A CHARACTERIZATION OF BORDA'S RULE VIA OPTIMIZATION

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Abstract

It is shown that Borda's social welfare rule coincides with a social welfare function resulting from a well-defined optimization principle applied to a collection of individual binary preferences.

1. Introduction

By Borda's rule we mean that social welfare function which deduces a social (not necessarily strict) preference ordering on n alternatives from a collection of k individual (not necessarily strict) preference orderings by assigning "Borda points" to alternatives in each individual preference ordering in the following way. The lowest ranking alternative in a given preference ordering is assigned 0 points, the next lowest 1 point, the next 2 points and so on, artihmetically, until the highest ranking alternative is assigned $n-1$ points. Preferential ties are handled by averaging Borda point assignments, and, for each alternative, the Borda points resulting from each of the preference orderings are summed. The alternative with the greatest total number of Borda points is then ranked socially first, the next greatest total second, and so on with equal totals resulting in social indifference.

Borda's rule is perhaps the most frequently considered voting procedure after simple majority voting. It is well known to fail the various versions of the independence of irrelevant alternatives axiom, reflecting its cardinal flavor, and it is, therefore, subject to strategic voting. On the other hand, Borda's rule satisfies virtually every other axiom one might naturally require of a social welfare function.

An axiomatic characterization of Borda's rule has been given [1]. Our purpose here is to note, in a certain setting to be described below, that Borda's rule coincides with a natural optimization criterion for social welfare. This characterizes Borda's rule, alternately, in terms of optimization (or, in a sense, in terms of a single axiom) and underlines its cardinal character. Since the optimization criterion requires only binary comparison inputs, it also facilitates extension of Borda's rule to not necessarily transitive preference relations.

It is useful for our purposes here to think of the assignment of Borda points in Borda's rule in a slightly different (equivalent) way. The assignment of points to a given alternative in a given preference ordering is simply the number of alternatives below the given one in the preference ordering with ties counting one-half. The Borda vote of individual t, $1 \leqslant t \leqslant k$, may be thought of as an n-vector b_t, in which the i-th component is the assignment of Borda points from t's preference to alternative i. The sum $b = \sum_{t=1}^{k} b_t$ is then the vector of social Borda totals from which the alternatives are ranked via: i above j if and only if $b_i > b_j$.

2. The Setting

Summarize the preferences of our k individuals over the n alternatives in an n-by-n matrix $A = (a_{ij})$ as follows. For $i = 1, \ldots, n$, let $a_{ii} = 0$, and, for $i \neq j$, let a_{ij} be the number of individuals (strictly) preferring i to j minus the number (strictly) preferring j to i, i.e., the net number of individuals preferring i to j. Thus, A is skew-symmetric and $A + A = 0$. (Alternatively, A may be thought of as a sum of k individual 0, 1, -1 preference matrices A_t, $1 \leqslant t \leqslant k$, in which the i, j entry is 1 if the individual prefers i to j, -1 if the individual prefers j to i and 0 otherwise.) Now, determine a social preference ordering from A in the following way. Consider n-by-n matrices X built from vectors $x \in R^n$ via $X = (x_i - x_j)$, and determine the (to be justified later) X which most closely approximates A in the least squares or Frobenius norm sense, i.e.,

$$\underset{x \in R^n}{\text{minimize}} \sum_{i,j} (a_{ij} - x_i + x_j)^2 \quad .$$

Then rank the alternatives according to (algebraic) size of the components of x, i.e., i above j if and only if $x_i > x_j$. This is clearly complete and transitive with ties corresponding to indifference. Of course, x is not unique, but it will be up to translation by a vector with equal components; thus the ordering is uniquely determined by A. Call this social welfare function the net preference optimization rule. Intuitively, it asks for an assignment of scalars to the alternatives so that the differences approximate net preferences. Since the matrices X are also skew-symmetric, we are simply approximating certain skew-symmetric matrices by skew-symmetric matrices of a special type in the least squares or Frobenius norm sense.

3. Main Result

Our principal observations are that the net preference optimization rule is well defined (i.e., it produces a unique social preference ordering) and that its output necessarily coincides with that of Borda's rule. We see this by determining a matrix X of the special form, which most closely approximates A in the least squares sense, showing that X is unique, and showing that the ranking associated with X is the same as that of Borda's rule.

First of all, it is clear that the problem

$$\underset{x \in R^n}{\text{minimize}} \sum_{i,j} (a_{ij} - x_i + x_j)^2$$

has a solution and that any solution occurs at a critical point of

$$f(x) = \sum_{i,j} (a_{ij} - x_i + x_j)^2 \quad .$$

We first determine the critical points of f. We have

$$\frac{\partial f}{\partial x_i} = 4 \sum_{j \neq i} a_{ij} - 2(n-1)x_i + 2 \sum_{j \neq i} x_j$$

because of the chain rule and the skew-symmetry of A. Thus x is a critical point of f if and only if x is a solution to the system of linear equations:

$$(n-1)x_i = 2 \sum_{j \neq i} a_{ij} + \sum_{j \neq i} x_j \quad ,$$

which may be rewritten in matrix-vector form as

$$(nI - J)x = 2Ae \quad ,$$

in which we use e to denote the n-vector of all 1's and J to denote the n-by-n matrix of all 1's. The value of Ae may be determined in a simple way. First, observe that $\frac{1}{2}(A_t e + (n-1)e) = b_t$, $1 \leqslant t \leqslant k$, because the individual Borda points associated with a given alternative in t's preferences are just the number of alternatives ranking below the given one. Summing over individuals gives

$$2b = \sum_{t=1}^{k} 2b_t = \sum_{t=1}^{k} \left(A_t e + (n-1)e\right) = \sum_{t=1}^{k} A_t e + k(n-1)e = Ae + k(n-1)e \ .$$

Thus,

$$Ae = 2b - k(n-1)e \ .$$

The linear system we must solve for critical points is then

$$(nI - J)x = 2\left(2b - k(n-1)e\right) \ .$$

Note that the symmetric coefficient matrix, with eigenvalues n (occurring $n-1$ times) and 0 (occurring once) has rank exactly $n-1$. It is easy to verify by substitution that $\frac{4}{n}b$ is a solution and, since the null space of $nI - J$ is spanned by e, that all solutions are of the form

$$x = \frac{4}{n}b + re \ , \quad r \in R \ .$$

Each of these, then, produces the minimum for f, the same matrix X and the same social ranking as b, i.e., the same as Borda's rule. We record this as a

Theorem. For each set of transitive (not necessarily strict) individual preferences, the net preference optimization rule produces a unique (transitive, not necessarily strict) social preference ordering, which coincides with the result of Borda's rule.

4. Remarks

Alternatively, the result of the last section may be viewed as indicating that Borda's rule arises in a natural way from an optimization view of preference aggregation. Importantly, this optimization approach only requires binary preferences from each individual, as the construction of each skew-symmetric A_t only requires an asymmetric (not necessarily strict) preference relation from t. Thus, a skew-symmetric A could be constructed from nontransitive preferences and the optimization carried out in the same way, producing a unique social outcome which extends Borda's rule.

Reference

[1] H. P. Young, An Axiomatization of Borda's Rule, Journal of Economic Theory 9 (1974), 43-52.

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