(x,t), \partial n(x,t)/\partial t\) continuous for \(x \geq 0, t \geq 0\), such that, for each \(T > 0\),

\[
(2.3) \quad \sup_{x \geq 0} |n(x,t)| + \int_0^\infty |n(x,t)|dx \leq C(T) < \infty \text{ if } 0 \leq t \leq T.
\]

The solution has the following additional properties:

\[(2.4) \quad n(x,t) \text{ is analytic in } t, \quad t \geq 0,\]

\[(2.5) \quad n(x,t) \geq 0.\]

This theorem is due to Melzak [10]. We note that if \(K\) and \(B\) have compact support and \(n_0(x) > 0\) for all \(x > 0\), then the proof of \((2.5)\) is much simpler than the proof given in [10] (or [3]). Indeed, if \(x\) is large then from \((1.2)\) we see that \(dn(x,t)/dt = 0\) so that \(n(x,t) = n_0(x) > 0.\) Hence if \((2.5)\) is not true, with strict inequality, then there is a smallest \(t = t_0 > 0\) such that \(n(x,t) > 0\) if \(t < t_0\) for all \(x \geq 0\) and \(n(x_0,t_0) = 0\) for a finite point \(x_0\). From \((1.2)\) we then deduce that

\[
\frac{dn(x_0,t_0)}{dt} > 0,
\]

which is a contradiction. Finally, the solution \(n(x,t)\) for general \(K, B, n_0\), can be obtained as a pointwise limit of solutions with compact \(K, B\) and strictly positive \(n_0(x)\) (cf. [3] or [4]), so that \((2.5)\) is true in general.

THEOREM 2.2. If \((1.1), (2.2)\) hold then there exists a unique solution of \((1.4), (1.5)\) with \(n(x,c,t), \partial n(x,c,t)/\partial t\) continuous for \(x \geq 0, c \geq 0, t \geq 0\), such that, for each \(T > 0\),

\[
(2.6) \quad \sup_{x \geq 0,c \geq 0} |n(x,c,t)| + \int_0^\infty \int_0^\infty |n(x,c,t)|dxdc \leq C(T) < \infty \text{ if } 0 \leq t \leq T.
\]

The solution has the following additional properties:

\[(2.7) \quad n(x,c,t) \text{ is analytic in } t, \quad t \geq 0,\]

\[(2.8) \quad n(x,c,t) \geq 0.\]

Proof. The proof is similar to the proof of Theorem 2.1 provided we can treat the first term on the right-hand side of \((1.4)\) analogously to the first term on the right-hand side of \((1.2)\); this means that we have to control the factor \(x/(x-\xi)\) in the integrand.

Introduce the operator

\[
J(m,n) = \frac{1}{2} \int_0^\infty \int_0^\infty K(x-\xi,x)m \left( x-\xi, \frac{xc-\xi\gamma}{x-\xi} \right) \frac{x}{x-\xi} n(\xi,\gamma)d\gamma d\xi
\]

and break up the integral into

\[
\int_0^{x/2} \int_0^x + \int_0^{x/2} \int_0^\infty
\]

and
In the second integral substitute

\[
\gamma \rightarrow \gamma' = \frac{xc - \xi \gamma}{x - \xi}, \quad \text{or} \quad \gamma = \frac{xc - (x - \xi)\gamma'}{\xi} , \quad d\gamma = \frac{x - \xi}{\xi} d\gamma'.
\]

We then obtain

\[
J(m, n) = \frac{1}{2} \int_0^{x/2} \int_0^{x/2} K(x - \xi, \xi) m \left( x - \xi, \frac{xc - \xi \gamma}{x - \xi}, t \right) n(\xi, \gamma, t) \frac{x}{x - \xi} d\gamma d\xi - \frac{1}{2} \int_0^{x/2} \int_0^{x/2} K(x - \xi, \xi) n \left( \xi, \frac{xc - (x - \xi)\gamma'}{\xi}, t \right) m(x - \xi, \gamma', t) \frac{x}{\xi} d\gamma' d\xi.
\]

\[
= J_1 - J_2.
\]

Since the factors \(\frac{x - \xi}{x - \xi}\) and \(\frac{x}{\xi}\) in \(J_1\) and \(J_2\) respectively are uniformly bounded, we can easily establish, as in [10], the bound

\[
|J(m, n)|_{L^\infty} \leq C |n|_{L^\infty} \int \int |m| + C |m|_{L^\infty} \int \int |n|
\]

where

\[
|f|_{L^\infty} = \sup_{x \geq 0, c \geq 0} |f(x, c, t)|,
\]

\[
\int \int |f| = \int_0^\infty \int_0^\infty |f(x, c, t)| dc dx.
\]

Next we estimate

\[
\int \int |J(m, n)| dc dx
\]

by substituting

\[
c \rightarrow c' = \frac{xc - \xi \gamma}{x - \xi}, \quad dc' = \frac{x}{x - \xi} dc.
\]

we obtain

\[
|J(m, n)| \leq C \int \int |m| \cdot \int \int |n|
\]

With (2.9), (2.10) at hand, we can now proceed as in [10], with just minor changes, to establish Theorem 2.2. The positivity of \(n(x, c, t)\) can be proved, more simply, by the argument following Theorem 2.1.

**Theorem 2.3.** If (1.1), (2.1), (2.2) hold and, in addition,

\[
n_0(x) = \int_0^\infty n_0(x, c) dc,
\]
then

\[(2.12) \quad \int_0^\infty n(x, c, t)dc = n(x, t).\]

where \(n(x, c, t)\) and \(n(x, t)\) are the solutions of (1.4), (1.5) and (1.2), (1.3), respectively.

**Proof.** We first proceed formally to integrate (1.4) with respect to \(c\). In the first integral on the right-hand side we substitute

\[(2.13) \quad c \to c' = \frac{xc - \xi y}{x - \xi}, \quad dc' = \frac{x}{x - \xi}dc\]

to get

\[(2.14) \quad \frac{1}{2} \int_0^\infty K(x - \xi, \xi)n(x - \xi, t)n(\xi, t)d\xi.\]

where \(n(x, t)\) is the function defined by the left-hand side of (2.12). All the other integrals yield immediately the corresponding integrals of (1.2). Hence, by uniqueness, the left-hand side of (2.12) is the solution of (1.2), (1.3).

To prove the theorem rigorously we rewrite (1.4) in integrated form:

\[(2.15) \quad n(x, c, t) - n_0(x, c) = \int_0^t dt \left\{ \frac{1}{2} \int_0^x \int_0^\infty K \cdots + \cdots \right\}.\]

Since

\[\int_0^\infty \int_0^\infty n(x, c, t)dcdx \leq C(T) < \infty \text{ if } 0 \leq t \leq T,\]

the integral

\[\int_0^\infty n(x, c, t)dc\]

is finite a.e. in \(x\), for each \(t\), and we may integrate (2.15) with respect to \(c\), \(0 \leq c < \infty\). Each of the four multiple integrals thus obtained is well defined and is finite. We now proceed with the substitution (2.13) to obtain the term (2.14, and the proof of the theorem then follows as before.


**Theorem 3.1.** (Conservation law). If (1.1), (2.1) hold and

\[(3.1) \quad \int_0^\infty x n_0(x)dx < \infty,\]
then

\[ (3.2) \quad \int_0^\infty x^n(x,t)dx = \int_0^\infty x^{n_0}(x)dx \]

for all \( t > 0 \).

**Proof.** The assertion (3.2) means that

\[ (3.3) \quad \frac{d}{dt} \int_0^\infty x^n(x,t)dx = 0. \]

This relation follows formally as a special case of Theorem 4.1 with \( f(x,c) = x \). However, in order to apply the proof rigorously we need to know a priori that, for any \( T > 0 \),

\[ (3.4) \quad \int_0^\infty x^n(x,t)dx \leq C(T) < \infty \quad \text{if} \quad 0 \leq t \leq T. \]

If \( K, B \) and \( n_0 \) have compact support then (3.4) is true and then so is (3.2). We now approximate general \( K, B, n_0 \) by \( K_j, B_j, n_{0,j} \) with compact support and denote the corresponding solutions by \( n_j(x,t) \). Then, on one hand we have

\[ \int_0^\infty x^n_j(x,t)dx = \int_0^\infty x^{n_{0,j}}(x)dx \leq C \]

for all \( j \), and on the other hand \( n_j(x,t) \to n(x,t) \) pointwise (cf. [4]). It follows that \( n(x,t) \) satisfies (3.4) and consequently (3.3), or (3.2), holds.

**Theorem 3.2.** If (1.1), (2.2) hold and

\[ (3.5) \quad n_0(x,c) = 0 \quad \text{for} \quad x \geq 0, \quad c > c_* \quad (c_* > 0), \]

then

\[ (3.6) \quad n(x,c,t) = 0 \quad \text{for} \quad x \geq 0, \quad c \geq c_* \quad t \geq 0. \]

**Proof.** If \( c > c_* \) then in the first integral on the right-hand side of (1.4)

\[ \text{either} \quad \gamma > c_* \quad \text{or} \quad \frac{xc - \xi \gamma}{x - \xi} > c_* \]

Assuming that \( K \) has compact support and using (2.3), we conclude that the integral is bounded by \( CN(t) \), where

\[ N(t) = \sup_{x \geq 0,c \geq c_*} n(x,c,t). \]

If \( B \) has compact support then the second integral on the right-hand side of (1.4) is bounded by

\[ C \sup_{\xi \geq x} n(\xi,c,t), \quad \text{or by} \quad CN(t). \]
From (1.4) it then follows that

$$\frac{dn(x, c, t)}{dt} \leq CN(t).$$

If $x \geq 0$, $c > c_*$, so that

$$N(t) \leq N(0) + C \int_0^t N(t') dt'.$$

Since $N(0) = 0$, the assertion (3.6) follows.

Having proved (3.6) in the case where $K$ and $B$ have compact support, we can now establish the same result for general $K$, $B$ by approximation (cf. the proof of Theorem 3.1).

For convenience we collect all the assumptions made so far on $n_0(x, c)$ and $n_0(x)$:

$$\begin{cases}
    n_0(x) \text{ and } n_0(x, c) \text{ are continuous functions for } x \geq 0, c \geq 0, \\
    0 \leq n_0(x) \leq A_0, \quad \int_0^\infty (1 + x)n_0(x)dx < \infty, \\
    0 \leq n_0(x, c) \leq A_1, \quad \int_0^\infty n_0(x, c)dc = n_0(x), \\
    n_0(x, c) = 0 \text{ if } c \geq c_* \quad (c_* > 0). \tag{3.7}
\end{cases}$$

**Corollary 3.3.** If (1.1) and (3.7) are satisfied then (3.2) and (3.6) hold.

4. **A formula for $dS/dt$.** Let $f(x, c)$ be any function satisfying

$$|f(x, c)| \leq C(1 + x) \quad \text{if } x \geq 0, c \leq c_* \tag{4.1}$$

and arbitrary otherwise. By (2.6) and Corollary 3.3 it follows that the integral

$$S(t) = \int_0^\infty \int_0^\infty f(x, c)n(x, c, t)dxdc \tag{4.2}$$

exists for all $t \geq 0$.

**Theorem 4.1.** If (1.1), (3.7) hold and $f$ satisfies (4.1), then

$$\frac{dS(t)}{dt} = \frac{1}{2} \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma K(x, \xi)n(x, c, t)n(\xi, \gamma, t)$$

$$\times \left[ f \left( x + \xi, \frac{x \xi + \xi \gamma}{x + \xi} \right) - f(x, c) - f(\xi, \gamma) \right]$$

$$+ \int_0^\infty dx \int_0^\infty dc \int_0^\infty \frac{B(\xi, x)n(\xi, c, t)}{x} \left[ f(x, c) - \frac{1}{\xi} f(\xi, c) \right] d\xi. \tag{4.3}$$

Note that, by (2.6) and Corollary 3.3, all the integrals on the right-hand side of (4.3) are absolutely convergent.
Proof. Formally

\begin{equation}
\frac{dS(t)}{dt} = \int_0^\infty \int_0^\infty f(x,c) \frac{dn(x,c,t)}{dt} dx dc.
\end{equation}

By substituting \(dn/dt\) from (1.4) and using (2.6), (3.2), (4.1), we find that each of the resulting four integrals is absolutely convergent.

A standard argument then shows that \(S(t)\) is differentiable and \(dS/dt\) is given by (4.4).

We can next write

\begin{equation}
\frac{dS}{dt} = I_1 + I_2
\end{equation}

where

\begin{align*}
I_1 &= \int_0^\infty dc \left[ \int_0^\infty f(x,c) dx \int_x^\infty B(\xi, x)n(\xi, c, t)d\xi \\
&\quad - \int_0^\infty f(x,c)n(x,c,t)dx \int_x^\infty B(x, \xi)d\xi \right], \\
I_2 &= \frac{1}{2} \int_0^\infty dx \int_0^\infty f(x,c)dc \int_0^\infty \int_0^\infty K(x - \xi, \xi)n \left( x - \xi, \frac{xc - \xi \gamma}{x - \xi}, t \right) n(\xi, \gamma, t) \frac{x}{x - \xi} d\gamma d\xi \\
&\quad - \int_0^\infty dc \int_0^\infty dx \int_0^\infty n(x,c,t)K(x, \xi)n(\xi, \gamma, t)f(x,c)d\gamma d\xi \\
&= I_{21} - I_{22}.
\end{align*}

To evaluate \(I_1\) we change the notation of the variables \(x \to \xi\) and \(\xi \to x\) in the second integral in \([\cdots]\) to get

\[\int_0^\infty d\xi \int_0^\xi^\infty \frac{x}{x - \xi} B(\xi, x)f(\xi, c)n(\xi, c, t)dx.\]

This, in turn, is equal, by changing the order of integration, to

\[\int_0^\infty dx \int_0^\xi^\infty \frac{x}{x - \xi} B(\xi, x)f(\xi, c)n(\xi, c, t)d\xi.\]

Hence

\begin{equation}
I_1 = \int_0^\infty dc \int_0^\infty dx \int_0^\xi^\infty B(\xi, x) \left[ f(x,c) - \frac{x}{\xi} f(\xi, c) \right] n(\xi, c, t)d\xi
\end{equation}

Next we turn to \(I_{21}\) and change variables \(x \to x'\) where \(x - \xi = x'\) (for each fixed \(\xi\)) and \(c \to c'\) by

\[c = \frac{x'c' + \xi \gamma}{x' + \xi}, \quad \text{or} \quad \frac{xc - \xi \gamma}{x - \xi} = c'.\]
We get

\[ I_{21} = \frac{1}{2} \int_0^\infty dx' \int_0^\infty d\xi \int_0^{x'} dc' \int_0^{\xi} d\gamma K(x', \xi)n(x', c, t)n(\xi, \gamma, t)f \left( \xi + x', \frac{x'c' + \xi \gamma}{x' + \xi} \right). \]

We shall use the relations

\[ \int dc' \int d\gamma \Phi = \int dc' \int d\gamma \Phi + \int dc' \int d\gamma \Phi \]

\[ = \int dc' \int d\gamma \Phi + \int d\gamma \int dc' \Phi, \]

which is valid with any integrand \( \Phi \). Since after integration, in \( I_{21} \), with respect to \( x' \) and \( \xi \) we get an expression \( \Psi(\gamma, c') \) which is symmetric in \( \gamma, c' \), we can write

\[ \int_0^\infty d\gamma \int dc' \Psi(\gamma, c') = \int dc' \int d\gamma \Psi(\gamma, c'). \]

Hence for the purpose of calculating \( I_{21} \) we may replace the right-hand side of (4.7) by

\[ 2 \int dc' \int d\gamma \Phi. \]

Consequently

\[ I_{21} = \int_0^\infty dx' \int_0^\infty d\xi \int dc' \int d\gamma K(x', \xi)n(x', c', t)n(\xi, \gamma, t)f \left( \xi + \frac{x'c' + \xi \gamma}{x' + \xi}, t \right). \]  

Next we can write

\[ I_{22} = \int_0^\infty dx \int_0^\infty d\xi \int dc \int d\gamma K(x, \xi)n(x, c, t)n(\xi, \gamma, t)f(x, c) \]

\[ + \int_0^\infty dx \int_0^\infty d\xi \int dc \int d\gamma K(x, \xi)n(x, c, t)n(\xi, \gamma, t)f(x, c) \]

\[ = L_1 + L_2 \]

In \( L_2 \) we write

\[ \int dc \int d\gamma = \int d\gamma \int dc \]

to get

\[ L_2 = \int_0^\infty dx \int_0^\infty d\xi \int d\gamma \int dcK(x, \xi)n(x, c, t)n(\xi, \gamma, t)f(x, c). \]
Changing the notation of variables \( \gamma \leftrightarrow c \) and \( x \leftrightarrow \xi \) we find, using the symmetry of \( K(x, \xi) \), that

\[
L_2 = \int_0^\infty d\xi \int_0^\infty dx \int_0^\infty dc \int_0^\infty d\gamma K(x, \xi) n(\xi, \gamma, t)n(x, c, t)f(\xi, c).
\]

We substitute this expression into (4.9) and, upon renaming \( x', c' \) in (4.8) by \( x, c \), obtain

\[
I_2 = I_{21} - I_{22} = \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma K(x, \xi)n(x, c, t)n(\xi, \gamma, t)
\times \left[ f \left( x + \xi, \frac{xc + \xi \gamma}{x + \xi} \right) - f(x, c) - f(\xi, \gamma) \right].
\]

Next we write in \( I_2 \)

\[
\cdots \int_0^\infty dc \int_0^\infty d\gamma \text{ as } \cdots \int_0^\infty d\gamma \int_0^\infty dc
\]

and change notation \( c \leftrightarrow \gamma, x \leftrightarrow \xi \). We find that \( I_2 \) is equal to a similar integral in which

\[
\int_0^\infty dc \int_0^\infty d\gamma \text{ is replaced by } \int_0^\infty dc \int_0^\infty d\gamma
\]

(here we again use the symmetry of \( K(x, \xi) \)). It follows that \( I_2 \) is equal to \( \frac{1}{2} \) times the first integral on the right-hand side of (4.3). Recalling also (4.6), (4.5), the assertion (4.3) follows.

Formula (4.3) will play a key role in the subsequent sections. Note that the choice \( f = x \) gives the conservation law (3.2) which was already proved. On the other hand the choice \( f = xc \) gives a new conservation law:

\[
(4.10) \quad \int_0^\infty \int_0^\infty xc_n(x, c, t)dx dc = \int_0^\infty \int_0^\infty xcn_0(x, c)dx dc \text{ for all } t \geq 0.
\]

**Corollary 4.2.** If \( g(c) \) is a convex function then

\[
\frac{d}{dt} \int_0^\infty \int_0^\infty g(c)xn(x, c, t)dx dc \leq 0.
\]

Indeed, the second integral on the right-hand side of (4.3) is zero whereas the first integral is \( \leq 0 \)

**Corollary 4.3.** If \( n_0(x, c) = 0 \) for \( 0 \leq c \leq \bar{c} \), \( x \geq 0 \) then \( n(x, c, t) = 0 \) for \( 0 \leq c \leq \bar{c} \), \( x \geq 0 \), \( t \geq 0 \).

Indeed, taking \( g(c) \) to be the convex function \((\bar{c} - c)^+\) we conclude that

\[
\int_0^\infty \int_0^\infty (\bar{c} - c)^+ xn(x, c, t)dx dc \leq \int_0^\infty \int_0^\infty (\bar{c} - c)^+ xcn_0(x, c)dx dc = 0
\]
and the assertion follows.

The proof of Theorem 4.1 can also be applied (or, rather, specialized) to functions of the form

\[(4.11) \quad \tilde{S}(t) = \int_0^\infty f(x)n(x,t)dx.\]

We then get:

**Theorem 4.4.** If (1.1), (3.7) hold and \( f(x) \) satisfies

\[(4.12) \quad |f(x)| \leq C(1 + x),\]

then

\[(4.13) \quad \frac{d\tilde{S}(t)}{dt} = \frac{1}{2} \int_0^\infty \int_0^\infty K(x,\xi)n(x,t)n(\xi,t)[f(x + \xi) - f(x) - f(\xi)]\,dx\,d\xi \]

\[+ \int_0^\infty dx \int_0^\infty B(\xi, x)n(\xi,t)[f(x) - \frac{x}{\xi}f(\xi)]\,d\xi.\]

**5. \( D(t) \to 0 \) if \( t \to \infty. \)** In this section we impose additional mild assumptions on \( K, B \) and \( n_0 \):

\[(5.1) \quad \int_0^\infty x^2n_0(x)dx < \infty\]

and

(a) \( K(x,\xi) \leq C(x + \xi); \)
(b) \( \partial_\xi K(x,\xi) \geq -C; \)
(c) \( \exists \) a positive constant \( \gamma_1 \) such that, for any \( N \gg 1, \)

\[(5.2) \quad K(x,\xi) \geq \frac{\gamma_1}{N}(x + \xi) \text{ if } 0 \leq x, \xi \leq N; \]

(d) \( \int_0^\xi B(\xi, x)x(\xi - x)dx \geq \gamma_2 \xi^2 \quad \forall \xi > 0, \text{ for some } \gamma_2 > 0. \)

Remark 1.1 indicates that condition (5.2)(d) is quite natural. Note also that (5.2)(c) is satisfied if

\[K(x,\xi) \geq \begin{cases} \delta(x + \xi) & \text{for } x + \xi < 1 \\ \delta & \text{for } x + \xi > 1 \end{cases}\]

where \( \delta \) is a small positive constant.

Set

\[(5.3) \quad M_0 = \int_0^\infty \int_0^\infty xn_0(x,c)dxdc, \quad M_1 = \int_0^\infty \int_0^\infty xcn_0(x,c)dxdc, \]

\[(5.4) \quad c_\infty = \frac{M_1}{M_0}. \]
In this section we prove:

**Theorem 5.1.** If (1.1), (3.7) and (5.1), (5.2) hold, then the function

\[ D(t) \equiv \int_0^\infty \int_0^\infty x(c - c_\infty)^2 n(x, c, t) dx dc \]

converges to zero as \( t \to \infty \).

We need several lemmas.

**Lemma 5.2.** There exists a constant \( C \) independent of \( t \) such that

\[ \int_0^\infty x^2 n(x, t) dx \leq C \quad \forall t > 0. \]

**Proof.** Consider first the case where

\[ K, B \text{ and } n_0 \text{ have compact support.} \]

Then, for any \( T > 0 \),

\[ \tilde{S}(t) \equiv \int_0^\infty x^2 n(x, t) dx \leq C(T) < \infty \quad \text{if} \quad 0 \leq t \leq T, \]

and we may apply (4.13) with \( f(x) = x^2 \) (The proof is the same as that of Theorem 4.4.). We get

\[ \frac{d\tilde{S}}{dt} \leq C \int_0^\infty \int_0^\infty n(x, t) n(\xi, t) x\xi dx d\xi \]

\[ + \int_0^\infty n(\xi, t) d\xi \int_0^\xi B(\xi, x) (x^2 - x\xi) dx, \]

since \( K(x, \xi) \leq C \). Using Theorem 3.1 and (5.2) (d) we obtain

\[ \frac{d\tilde{S}}{dt} \leq C - \gamma_2 \int_0^\infty \xi^2 n(\xi, t) d\xi = C - \gamma_2 \tilde{S}(t). \]

Consequently

\[ \tilde{S}(t) \leq \tilde{S}(0) e^{-\gamma_2 t} + \frac{C}{\gamma_2}. \]

and (5.6) follows with a constant which is independent of the assumptions in (5.7). Approximating general \( K, B, n_0 \) by functions with compact support and applying (5.8) to the corresponding solutions, we obtain, in the limit, the assertion (5.6).

Introduce the function

\[ S_2(t) = \int_0^\infty \int_0^\infty x^2 n(x, c, t) dx dc. \]
Then
\[(5.10) \quad D(t) = S_2(t) - 2c_\infty M_1 + C_\infty^2 M_0.\]

**Lemma 5.3.** The function \(S_2(t)\) satisfies:
\[(5.11) \quad \frac{dS_2(t)}{dt} \leq 0,\]
\[(5.12) \quad \frac{d^2S_2(t)}{dt^2} \leq C\]

for all \(t > 0\), and
\[(5.13) \quad \frac{dS_2(t)}{dt} \to 0 \quad \text{if} \quad t \to \infty.\]

**Proof.** Using (4.3) with \(f(x, c) = xc^2\) and noting that
\[(5.14) \quad f \left( x + \xi, \frac{xc + \xi \gamma}{x + \xi} \right) - f(x, c) - f(\xi, \gamma) = -\frac{x \xi}{x + \xi}(c - \gamma)^2,\]
we get
\[(5.15) \quad \frac{dS_2}{dt} = -\frac{1}{2} \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma \frac{K(x, \xi)}{x + \xi} x \xi (c - \gamma)^2 n(x, c, t)n(\xi, \gamma, t),\]
so that \(dS_2/dt \leq 0\).

To prove (5.12) we want to apply Theorem 4.1 with
\[(5.16) \quad f(x, c) = \frac{K(x, \xi)}{x + \xi} x(c - \gamma)^2; \quad \xi, \gamma \text{ are parameters.}\]

for simplicity we first take \(B = 0\). Then
\[
\frac{d^2S_2}{dt^2} = -\frac{1}{2} \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma \xi n(\xi, \gamma, t) \int_0^\infty d\zeta \int_0^\infty d\sigma K(x, \xi)n(x, c, t)n(\xi, \sigma, t)
\times \left[ \frac{K(x + \zeta, \xi)}{x + \zeta + \xi}(x + \zeta) \left( \frac{xc + \zeta \sigma}{x + \zeta} - \gamma \right)^2 - \frac{K(x, \xi)}{x + \xi} x(c - \gamma)^2 \right.
\quad - \frac{K(\xi, \xi)}{\zeta + \xi} \zeta(\sigma - \gamma)^2 \right].
\]

Using
\[
\frac{xc + \zeta \sigma}{x + \zeta} - \gamma = \frac{x(c - \gamma) + \zeta(\sigma - \gamma)}{x + \zeta}
\]
and a calculation similar to (5.14), we find that the expression in brackets is equal to
\[
\frac{K(x + \zeta, \xi)}{x + \zeta + \xi} \left( \frac{-x \xi(c - \sigma)^2}{x + \xi} \right) + \left( \frac{K(x + \zeta, \xi)}{x + \zeta + \xi} - \frac{K(x, \xi)}{x + \xi} \right) x(c - \gamma)^2
\quad + \left( \frac{K(x + \zeta, \xi)}{x + \zeta + \xi} - \frac{K(\xi, \xi)}{\zeta + \xi} \right) \zeta(\sigma - \gamma)^2.
\]
By (5.2)(a), $K(x, \zeta)$ times the first term is bounded by $cx\zeta$, and, by (5.2)(a) and (5.2)(b), $K(x, \zeta)$ times each of the remaining two terms is bounded from below by $-Cx\zeta$.

It follows that $d^2S_2/dt^2$ is bounded from above by

$$C \left( \int_0^\infty \int_0^\infty x n(x, c, t) dx dt \right)^3 \leq \text{const.}$$

If $B \neq 0$ then we have to estimate the corresponding positive contribution to $d^2S_2/dt^2$:

$$\frac{1}{2} \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma \xi n(\xi, \gamma, t) f(x, c) n(x, c, t) \int_0^\xi x B(x, \zeta) d\zeta \tag{5.17}$$

where $f(x, c)$ is defined as in (5.16). Since $K(x, \xi) \leq C(x + \xi)$, we have $f(x) \leq Cx$. Recalling also the last condition in (1.1) we find that the expression in (5.17) is bounded by

$$C \left( \int_0^\infty \int_0^\infty x n(x, c, t) dx dc \right)^2 \leq \text{const.}$$

This completes the proof of (5.12).

It remains to prove (5.13). If the assertion is not true then there is a sequence $t_n \to \infty$ such that

$$0 \geq S'_2(t_n) \to -\alpha < 0 \text{ if } n \to \infty.$$ 

Using (5.12) we get

$$S_2(t_n + \epsilon) - S_2(t_n) = \epsilon S'_2(t_n) + \frac{\epsilon^2}{2} S''_2(t_n) \leq -\frac{\alpha \epsilon}{2}$$

if $\epsilon$ is positive and small enough. It follows that we have $f(x, c) \leq Cx$.

$$S_2(t_n + \epsilon) \leq S_2(t_1) - \frac{n \epsilon}{2} < 0$$

if $n$ is sufficiently large, a contradiction.

For any $N \gg 1$ introduces the quantities

$$M_{0,N} = \int_0^N dx \int_0^\infty x n(x, c, t) dc,$$

$$M_{1,N}(x) = \int_0^N dx \int_0^\infty x^2 n(x, c, t) dc$$

and

$$S_{2,N}(t) = \int_0^n dx \int_0^\infty c^2 x n(x, c, t) dc.$$
By Lemma 5.2,

\begin{equation}
|S_2(t) - S_{2,N}(t)| \leq \frac{C}{N} \quad \text{for all } t \geq 0,
\end{equation}

(5.18)

\begin{align*}
|M_{0,N} - M_0| + |M_{1,N} - M_1| & \leq \frac{C}{N}.
\end{align*}

**Lemma 5.4.** There holds:

\begin{equation}
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\xi d\eta (c - \gamma)^2 n(x, c, t)n(\xi, \gamma, t) = 2 \left( S(t)M_0 - M_1^2 \right)
\end{equation}

(5.19)

and, in particular,

\begin{equation}
S(t)M_0 \geq M_1^2.
\end{equation}

(5.20)

The proof is obvious.

**Proof of Theorem 5.1.** by (5.15)

\[ \frac{dS_2(t)}{dt} = -\frac{1}{2} \int_{x<N,\xi<N} dx dc d\gamma \frac{K(x,\xi)}{x+\xi} x\xi(c-\gamma)^2 n(x,c,t)n(\xi,\gamma,t) + J_N \]

where, since \( K(x,\xi) \leq C \),

\[ |J_N| \leq C \int_{\xi>N} dx dc d\gamma x\xi(n(x,c,t)n(\xi,\gamma,t) \leq \frac{C}{N^2}; \]

the last inequality follows by Lemma 5.2. Using (5.2)(c) we conclude that

\[ \frac{dS_2}{dt} \leq -\frac{\gamma_1}{N} \int_{x<N,\xi<N} dx dc d\gamma x\xi(c-\gamma)^2 n(x,c,t)n(\xi,\gamma,t) + \frac{C}{N^2}. \]

Evaluating the last integral as in (5.19), we get

\[ \frac{dS_2}{dt} \leq -\frac{2\gamma_1}{N} (S_{2,N}(t)M_{0,N} - M_{1,N}^2) + \frac{C}{N^2} \]

so that

\[ S_{2,N}(t)M_{0,N} - M_{1,N}^2 \leq -\frac{N}{2\gamma_1} S'_2(t) + \frac{C}{N}. \]

Using (5.18) we deduce that

\[ S_2(t)M_0 - M_1^2 \leq -\frac{N}{2\gamma_1} S'_2(t) + \frac{C}{N}. \]

Taking \( t \to \infty \) and using (5.13) we find that

\[ \lim_{t \to \infty} (S(t)M_0 - M_1^2) \leq \frac{C}{N}, \]
Taking $N \to \infty$ and recalling (5.20), we find that

$$S(t)M_0 - M_1^2 \to 0 \quad \text{if} \quad t \to \infty.$$ 

Consequently, as $t \to \infty$,

$$D(t) \to \frac{M_1^2}{M_0} - 2C_\infty M_1 + C_\infty^2 M_0 = 0, \quad \text{by (5.10) and (5.4)},$$

and the proof of Theorem 5.1 is complete.

Remark 5.5. Dubovskii and Stewart [4] proved existence and uniqueness for (1.2), (1.3) also in case $K(x, \xi)$ is unbounded, provided

$$K(x, \xi) \leq C(x + \xi),$$

$$n_0(x) \leq Ce^{-\lambda x} \quad (\lambda > 0);$$

The solution is exponentially decreasing, and uniqueness is in the class of exponentially decreasing functions. Their result can be extended to (1.4), (1.5). Suppose now that $K$ also satisfies:

$$K(x, \xi) \geq \gamma_3(x + \xi), \quad \gamma_3 > 0.$$ 

Then the proof of Theorem 5.1 remains valid without actually splitting integrals by $x \geq N$ or $\xi \geq N$. Thus we get, in this case,

$$\frac{dS_2}{dt} \leq -\gamma_3 \left(S_2(t)M_0 - M_1^2\right).$$

The non-negative function

$$T(t) = S_2(t) - \frac{M_1^2}{M_0}$$

then satisfies

$$\frac{dT}{dt} + 2\gamma_3 T \leq 0$$

so that

$$0 \leq T(t) \leq T(0)e^{-2\gamma_3 t}.$$ 

We conclude that, as $t \to \infty$,

$$D(t) \to 0 \text{ exponentially fast.}$$

Note that in this case we do not require the assumption (5.2)(d).
6. Asymptotic behavior of $xn(x, c, t)dx dc$. In this section we add one more assumption on $K, B, n_0$:

\begin{equation}
K(x, \xi) \leq \eta(x + \xi) \text{ where } 2M_0\eta < \gamma_2, \quad \gamma_2 \text{ as in } (5.2)(d).
\end{equation}

**Theorem 6.1.** Let (1.1), (3.7), (5.1), (5.2) and (6.1) hold. Then there exists a non-negative measure $dL(x)$ such that

\begin{equation}
\lim_{t \to \infty} \int_0^\infty \int_0^\infty g(x, c)xn(x, c, t)dx dc = \int_0^\infty g(x, c_\infty)dL(x)
\end{equation}

for any bounded continuous function $g(x)$.

Introducing the measures

\[ d\mu_t(x, c) = xn(x, c, t)dx dc \]

we can restate (6.2) in the form:

\begin{equation}
d\mu_t(x, c) \to \delta(c - c_\infty)dL(x) \text{ as } t \to \infty, \text{ in the sense of weak convergence,}
\end{equation}

where $\delta(y)$ is the Dirac measure.

The proof requires several lemmas.

**Lemma 6.2.** Under the assumptions of Theorem 6.1.

\begin{equation}
\lim_{t \to \infty} \int_0^\infty g(x)xn(x, t)dx \text{ exists}
\end{equation}

for any bounded continuous function $g(x)$.

**Proof.** Consider first the function

\[ S_0(t) = \int_0^\infty x^2n(x, t)dx \]

By (4.13)

\[
\frac{dS_0}{dt} = \int_0^\infty \int_0^\infty K(x, \xi)n(x, t)n(\xi, t)x\xi dx d\xi
+ \int_0^\infty dx \int_0^\infty B(\xi, x)n(\xi, t)(x^2 - x\xi) d\xi \\
\leq \eta \int_0^\infty \int_0^\infty (x + \xi)n(x, t)n(\xi, t)x\xi dx d\xi
+ \int_0^\infty n(\xi, t)d\xi \int_0^\infty B(\xi, x)(x^2 - x\xi) dx \\
\leq 2\eta S_0(t)M_0 - \gamma_2 S_0(t)
\]
so that

\[ \frac{dS_0}{dt} \leq -\delta S_0(t), \quad \delta = \gamma_2 - 2\eta M_0 > 0. \]

Suppose now that \( g(x) \) is a uniformly Lipschitz continuous function and write

\[ \int_0^\infty g(x) x n(x,t) dx = \int_0^\infty (g(x)x + Nx^2)n(x,t) dx \]

(6.6)

\[ -N \int_0^\infty x^2 n(x,t) dx \equiv \hat{S}_1(t) - NS_0(t). \]

By (6.5), \( S_0(t) \) is monotone decreasing, and hence

(6.7)

\[ \lim_{t \to \infty} S_0(t) \text{ exists.} \]

Next, by Theorem 4.4 with \( f(x) = xg(x) \),

\[ \frac{d}{dt} \int_0^\infty g(x) x n(x,t) dx \]

\[ \leq \frac{1}{2} \int_0^\infty dx \int_0^\infty d\xi \eta(x + \xi)n(x,t) n(\xi,t)x \xi \]

\[ \times \left[ \frac{g(x + \xi) - g(x)}{\xi} + \frac{g(x + \xi) - g(\xi)}{x} \right] \]

\[ + \int_0^\infty dx \int_0^\infty B(\xi, x)n(\xi,t)x[g(x) - g(\xi)] d\xi. \]

The first integral on the right-hand side is bounded by

\[ C \int_0^\infty x^2 n(x,t) dx \cdot \int_0^\infty \xi n(\xi,t) d\xi \]

and the second integral is bounded by

\[ \int_0^\infty n(\xi,t) d\xi \int_0^\infty B(\xi, x)xC|x - \xi| dx \leq C \int_0^\infty \xi^2 n(\xi,t) d\xi. \]

Recalling (6.5), it follows that

\[ \frac{d\hat{S}_1}{dt} \leq -\delta NS_0(t) + C_0 S_0(t) < 0 \quad \text{if } N > \frac{C_0}{\delta}, \]

and consequently

\[ \lim_{t \to \infty} \hat{S}_1(t) \text{ exists.} \]
Combining this with (6.7) we deduce from (6.6) that (6.4) holds for any uniformly Lipschitz continuous function $g(x)$.

Consider next the case where $g(x)$ is any bounded continuous function and let $g_\epsilon$ be a mollifier of $g$:

$$|g_\epsilon(x) - g(x)| < \epsilon \text{ and } |g_\epsilon'(x)| \leq \frac{C}{\epsilon} \text{ for all } x \geq 0.$$

Write

$$\int_0^\infty g(x)xn(x,t)dx = \int_0^\infty g_\epsilon(x)xn(x,t)dx + \int_0^\infty (g(x) - g_\epsilon(x))xn(x,t)dx \equiv T_1(t) + T_2(t).$$

Clearly

$$|T_2(t)| < \epsilon.$$

By what was already proved above

$$A_\epsilon \equiv \lim_{t \to \infty} T_1(t) \text{ exists.}$$

Hence

$$\lim_{t \to \infty} \left| \int_0^\infty g(x)xn(x,t)dx - A_\epsilon \right| \leq \epsilon.$$

Since the $A_\epsilon$ are uniformly bounded, we can take a sequence $\epsilon = \epsilon_m \to 0$ such that $A_{\epsilon_m} \to A$, and obtain

$$\lim_{t \to \infty} \left| \int_0^\infty g(x)xn(x,t)dx - A \right| = 0.$$

This completes the proof of Lemma 6.2.

**Lemma 6.3.** Under the assumptions of Theorem 6.1,

$$\lim_{t \to \infty} \int_0^\infty \int_0^\infty g(x,c)xn(x,c,t)dx dc \text{ exists}$$

for any bounded continuous function $g(x,c)$.

The proof is similar to the proof of Lemma 6.2 and will be omitted. Note that here we work (cf.(6.6)) with

$$(g(x,c) + Nx^2) \text{ and } -Nx^2.$$ 

The limit in (6.8) is a bounded linear functional $\Phi$ on the space of bounded continuous functions $g$ equipped with the $L^\infty$ norm. By [6, pp. 261–262] we can write

$$\lim_{t \to \infty} \int_0^\infty \int_0^\infty g(x,c)xn(x,c,t)dx dc$$

$$= \int \int g(x,c)dN(x,c) \equiv \Phi(g)$$
where $dN$ is a bounded finitely additive regular set function. It will be also useful to view the 
right-hand side of (6.9) as a distribution, and use the properties

\begin{equation}
(6.10) \quad |\Phi(g)| \leq C|g|_{L^\infty}, \quad \Phi(g) \geq 0 \text{ if } g \geq 0.
\end{equation}

From Theorem 5.1 it follows that
\[
\int \int (c - c_\infty)^2 dN(x, c) = 0
\]
and therefore also
\begin{equation}
(6.11) \quad \int \int g(x, c)(c - c_\infty)^2 dN(x, c) = 0
\end{equation}
for any bounded continuous function $g(x, c)$.

If $g(x, c)$ vanishes in a neighborhood of the line $\{c = c_\infty\}$ then we can apply (6.11) to the function

\[\tilde{g}(x, c) = \begin{cases} 
g(x, c) & \text{on supp} \ g \\
(c - c_\infty)^2 & \text{elsewhere} \end{cases}\]

and conclude that

\[\Phi(g) = \int \int \tilde{g}(x, c)(c - c_\infty)^2 dN(x, c) = 0.\]

It follows that the support of the distribution $\Phi$ lies in the set $\{c = c_\infty\}$.

By a standard result in the theory of distributions (e.g. [7, p. 69, Theorem 31]), $\Phi$ must then have the form

\[\Phi(g) = \sum_{0 \leq i, j \leq l} \frac{\partial^{i+j}}{\partial c^i \partial x^j} \int g(x, c_\infty) dk_{ij}(x)\]

where the derivatives $\partial^{i+j}/\partial c^i \partial x^j$ are taken in the sense of distributions, and $dk_{ij}$ are measures.

In view of (6.10) we must have $l = 0$ and, furthermore, $dk_{00}$ is a non-negative measure. This completes the proof of Theorem 6.1.

The equilibrium equation (1.1) is

\begin{equation}
(6.12) \quad \frac{1}{2} \int_0^\infty K(x - \xi, \xi) n_\infty(x - \xi)n_\infty(\xi) d\xi + \int_0^\infty B(\xi, x)n_\infty(\xi) d\xi
\end{equation}

\[-n_\infty(x) \int_0^\infty K(x, \xi)n_\infty(\xi) d\xi - n_\infty(x) \int_0^\infty B(x, \xi) d\xi = 0.\]

If $n_\infty(x)$ is a bounded, non-negative, $L^1(0, \infty)$-solution of (6.12), then the measure $dL(x) = xn_\infty(x) dx$ satisfies:

\begin{equation}
(6.13) \quad \frac{1}{2} \int_0^\infty dL(\xi) \int_0^\infty dL(x) K(x, \xi) \left[ \frac{g(x + \xi) - g(x)}{\xi} + \frac{g(x + \xi) - g(\xi)}{x} \right] 
\end{equation}

\[+ \int_0^\infty dL(\xi) \int_0^\infty B(\xi, x) \frac{x}{\xi}(g(x) - g(\xi)) dx = 0\]
for any Lipschitz function $g$. Indeed, this follows from Theorem 4.4 applied to $f(x) = xg(x)$.

**Definition.** A bounded non-negative measure $dL(x)$ is called a weak solution of (6.12) if (6.13) holds for any continuously differentiable function $g(x)$ with uniformly bounded derivative.

**Theorem 6.4.** Under the assumptions of Theorem 6.1 the measure $dL(x)$ is a weak solution of (6.12).

**Proof.** We first establish (6.13) in case

(6.14) \[ |g'(x)| + |g''(x)| \leq C, \quad g(x) \text{ monotone increasing.} \]

The starting point is the assertion that

(6.15) \[ \frac{d}{dt} \int_0^\infty xg(x)n(x,t) \to 0 \text{ if } t \to \infty. \]

To prove it we set

\[ \tilde{S}_2(t) = \int_0^\infty xg(x)n(x,t)dx \]

and proceed as in the proof of Lemma 5.3. By Lemma 6.2

(6.16) \[ \lim_{t \to \infty} \tilde{S}_2(t) \text{ exists.} \]

We next claim that

(6.17) \[ \frac{d^2 \tilde{S}_2(t)}{dt^2} \leq C. \]

To prove it we use Theorem 4.4 with $f(x) = xg(x)$ to get:

(6.18) \[ \frac{d \tilde{S}_2(t)}{dt} = \frac{1}{2} \int_0^\infty dx \int_0^\infty d\xi K(x,\xi) x\xi n(x,t)n(\xi,t) \]

\[ \times \left[ \frac{g(x+\xi) - g(x)}{\xi} + \frac{g(x+\xi) - g(\xi)}{x} \right] \]

\[ + \int_0^\infty dx \int_0^\infty B(\xi, x)n(\xi,t)x(g(x) - g(\xi))d\xi \equiv J_1(t) + J_2(t), \]

and continue to differentiate once more in $t$. Consider first the case $B = 0$. Then we can proceed similarly to the proof of (5.13) with $f(x, c)/x$ in (5.16) replaced by

\[ f(x) = K(x, \xi) \left[ \frac{g(x+\xi) - g(x)}{\xi} + \frac{g(x+\xi) - g(\xi)}{x} \right] \]

where $\xi$ is viewed as a parameter. Using (5.2)(b) and the assumptions on $g$ in (6.14) we easily deduce that

\[ \frac{d^2 \tilde{S}_2}{dt^2} \leq C. \]
If \( B(x, \xi) \neq 0 \) then the additional terms in \( d^2 \tilde{S}_2 / dt^2 \) are also bounded, as can easily be shown.

Having proved (6.16), (6.17), we can now quickly prove (6.15). Indeed, if (6.15) is not true then there is a sequence \( t_n \rightarrow \infty \) such that

\[
\beta_n \equiv \frac{d\tilde{S}_2(t_n)}{dt} \rightarrow \beta \neq 0 \quad \text{if} \quad n \rightarrow \infty.
\]

It follows that

\[
\tilde{S}_2(t_n - \beta_n \varepsilon) - \tilde{S}_2(t_n) = -\beta_n^2 \varepsilon + \frac{1}{2} \beta_n^2 \varepsilon^2 \tilde{S}_2''(\tilde{t}_n) \\
\leq -\beta_n^2 \varepsilon + C\varepsilon^2 < -\frac{1}{2} \beta^2 \varepsilon
\]

if \( \varepsilon \) is positive and small and \( n \) is large; this is a contradiction to (6.16).

So far we have proved that the left-hand side of (6.18) converges to zero as \( t \rightarrow \infty \). Next we evaluate the right-hand side of (6.18) as \( t \rightarrow \infty \). By Theorem 6.1

\[
J_2(t) = \int_0^\infty \xi n(\xi, t) d\xi \left[ \frac{1}{\xi} \int_0^\xi B(\xi, x) x(g(x) - g(\xi)) dx \right] \\
\rightarrow \int_0^\infty dL(\xi) \int_0^\xi B(\xi, x) \frac{1}{\xi} (g(x) - g(\xi)) dx.
\]

(6.19)

To evaluate \( J_1(t) \) introduce the functions

\[
F(x, t) = \int_0^\infty d\xi K(x, \xi) \xi n(\xi, t) \frac{g(x + \xi) - g(x)}{\xi},
\]

\[
F_\infty(x) = \int_0^\infty dL(\xi) K(x, \xi) \frac{g(x + \xi) - g(x)}{\xi}.
\]

The family of functions \( \{F(x, t), t > 0\} \) is uniformly bounded and equicontinuous, and, by Theorem 6.1,

\[
F(x, t) \rightarrow F_\infty(x) \quad \text{if} \quad t \rightarrow \infty.
\]

It follows that the convergence is uniform in any boundary interval \( 0 \leq x \leq R \). Using also Lemma 5.2, we easily deduce that

\[
\int_0^\infty x n(x, t) [F(x, t) - F_\infty(x)] dx \rightarrow 0 \quad \text{if} \quad n \rightarrow \infty.
\]

Since, by Theorem 6.1,

\[
\int_0^\infty dx n(x, t) F_\infty(x) \rightarrow \int_0^\infty dL(x) F_\infty(x)
\]

\[
25
\]
we conclude that
\[ \int_0^\infty xn(x,t)F(x,t)dx \to \int_0^\infty dL(x)F_\infty(x) \]
if \( t \to \infty \). Combining this result with (6.19), we deduce that the right-hand side of (6.18) converges to the left-hand side of (6.13), and since the left-hand side of (6.18) converges to zero, the equality (6.13) follows.

So far we have assumed that \( g \) satisfies (6.14). However by approximation we can establish (6.13) for any monotone increasing \( g \) with first bounded and continuous derivative. Finally, if \( g \) is not monotone increasing, we can decompose it into a difference of two monotone functions, and this completes the proof of Theorem 6.4.

Remark 6.1. If \( n_0(x) \) is a stationary solution then \( n(x,t) = n_0(x) \) and so
\[ \int_0^\infty n(x,c,t)dc = n_0(x). \]
It follows that
\[ \int_0^\infty \int_0^\infty g(x)xn(x,c,t)dx dc = \int_0^\infty g(x)xn_0(x)dx \]
and
\[ xn(x,c,t)dx dc \to \delta(c-c_\infty)(xn_0(x)dx). \]

Remark 6.2. It was observed by Carr [2] that if \( K \) and \( B \) are related by
\[ K(x,y)Q(x)Q(y) = B(x+y,y)Q(x+y) \]
where \( Q \) is a positive function, then
\[ n_\infty(x) = e^{xQ(x)} \]
is an equilibrium solution. Dubovskii and Stewart [5] proved that if \( K(x,y) \) and \( B(x,y) \) are linear functions then \( n(x,t) \) converges to an equilibrium solution \( n_\infty(x) \); their methods are entirely different from the methods of the present paper.

Acknowledgement. (1) We would like to thank David Ross from Eastman Kodak for suggesting the problems studied in this paper.

(2) The first author is partially supported by the National Science Foundation Grant DMS #94-01251. The second author is partially supported by AFOSR through contract number F49620-95-1-0113 and by NSF through grant number DMS-9622555.
Acknowledgement and Disclaimer. Effort sponsored by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant no. F49620-95-1. The US Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation thereby. The view and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or interpreted, of the Air Force Office of Scientific Research or the US Government.

REFERENCES


27
<table>
<thead>
<tr>
<th>#</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1398</td>
<td>B. Brighi &amp; M. Chipot</td>
<td>Approximation of infima in the calculus of variations</td>
</tr>
<tr>
<td>1399</td>
<td>G. Avalos</td>
<td>Concerning the well-posedness of a nonlinearly coupled semilinear wave and beam-like equation</td>
</tr>
<tr>
<td>1400</td>
<td>R. Lipton</td>
<td>Variational methods, bounds and size effects for composites with highly conducting interface</td>
</tr>
<tr>
<td>1401</td>
<td>B.T. Hayes &amp; P.G. LeFloch</td>
<td>Non-classical shock waves in scalar conservation laws</td>
</tr>
<tr>
<td>1402</td>
<td>K.T. Joseph &amp; P.G. LeFloch</td>
<td>Boundary layers in weak solutions to hyperbolic conservation laws</td>
</tr>
<tr>
<td>1403</td>
<td>Y. Diao, C. Ernst, &amp; E.J.J. Van Rensburg</td>
<td>Energies of knots</td>
</tr>
<tr>
<td>1404</td>
<td>Xiaofeng Ren</td>
<td>Multi-layer local minimum solutions of the bistable equation in an infinite tube</td>
</tr>
<tr>
<td>1405</td>
<td>Vlastimil Pták</td>
<td>Krylov sequences and orthogonal polynomials</td>
</tr>
<tr>
<td>1406</td>
<td>T. Aktosun, M. Klaus, &amp; C. van der Mee</td>
<td>Factorization of scattering matrices due to partitioning of potentials in one-dimensional Schrödinger-type equations</td>
</tr>
<tr>
<td>1407</td>
<td>C.-S. Man &amp; R. Paroni</td>
<td>On the separation of stress-induced and texture-induced birefringence in acoustoelasticity</td>
</tr>
<tr>
<td>1408</td>
<td>D.N. Arnold, R.S. Falk, &amp; R. Winther</td>
<td>Preconditioning discrete approximations of the Reissner-Mindlin plate model</td>
</tr>
<tr>
<td>1409</td>
<td>M.A. Kouritzin</td>
<td>On exact filters for continuous signals with discrete observations</td>
</tr>
<tr>
<td>1410</td>
<td>R. Lipton</td>
<td>The second Stekloff eigenvalue and energy dissipation inequalities for functionals with surface energy</td>
</tr>
<tr>
<td>1411</td>
<td>R. Lipton</td>
<td>The second Stekloff eigenvalue of an inclusion and new size effects for composites with imperfect interface</td>
</tr>
<tr>
<td>1412</td>
<td>W. Littman &amp; B. Liu</td>
<td>The regularity and singularity of solutions of certain elliptic problems on polygonal domains</td>
</tr>
<tr>
<td>1413</td>
<td>C.R. Collins</td>
<td>Spurious oscillations are not fatal in computing microstructures</td>
</tr>
<tr>
<td>1414</td>
<td>M.A. Horn</td>
<td>Sharp trace regularity for the solutions of the equations of dynamic elasticity</td>
</tr>
<tr>
<td>1415</td>
<td>A. Friedman, B. Hu &amp; Y. Liu</td>
<td>A boundary value problem for the Poisson equation with multi-scale oscillating boundary</td>
</tr>
<tr>
<td>1416</td>
<td>P. Bauman, D. Phillips &amp; Q. Tang</td>
<td>Stable nucleation for the Ginzburg-Landau system with an applied magnetic field</td>
</tr>
<tr>
<td>1417</td>
<td>J.M. Berg</td>
<td>A strain profile for robust control of microstructure using dynamic recrystallization</td>
</tr>
<tr>
<td>1418</td>
<td>P. Klouček</td>
<td>Toward the computational modeling of nonequilibrium thermodynamics of the Martensitic transformations</td>
</tr>
<tr>
<td>1419</td>
<td>S. Chawla &amp; S.M. Lenhart</td>
<td>Application of optimal control theory to in Situ bioremediation</td>
</tr>
<tr>
<td>1420</td>
<td>B. Li &amp; M. Luskin</td>
<td>Nonconforming finite element approximation of crystalline microstructure</td>
</tr>
<tr>
<td>1421</td>
<td>H. Kang &amp; J.K. Seo</td>
<td>Inverse conductivity problem with one measurement: Uniqueness of balls in $\mathbb{R}^3$</td>
</tr>
<tr>
<td>1422</td>
<td>Avner Friedman &amp; Robert Gulliver</td>
<td>Organizers, Mathematical modeling for instructors, July 29 – August 16, 1996</td>
</tr>
<tr>
<td>1423</td>
<td>G. Friesecke</td>
<td>Pair correlations and exchange phenomena in the free electron gas</td>
</tr>
<tr>
<td>1424</td>
<td>Y.A. Li &amp; P.J. Olver</td>
<td>Convergence of solitary-wave solutions in a perturbed Bi-Hamiltonian dynamical system</td>
</tr>
<tr>
<td>1425</td>
<td>C. Huang</td>
<td>On boundary regularity of vortex patches for 3D incompressible euler systems</td>
</tr>
<tr>
<td>1426</td>
<td>C. Huang</td>
<td>A free boundary problem with nonlinear jump and kinetics on the free boundaries</td>
</tr>
<tr>
<td>1427</td>
<td>X. Chen, C. Huang &amp; J. Zhao</td>
<td>A nonlinear parabolic equation modeling surfactant diffusion</td>
</tr>
<tr>
<td>1428</td>
<td>A. Friedman &amp; B. Hu</td>
<td>Optimal control of chemical vapor deposition reactor</td>
</tr>
<tr>
<td>1429</td>
<td>A. Friedman &amp; B. Hu</td>
<td>A non-stationary multi-scale oscillating free boundary for the Laplace and heat equations</td>
</tr>
<tr>
<td>1430</td>
<td>X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations</td>
<td></td>
</tr>
<tr>
<td>1431</td>
<td>J. Yong</td>
<td>Finding adapted solutions of forward-backward stochastic differential equations – Methods of continuation</td>
</tr>
<tr>
<td>1432</td>
<td>J. Yong</td>
<td>Linear forward-backward stochastic differential equations</td>
</tr>
<tr>
<td>1433</td>
<td>D.A. Dawson &amp; M.A. Kouritzin</td>
<td>Invariance principles for parabolic equations with random coefficients</td>
</tr>
<tr>
<td>1434</td>
<td>R. Lipton</td>
<td>Energy minimizing configurations for mixtures of two imperfectly bonded conductors</td>
</tr>
<tr>
<td>1435</td>
<td>D.C. Dobson &amp; F. Santos</td>
<td>Nondestructive evaluation of plates using Eddy current methods</td>
</tr>
<tr>
<td>1436</td>
<td>W. Littman &amp; B. Liu</td>
<td>On the spectral properties and stabilization of acoustic flow</td>
</tr>
<tr>
<td>1437</td>
<td>S. Sarkar &amp; S. Sundar Sarkar</td>
<td>Normal distribution as a method for data replication in a parallel data server</td>
</tr>
<tr>
<td>1438</td>
<td>S. Sarkar &amp; S. Sundar Sarkar</td>
<td>Parallel view materialization with dynamic load balancing: A graph theoretic approach</td>
</tr>
<tr>
<td>1439</td>
<td>S. Sarkar &amp; S. Sundar Sarkar</td>
<td>Internet and relational databases in a multi-tier client/server model</td>
</tr>
<tr>
<td>1440</td>
<td>J. Liang &amp; S. Subramaniam</td>
<td>Numerical computing of molecular electrostatics through boundary integral equations</td>
</tr>
<tr>
<td>1441</td>
<td>J. Wu</td>
<td>Inviscid limits and regularity estimates for the solutions of the 2-D dissipative quasi-geostrophic equations</td>
</tr>
<tr>
<td>1442</td>
<td>P. Constantin &amp; J. Wu</td>
<td>Statistical solutions of the Navier-Stokes equations on the phase space of vorticity and the inviscid limits</td>
</tr>
<tr>
<td>1443</td>
<td>M.A. Kouritzin</td>
<td>Stochastic processes and perturbation problems defined by parabolic equations with a small parameter</td>
</tr>
<tr>
<td>1444</td>
<td>M.A. Kouritzin</td>
<td>Approximations for singularly perturbed parabolic equations of arbitrary order</td>
</tr>
<tr>
<td>1445</td>
<td>A. Novick-Cohen</td>
<td>Triple junction motion for Allen-Cahn/Cahn-Hilliard systems</td>
</tr>
<tr>
<td>1446</td>
<td>P. Klouček</td>
<td>Approximations of the laminated microstructures</td>
</tr>
</tbody>
</table>
S. Sarkar & S.S. Sarkar, A graph theoretic approach for parallel view materialization with dynamic load balancing
S. Chawla, A minimax problem for parabolic systems with competitive interactions
B. Luong & F. Santosa, Quantitative imaging of corrosion in plates by Eddy current methods
R. Jordan & B. Turkington, Ideal magnetofluid turbulence in two dimensions
M. Fels & P.J. Olver, Moving coframes. I. A practical algorithm
S.Y. Malissov, On the Schwarz alternating method for eigenvalue problems
R. Lipton, Design of particle reinforced heat conducting composites with interfacial thermal barriers
J. Berg, A. Yezzi, & A. Tannenbaum, Phase transitions, curve evolution, and the control of semiconductor manufacturing processes
G. Avalos & I. Lasiecka, Uniform decay rates of solutions to a structural acoustics model with nonlinear dissipation
M. Nitsche, Siemens/IMA technical report
L. Wang, J.A. Cox, & A. Friedman, Model analysis of homogeneous optical waveguides by boundary integral method
C.P. Fung & S. Lototsky, Nonlinear filtering: Separation of parameters and observations using Galerkin approximation and Wiener chaos decomposition
S. Northshield, Several proofs of Ihara’s theorem
T. Aktosun, M. Klaus & C. van der Mee, Wave scattering in one dimension with absorption
F. Santosa, M. Vogelius, & J.-M. Xu, An effective nonlinear boundary condition for a corroding surface. Identification of the damage based on electrostatic data
J. Wu, Well-posedness of a semilinear heat equation with weak initial data
J. Wu, Quasi-geostrophic type equations with weak initial data
J. Ma & J. Yong, Approximate solvability of forward-backward stochastic differential equations
T.-P. Tsai, On Leray’s self-similar solutions of the Navier-Stokes equations satisfying local energy estimates
M.K. Gobbert, T.P. Merchant, L.J. Borucki, & T.S. Cale, A multiscale simulator for low pressure chemical vapor deposition
M.C. Tesi, E.J. Janse van Rensburg, E. Orlandini, & S.G. Whittington, Torsion of polygons in $\mathbb{Z}^3$
M. Grinfeld & A. Novick-Cohen, The viscous Cahn-Hilliard equation: Morse decomposition and structure of the global attractor
M. Mascagni, Polynomial versus matrix methods for leap-ahead in shift-register type pseudorandom number generators
M. Mascagni, Parallel linear congruential generators with prime moduli
B. Li & M. Luskin, Approximation of a Martensitic laminate with varying volume fractions
D. Yang, Stabilized schemes for mixed finite element methods with applications to elasticity and compressible flow problems
P.J. Olver & V.V. Sokolov, Integrable evolution equations on associative algebras
A. Bondarenko, Singular structure of the fundamental solution of the transport equation
S. Sarkar & S.S. Sarkar, Views and data mining in a parallel data server
P.E. Bigeleisen & M. Cheney, Models for an anesthesia breathing circuit
R. Lipton, Influence of interfacial surface conduction on the DC electrical conductivity of particle reinforced composites
B. Cockburn & C.-W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems
A. Friedman & F. Reitich, Asymptotic Behavior of Solutions of Coagulation-Fragmentation Models
A. Friedman & C. Huang, Averaged Motion Approach to Brownian Dynamics
T. Aktosun, M. Klaus & C. van der Mee, Inverse Scattering in One-Dimensional Nonconservative Media
X. Feng, Absorbing Boundary Conditions for Electromagnetic Wave Propagation
D. Hömberg & J. Sokolowski, Optimal Control of Laser Hardening
M. Kruzik, Numerical Approach to Double Well Problems
S. Lototsky, Parameter Estimation for Stochastic Parabolic Equations: Asymptotic Properties of a Two-Dimensional Projection Based Estimate
S. Sarkar & S.S. Sarkar, Data Mining on Views in Relational Databases Using Bayesian Networks
D. Yang, Dynamic Finite Element Methods for Second Order Parabolic Equations
S. Evje & K.H. Karlsen, A Note on Viscous Splitting of Degenerate Convection-Diffusion Equations
X. Feng, Analysis of A Domain Decomposition Method for The Nearly Elastic Wave Equations Based on Mixed Finite Element Methods
P. Korman & Y. Li, On The Exactness of an S-Shaped Bifurcation Curve
M. Kruzik, DiPerna—Majda Measures and Uniform Integrability