

Symmetry of Constant Mean Curvature
Hypersurfaces in Hyperbolic Space.

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Introduction

In a recent paper, M. Do Carmo and B. Lawson studied hypersurfaces M of constant mean curvature in hyperbolic space [2]. They use the Alexandrov reflection technique to study M given the asymptotic boundary $\partial_\infty M$. For example, one of their theorems says M is a horosphere when $\partial_\infty M$ reduces to a point. They also prove a Bernstein type theorem for minimal graphs.

In this paper we shall extend their results to other boundary conditions. We prove an embedded M , of constant mean curvature, with $\partial_\infty M$ a subset of a codimension one sphere S , either is invariant by reflection in the hyperbolic hyperplane containing S or is a hypersphere. In the former case M is a "bigraph" over H : it meets any geodesic orthogonal to H either not at all or transversally in two points (one on each side of H) or tangentially on H .

As a corollary of this, when $\partial_\infty(M)$ equals two points p and q , then M is a hypersurface of revolution about the geodesic joining p to q .

We also consider minimal immersed hypersurfaces $M \subset H^n$ with M regular at ∞ . When $\partial_\infty M$ consists of two disjoint spheres S_1, S_2 we prove M is a catenoid or M is the two hyperbolic planes containing S_1 and S_2 .

The principal techniques we use to obtain these results are the Alexandrov reflection principle and R. Schoen's adaptation of this to complete minimal surfaces [4].

I. Definition and Notation.

When we refer to plane, distance, line, etc. we always mean the hyperbolic object in H^n . We work with the Poincare model of H^n : H^n is the interior of the unit ball in R^n . The asymptotic boundary of H^n is identified with the boundary

of the unit ball and denoted by $S(\infty)$. Given $A \subset H^n$, we denote by $\partial_\infty A$ the set of accumulation points of A in $S(\infty)$ and call it the asymptotic boundary of A . When the context is clear, we will omit the subscript ∞ .

Fix a hyperplane in H^n . We have two natural coordinate systems. First, one can use the geodesics orthogonal to P_0 to give each point coordinates (x,t) where $x \in P_0$ and t is the distance from x to (x,t) . This system does not suit our purposes since translation along one geodesic orthogonal to P_0 does not leave invariant another such geodesic. Also this does not extend to a coordinate system on $S(\infty)$.

Instead we shall use the latitude-longitude system. More precisely, choose coordinates in P_0 and let γ be the geodesic orthogonal to an origin $0 \in P_0$. Let γ_t be the 1-parameter group of isometries of H^n which along γ is translation by a distance t and such that the curves $t \rightarrow \gamma_t(x)$ are orthogonal to P_0 for each $x \in P_0$ (a positive sense along γ is chosen once and for all). Then each point of H^n has coordinates (x,t) where $x \in P_0$ and $\gamma_t(x) = (x,t)$.

Denote by P_t the plane $\gamma_t(P_0)$. We refer to P_t as a horizontal plane and the curve $t \rightarrow \gamma_t(x)$ as the vertical curve through x . Notice that for each s the reflection of H^n through the plane P_s is given by the formula $(x,t) \rightarrow (x, 2s - t)$.

Let $S_t = \partial_\infty P_t$. Then the coordinate system (x,t) extends to a coordinate system on $S(\infty)$ where each point (except the two limit points of γ) has a unique coordinate (x,t) , $x \in S_0$, $t \in \mathbb{R}$. By a Moebius transformation we can send γ to the north pole - south pole geodesic and P_0 to the equatorial plane. Then the coordinates on $S(\infty)$ are the usual latitude-longitude coordinates.

We say $A \subset H^n$ is a graph over P_s if the vertical projection of A to P_s is injective, and A has locally bounded slope if the vertical field $v = (0,1)$ is not tangent to A at any interior point of A .

We say A is above B , $A > B$, if whenever a vertical curve meets both A and B , then every point of A (on this vertical) is above every point of B . These notions extend directly to $S(\infty)$ with respect to the horizontals S_t and the vertical curves.

For $A \subset H^n$ and $s \in R$, let $A_s^+ = \{(x,t) \in A / t > s\}$ and similarly let A_s^- be the set of points of A below P_s . Let $A_s^{*+} = \{(x, 2s - t) / (x,t) \in A_s^+\}$. Also let H_s^+ (resp. H_s^-) be the set of all points above P_s (resp. below P_s).

II. The Main Result.

Now let M be a complete properly embedded hypersurface in H^n of constant mean curvature. Our main result is:

Theorem 2.1. If $\partial_\infty M \subset S_0 = \partial_\infty(P_0)$, then M is a hypersphere (in which case $\partial_\infty M = S_0$) or M is invariant under the reflection through P_0 and M is a bigraph over P_0 (i.e. M_0^+ is a graph over P_0 of locally bounded slope and M_0^- as well).

Remarks

1. It would be interesting to have examples of M as in 2.1, having prescribed boundary. for example, given 3 (or n) points on the equator, does there exist a constant mean curvature M with boundary these points?
2. Suppose $\partial_\infty M = S_0$ and M satisfies 2.1. Then is M a hypersphere? Do Carmo and Lawson claim this in theorem B of [2], however their proof assumes S_0 is the homological boundary of M . Imagine a torus in H^3 , touching $S(\infty)$ along the equator. Perhaps such a surface can have constant mean curvature H (it could not be of revolution but we can not even rule out the possibility $H > 1$).

We shall use the following version of the maximum principle [4]:

1. Let M_1, M_2 be connected complete hypersurfaces of constant mean curvatures

C_1, C_2 . Suppose M_1 and M_2 are tangent at an interior point x and their mean curvature vectors both point in the vertical up direction. If M_1 is above M_2 in a neighborhood of x and $C_2 > C_1$, then $M_1 = M_2$.

2. Assume x is an interior point of ∂M_1 and ∂M_2 , $\partial M_1, \partial M_2$ are tangent at x and M_1, M_2 as well. Also suppose the mean curvature vectors of M_1, M_2 point in the vertical up direction at x . If M_1 is above M_2 in a neighborhood of x and $C_2 > C_1$, then $M_1 = M_2$.

Proof of 2.1. Let C and C' be the connected components of $H^n - M$ and assume the mean curvature vector X of M points into C . Clearly we can suppose $X \neq 0$ since $X = 0$ easily implies $M = P_0$. So there are points of M above or below P_0 ; assume above for the moment. For t sufficiently large, P_t is disjoint from M , so there is a largest $T > 0$ such that $P_T \cap M \neq \emptyset$.

Let $J = \{t \in [0, T] \mid M_t^+ \text{ is a graph of locally bounded slope over } P_t, M_t^* \supset M_t^-, \text{ and for } s > t, X \text{ points into } H_s^- \text{ at each point of } M \cap P_s\}$. We have $T \in J$ and if $t \in J, t < s < T$, then $s \in J$. We shall prove $0 \in J$ by showing J is open and closed in $[0, T]$.

First we see why J is closed. Suppose $(t, T] \not\subset J$. If M_t^+ is not a graph then two points of M_t^+ are on the same vertical, so there is an $s, t < s < T$ and $x \in P_0$, such that (x, t) and (x, s) are both in M . We choose s so there are no other points of M on the vertical L joining (x, t) to (x, s) . Now M is a graph in a neighborhood of (x, s) , never vertical, and X points into H_s^- at (x, s) . This implies $L \subset C$ and M is tangent to P_t at (x, t) and below P_t in a neighborhood of (x, t) . The reason for the latter property is the vertical curves meeting M in a neighborhood of (x, s) , descend to fill a neighborhood of (x, t) in P_t . So if any point of M near (x, t) were strictly above P_t we would not have M_τ^+ a graph for some τ slightly larger than t . This violates $(t, T] \subset J$.

Now since M is entirely below P_t in a neighborhood of (x, t) , X must

point into H_t^- at (x,t) . But $L \in C$ and X points into C along M so this is a contradiction, and M_t^+ is a graph over P_t . M_t^+ has locally bounded slope as well. Also $M_t^{*+} > M_t^-$ since were this not the case, it would already fail to be true for s slightly larger than t .

Finally, the mean curvature vector X points into H_t^- at each point of $M \cap P_t$, since at $(x,t) \in M$, M is either entirely below P_t at (x,t) (in which case X points down) or (x,t) is an accumulation point of points of M above P_t , and then X points into H_t^- by continuity. Notice that our convention allows X to be tangent to P_t and point into H_t^- . So we have proved J is closed.

Next we show J is open. Let $[t,T] \subset J$ with $t > 0$. Let $(x,t) \in M$ and let $D \subset M$ be a disc containing (x,t) . Notice that M is not vertical at (x,t) , for if this were so, consider the half discs D_t^{*+} and D_t^- . They have the same boundary, they are tangent at (x,t) and their mean curvature vectors are the same at (x,t) . Moreover, D_t^{*+} is not vertical at points strictly below P_t and $D_t^{*+} > D_t^-$ hence they do not cross at (x,t) . So by the maximum principle, $D_t^{*+} = D_t^-$ near (x,t) and by analytic continuation $M_t^{*+} = M_t^-$. But $t > 0$ so this contradicts $\partial M \subset S_0$.

This proves M is a graph in a neighborhood U of P_t and not vertical in U . This implies X points down in U as well (for $s \in (0,t)$, the part of M between P_s and P_t is compact). It remains to verify that $M_s^{*+} > M_s^-$ for s near t . This is done exactly as in [4] so we just sketch the argument here. Since $M \cap U$ is a graph, we have $M_s^{*+} - V > M_s^-$ for V a neighborhood of P_t , $V \subset U$, and s near t . Also $M_s^+ - V$ is compact and its image under reflection through P_t is disjoint from M_t^- so by continuity, for s near t , we have $(M_s^{*+} - V) > M_s^-$. This means $M_s^{*+} > M_s^-$ for s near t . This argument uses the fact that M_t^+ is strictly above M_t^- whenever $(t,T] \subset J$ and $t \neq 0$.

If not, there would be a largest $t_0 > t$ for which this fails. Then $M_{t_0}^* +$ and $M_{t_0}^-$ are tangent at some point q and their mean curvature vectors have the same direction at q (M separates into two connected components and if X points into C then the mean curvature vector of $M_{t_0}^* +$ also points into C). Thus M would be invariant under reflection through P_{t_0} by the maximum principle and this violates the hypothesis on $\partial_\infty M$. Thus we have proved $0 \in J$; in particular, X points into H_0^- at each point of $M \cap P_0$.

Suppose there are points of M strictly below P_0 . Then the same reasoning used to prove $J = [0, T]$, shows M_0^- is a graph of bounded slope and X points into H_0^+ at each point of $M \cap P_0$ (just turn H^n over). Hence X is horizontal along $M \cap P_0$, or equivalently M is vertical at each $x \in M \cap P_0$. Let $x \in M \cap P_0$ and D be a disc in M containing x . Since $D_0^* > D_0^-$ and D is vertical at x , not vertical at each $y \in D \cap P_s$, $s > 0$, we conclude, as before, $D^* = D$ and hence $M^* = M$ by analytic continuation. Thus when M has points on both sides of P_0 , we know M is a bigraph and invariant by reflection through P_0 .

To complete the proof of theorem 2.1, we must show that if $M \subset H_0^+$ then M is a hypersphere. First we show the mean curvature H of M must be between 0 and 1: $0 < H < 1$. Consider the family L_τ of horospheres tangent to $S(\infty)$ at the south pole, the parameter τ chosen so that $L_0 =$ the south pole and $L_\infty = S(\infty)$. For small positive τ , L_τ is disjoint from M so there is a smallest τ such that $L_\tau \cap M \neq \emptyset$. Let x be such an intersection point, clearly $x \in P_s \cap M$, $s > 0$ so X points into H_s^- at x . Now the mean curvature vector Y of L_τ at x also points into H_s^- and M and L_τ are tangent at x so X and Y have the same direction at x . Therefore H is less than the mean curvature of L_τ , which is one, and we have strict inequality by the maximum principle. Let N_0 be the hypersphere of H^n with boundary ∂P_0 and with mean curvature vector pointing down and of length H . Foliate H^n by the hyperspheres $\gamma_t N_0 = N_t$.

We claim $M = N_0$. If not, there are points of M above or below N_0 . Suppose there are points below. Then there is a smallest negative t such that $N_t \cap M \neq \emptyset$. N_t is tangent to M , on one side of M , and their mean curvature vectors have the same orientation. So by the maximum principle, $M = N_t$, which contradicts $\partial M \subset S_0$. A similar argument works if there are points of M above N_0 . Q.E.D.

Corollary 2.2. In addition to the hypothesis of 1.1, assume ∂M consists of two distinct points p and q . Then M is a hypersurface of revolution about the geodesic Γ joining p to q . More precisely. Let Q be any hyperplane orthogonal to Γ . Then M is transverse to Q and $M \cap Q$ is a round sphere centered at $\Gamma \cap Q$.

Remark. These hypersurfaces of revolution have been classified by Hsiang [3], and in this paper he obtains a special case of corollary 2.2.

Proof of 2.2. It follows immediately from 2.1 that M is transverse to Q except possibly at $y = \Gamma \cap Q$, and that $M \cap Q$ is either the empty set, or $\{y\}$, or a round sphere centered at y . It cannot be empty since Q separates p and q . If $M \cap Q = \{y\}$, then M is tangent to Q at y and lies on one side of Q near y . It follows that hyperplanes Q' close to Q , on the other side, do not meet M . This is a contradiction so $M \cap Q$ is a round sphere.

III. Minimal Hypersurfaces of H^n .

Let M be a minimal hypersurface of H^n . We say that M is regular at ∞ if the asymptotic boundary B of M is a C^2 codimension one submanifold of $S(\infty)$ and $\bar{M} = M \cup B$ is of class C^1 on B . M. Anderson has proved that any C^2 codimension one submanifold $B \subset S(\infty)$ bounds a minimal $M \subset H^n$ [1]. We do not

know if one has boundary regularity as in the euclidean category. In this section we adapt the work of R. Schoen to our context to obtain information about M given B .

Theorem 3.1. Let $B \in S(\infty)$ be a C^2 codimension one immersed boundary, not necessarily connected. Assume B_0^+ is a graph of locally bounded slope and $B_0^+ \succ B_0^-$. Let M be a minimal hypersurface immersed in H^n with $\partial M = B$ and M regular at ∞ . Then M_0^+ is a graph of locally bounded slope and $M_0^+ \succ M_0^-$.

Proof: First we remark that M is orthogonal to $S(\infty)$ along B . We see this as follows. Let $x \in B$ and $S_1, S_2 \in S(\infty)$ be round codimension one spheres passing through x such that each S_i is tangent to B at x and S_i and B are on one side of each other at x , and S_1 and S_2 are on opposite sides of B at x (This is where we need B to be C^2). Let D_1, D_2 be the disks on $S(\infty)$ with boundary S_1, S_2 . Choose the S_k small enough so that $\text{int } D_k$ is disjoint from B for $k = 1, 2$. Let $y \in \text{int } D_k$ and $S(y)$ be a small round sphere centered at y . Let $H(y)$ be the hyperbolic plane of H^n with boundary $S(y)$; $H(y)$ is a hemisphere orthogonal to $S(\infty)$. For $S(y)$ small, $H(y)$ is disjoint from M . As $S(y)$ grows to become S_k , $H(y)$ stays disjoint from M since if this were not so we would have $M = H(y)$ by the maximum principle. Thus M is forced between the two hyperbolic planes H_1, H_2 with boundaries S_1, S_2 , and is orthogonal to B .

Let $T > 0$ be the largest T such that $P_T \cap B \neq \emptyset$. Let $J = \{t \in [0, T] \mid M_t^+ \text{ is a graph of locally bounded slope and } M_t^+ \succ M_t^-\}$. As in the proof of 2.1, we see that J is open and closed in $[0, T]$ hence $0 \in J$ and 3.1 is proved. We will not go through the details, however some comments are in order. M need not be embedded to make the argument since the maximum principle applies without worrying about local orientations. Also M orthogonal to $S(\infty)$

along B implies M is a graph of locally bounded slope in a neighborhood of B_0^+ . One can prove, as in theorem 2 of [4], that if B is embedded and $B_0^+ = B_0^-$ then M is embedded and $M^* = M$.

Theorem 3.2. Let S_1, S_2 be disjoint round spheres in $S(\infty)$ and let M be a connected minimal hypersurface in H^n with $\partial_\infty M = S_1 \cup S_2$ and M regular at ∞ . Then M is a catenoid.

Remark. By a conformal transformation of $S(\infty)$ we can make S_1 and S_2 horizontal (just send the geodesic joining the centers of S_1 and S_2 , to the north pole-south pole geodesic). The radius R of this annular region between S_1 and S_2 is a conformal invariant. If R is small enough then there exists a catenoid with boundary $S_1 \cup S_2$.

Proof of 3.2. We can assume S_1, S_2 are horizontal and symmetric with respect to the equator. Then by 3.1, M is also symmetric with respect to the equatorial plane. It suffices to show M is a surface of revolution about the geodesic γ joining the north and south poles. Let P be a hyperplane containing γ ; we need show M is invariant by reflection in P . Rotate H^n by $\pi/2$ so that P becomes horizontal. Then $B = S_1 \cup S_2$ satisfies the hypothesis of 3.1 from above and below P so M is invariant by reflection through P .

Theorem 3.3. Let $B \subset S(\infty)$ be a C^2 codimension one submanifold and suppose B is a graph of locally bounded slope over S_0 . Let M be a constant mean curvature hypersurface embedded in H^n with the homological boundary of M equal to B . Then M is a graph over P_0 . If M is minimal and immersed with $\partial_\infty M = B$, then M is a graph as well and M is unique.

Remark. The existence of a minimal M with $\partial_\infty M = B$ is proved in [1]. When M is minimal and the homological boundary of M equals B , then unicity is proved in [2].

Proof of 3.3. Suppose M is not a graph. Let γ be a vertical curve such that M intersects γ in at least two points p and q and the distance T between p and q is maximum with respect to this property. Then $\gamma_T M$ and M are tangent at one of the points p, q and have the same mean curvature vectors. Thus $\gamma_T M = M$ which contradicts $\partial_\infty M$ a graph.

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