A NEW PROOF OF MOSER'S PARABOLIC HARNACK INEQUALITY

VIA THE OLD IDEAS OF NASH

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Dedicated to Jim Serrin on the occasion of his 60th birthday.

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Introduction

In 1958 Nash published his fundamental work on the local Hölder continuity of solutions of second order parabolic equations with non-smooth coefficients ([8]). The primary purpose of that work was to study the properties of the fundamental solution corresponding to the parabolic operator and to derive from these properties the regularity for a general solution. Though the work is often cited in the literature about weak solutions of elliptic and parabolic equations, one feels that Nash's ideas were never fully understood (and maybe still are not) and that because of this the more understandable and seemingly more fruitful ideas of DeGiorgi ([4]) and Moser ([6], [7]) were subsequently adopted.

In the present article, we return to Nash's ideas. In particular, by modifying and persuing his arguments, we establish directly what we feel is the logical goal of this line of reasoning, namely: the estiamtes for the fundamental solution first proved by D.G. Aronson. (See [1] and [2].) From Aronson's estimates the parabolic Harnack inequality of Moser ([7]) and, consequently (as was shown by Moser [7, p. 108]), Nash's local Hölder continuity of weak solutions to parabolic equations follow. That is, our approach reverses the chronological order in which these results were derived originally.

To make the above statements mathematically precise we introduce the basic notations and definitions to be used throughout this work. We will be studying parabolic operators of the form

$$L = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(t,x)D_{x_j}) - D_t$$

where t is a real number and $x = (x_1,\dots,x_n) \in R^n$. Our basic assumptions on the matrix $a(t,x) \equiv (a_{ij}(t,x))$ are symmetry, (ie. $a_{ij} = a_{ji}$), and the existence of a number $\lambda \in (0,1]$ such that for all $(t,x) \in R^{n+1}$ and all

$\xi \in R^n$

$$\lambda|\xi|^2 \leq a(t,x)\xi \cdot \xi \equiv \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2.$$

We may and do make the qualitative assumption that the matrix $a(t,x)$ is smooth; however, we emphasize that all quantitative estimates are only allowed to depend on the dimension and the number λ. Besides x, the letters y and ξ will be used to denote points in R^n and the letters t,s, and r will be reserved for real numbers.

We let $\Gamma(t,x;s,y) \equiv \Gamma_a(t,x;s,y)$ denote the fundamental solution of the parabolic operator L. As stated above the purpose of this paper is to use the ideas of Nash to obtain the following estimates: for $s < t$

$$(*) \qquad \frac{\exp\{-C\frac{|x-y|^2}{t-s}\}}{C(t-s)^{n/2}} \leq \Gamma(t,x;s,y) \leq \frac{C\exp\{-\frac{|x-y|^2}{C(t-s)}\}}{(t-s)^{n/2}}$$

where C depends only on λ and n. The inequalities * were first obtained by Aronson in [1]. (See also [2].) His proof, however, relied on Moser's parabolic Harnack inequality ([7]). Our point here is to first establish the estimates (*) and then derive the Harnack inequality as an easy consequence. The outline of the paper is simple; the upper bound is obtained in Section 1, the lower bound in Section 2, and Harnack's inequality in Section 3.

Section 1: The upper bound

Our proof of the upper bound for the fundamental solution is essentially due to Nash. In particular, Nash used the same proof to derive the right side of (*) without the exponential factor. There are various ways of passing from his result to the one including the exponential factor. The one which we have adopted is based on a method which was introduced in this context by E.B. Davies ([3]).

(We wish to point out that Aronson's original proof ([1]) of the upper bound in (*), like the one given here, does not depend on Harnack's inequality. Our reasons for presenting a proof here are completeness and unification of the arguments. At the same time, it should be emphasized that the upper bound itself is an important tool for our understanding and simplification of those ideas of Nash needed to obtain the lower bound in Section 2.)

Fix an element $\vec{\alpha} \in R^n$ and set $\psi(x) = \vec{\alpha} \cdot x \equiv \sum_{i=1}^{n} \alpha_i x_i$. Let $A_t = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(t,x)D_{x_j})$ and $A_t^{\psi} = \exp(-\psi)A_t \exp\psi$. If $f \in S(R^n;(0,\infty))$ (i.e. f is a positive function from the Schwartz test function space)

$$(A_t^{\psi}f,f^{2p-1}) \equiv \int A_t^{\psi}f(x)f^{2p-1}(x)dx$$

$$= \int(a(\vec{\alpha})\cdot\vec{\alpha})f^{2p}(x)dx - 2(1-\frac{1}{p})\int a(\nabla\psi f^p)\cdot\nabla(f^p)dx.$$

$$-(\frac{2}{p}-\frac{1}{p^2})\int a(\nabla f^p)\cdot\nabla f^p dx.$$

Hence, setting $\|f\|_p = (\int_{R^n} |f|^p)^{1/p}$,

$$(A_t^{\psi}f,f^{2p-1}) < -\frac{\lambda}{2p}\int|\nabla f^p|^2 + \frac{\alpha^2}{\lambda}p\|f\|_{2p}^{2p}, \quad p > 1,$$

where $\alpha = |\vec{\alpha}|$. (This observation is the key to Davies's analysis.) At the same time, $(2\pi)^n \|f\|_2^2 \le C[R^n \|f\|_1^2 + R^{-2} \|\nabla f\|_2^2]$ for all $R > 0$; and so

$$\|f\|_2^{2+\beta} \le C_n\left(\int |\nabla f|^2\right)\left(\int |f|\right)^\beta$$

with $\beta = 4/n$. (This inequality is the one from which Nash's upper bound comes.) Combining these, we arrive at

$$(1.1) \qquad (A_t^\psi f, f^{2p-1}) \le -\frac{\varepsilon}{2p} \frac{\|f\|_{2p}^{2p+\beta p}}{\|f\|_p^{\beta p}} + \frac{\alpha^2 p}{\lambda} \|f\|_{2p}^{2p}, \quad p \ge 1,$$

for some $\varepsilon > 0$ which depends only on n and λ. Finally, let $f \in S(R^n; (0,\infty))$ and define f_t by

$$f_t(x) = \exp(-\psi(x))\int f(y)\Gamma(t,x;0,y)\exp(\psi(y))dy,$$

where $\Gamma = \Gamma_a$. Then $t \in [0,\infty) \to f_t \in S(R^n;(0,\infty))$ is smooth and, for $p \ge 1$:

$$\frac{d}{dt} \|f_t\|_{2p}^{2p} = 2p(A_t^\psi f_t, f_t^{2p-1}).$$

Hence, by (1.1), for $p \in [1,\infty)$:

$$(1.2) \qquad \frac{d}{dt} \|f_t\|_{2p} \le -\frac{\varepsilon}{2p} \|f_t\|_{2p}^{1+\beta p} \|f_t\|_p^{-\beta p} + \frac{\alpha^2 p}{\lambda} \|f_t\|_{2p}, \quad t \ge 0.$$

In particular,

$$(1.3) \qquad \|f_t\|_2 \le e^{\alpha^2 t/\lambda} \|f\|_2, \quad t \ge 0.$$

(1.4) <u>Lemma</u>: Let $w: [0,\infty) \to [0,\infty)$ be a continuous non-decreasing function and suppose that $u \in C^1([0,\infty))$ is a positive function which satisfies

$$u'(t) \le -\frac{\varepsilon}{2p}\left(\frac{t^{(p-2)/\beta p}}{w(t)}\right)^{\beta p} u^{1+\beta p}(t) + \frac{\alpha^2 p}{\lambda} u(t), \quad t \ge 0.$$

where $p \in [2,\infty)$. Then, for each $\delta > 0$ there is a $K = K(\varepsilon,\delta) < \infty$ such that

$$u(t) \le (Kp^2)^{1/\beta p} w(t) e^{\delta\alpha^2 t/\lambda p}\, t^{(1-p)/\beta p}, \quad t > 0.$$

<u>Proof:</u> Set $v = e^{-\alpha^2 pt/\lambda} u$. Then

$$(v^{-\beta p})' \ge \frac{\varepsilon\beta}{2} \exp(\beta\alpha^2 p^2 t/\lambda) t^{p-2}/w^{\beta p} \;;$$

and so

$$\frac{\exp(\beta\alpha^2 p^2 t/\lambda)}{u(t)^{\beta p}} \ge \frac{\varepsilon\beta}{2w(t)^{\beta p}} \int_0^t \exp(\beta\alpha^2 p^2 s/\lambda) s^{p-2} ds.$$

Note that

$$\int_0^t \exp(\beta\alpha^2 p^2 s/\lambda) s^{p-2} ds = \left(\frac{\lambda t}{\beta\alpha^2 p^2}\right)^{p-1} \int_0^{\beta\alpha^2 p^2/\lambda} e^{ts} s^{p-2} ds$$

$$\ge \left(\frac{\lambda t}{\beta\alpha^2 p^2}\right)^{p-1} \exp[(\beta p^2\alpha^2 - \delta\beta\alpha^2)t/\lambda] \int_{\beta\alpha^2 p^2(1-\delta/p^2)/\lambda}^{\beta\alpha^2 p^2/\lambda} s^{p-2} ds$$

$$= \frac{t^{p-1}}{p-1}[1 - (1 - \delta/p^2)^{p-1}] \exp[(\beta p^2\alpha^2 - \delta\beta\alpha^2)t/\lambda].$$

Combining these, we get

$$\frac{e^{\beta\alpha^2 p^2 t/\lambda}}{u(t)^{\beta p}} \ge \frac{t^{p-1}}{Kp^2 w(t)^{\beta p}} \exp[(\beta p^2\alpha^2 - \delta\beta\alpha^2)t/\lambda],$$

where

$$\frac{1}{K} \equiv \frac{\varepsilon\beta}{2} \inf_{p \ge 2} \{p[1 - (1 - \delta/p^2)^{p-1}\} > 0.$$

Now set $p_k = 2^k$, $u_k(t) = \|f_t\|_{p_k}$, and $w_k(t) = \max\{s^{(p_k-2)/\beta p_k} u_k(s) : 0 \le s \le t\}$. By (1.3), if $\|f\|_2 = 1$, $u_1(t) \le \exp\{\alpha^2 t/\lambda\}$, and by (1.2) and the lemma,

$$\frac{w_{k+1}(t)}{w_k(t)} \leq (4^k K)^{1/\beta 2^k} \exp(\delta\alpha^2 t\ /2^k\lambda).$$

Hence, there is a $C < \infty$, depending only on n,λ and $\delta > 0$, such that

(1.5) $$\sup_k w_k(t) \leq C \exp[(1 + \delta)\alpha^2 t/\lambda].$$

(1.6) Theorem: There is a $C < \infty$, depending only on n and λ, such that

$$\Gamma_a(t,x;s,y) \leq \frac{C}{t^{n/2}} \exp(-|y - x|^2/Ct)$$

for all $0 \leq s < t < \infty$ and $x,y \in R^n$.

Proof: Since $\Gamma_a(t,x;s,y) = \Gamma_{a_s}(t - s,x;0,y)$ where $a_s(r,\xi) = a(s + r,\xi)$, we may and will take $s = 0$. Now define
$P_t^\psi f(x) = \exp(-\psi(x))\int f(y)\Gamma_a(t,x;0,y)\exp(-\psi(y))dy$ (i.e. $= f_t$). Then, from the preceeding, (with $\delta = 1$)

$$\|P_t^\psi f\|_\infty \leq \frac{C}{t^{n/4}} \exp(2\alpha^2 t/\lambda)\|f\|_2$$

for each $t > 0$. At the same time, it is clear that the adjoint $(P_t^\psi)^*$ of P_t^ψ is given by

$$(P_t^\psi)^* f(y) = \exp(\psi(y))\int \Gamma_{\tilde{a}}(t,y;0,x)\exp(-\psi(x))f(x)dx,$$

where $\tilde{a}(r,\xi) = a(t - r,\xi)$. Hence,

$$\|(P_t^\psi)^* f\|_\infty \leq \frac{C}{t^{n/4}} \exp(2\alpha^2 t/\lambda)\|f\|_2;$$

and so, by duality,

$$\|P_t^\psi f\|_2 \leq \frac{C}{t^{n/4}} \exp(2\alpha^2 t/\lambda)\|f\|_1.$$

Finally, note that $P_{2t} = Q_t^\psi \circ P_t^\psi$, where

$$Q_t^\psi f(x) = \exp(-\psi(x)) \int f(y)\ \Gamma_{a_t}(t,x;0,y)\exp(\psi(y))dy,$$

and $a_t(\cdot,\cdot) = a(\cdot + t,\cdot)$. Hence,

$$\|P_{2t}^\psi f\|_\infty < \frac{C^2}{t^{n/2}} \exp(\psi\alpha^2 t/\lambda)\|f\|_1,$$

which is equivalent to

$$\Gamma_a(2t,x;0,y) < \frac{C^2}{t^{n/2}} \exp(4\alpha^2 t/\lambda + \vec{\alpha} \cdot (x - y)).$$

We now get our estimate upon taking $\vec{\alpha} = \frac{\lambda}{8t} \frac{y - x}{|y - x|}$.

It is clear that we have not fully utilized the estimate (1.6) since we simply took $\delta = 1$. Had we carried δ through the proof of Theorem (1.6), we would have arrived at

$$\Gamma_a(t,x;s,y) < \frac{C(\delta)}{t^{n/2}} \exp(-|y - x|^2/(1 + \delta)\ \Lambda_t t)$$

for each $\delta > 0$, where $\Lambda_t \equiv \max\{\eta \cdot a(r,\xi)\eta\colon (r,\xi) \in [0,t] \times R^n$ and $\eta \in S^{n-1}\}$. It is the power of his method to get such precise exponential estimates that justifies the word "explicit" in the title of Davies's article [2].

Section 2. The Lower Bound

In this section we will establish the lower bound for the fundamental solution (i.e. the left hand side of (*)). Our procedure is, once again, basically due to Nash. However, the upper bound just established allows us to simplify his argument and to carry it to completion.

Lemma (2.1) (Nash's Lower Bound)

There is a constant $B < \infty$ depending only on λ such that for all $|x| \leq 1$

$$\int e^{-|y|^2/2} \log \Gamma_a(1,x;0,y)dy \geq -B.$$

Proof.

Observe that

$$\Gamma_a(t,x;t-s,y) = \Gamma_{a_t}(s,y;0,x)$$

where $a_t = a(t - \cdot,\cdot)$. In particular

$$\Gamma_a(t,x;0,y) = \Gamma_{a_1}(1,y;0,x).$$

Set $u(s,y) = \Gamma_{a_1}(s,y;0,x)$ and $G(s) = \int e^{-y^2/2} \log u(s,y)dy$.

Since $\int u(s,y)dy = 1$, $G(s) < 0$ and our goal is to estimate $G(1)$ from below. We will obtain this from a differential inequality satisfied by G. Namely:

$$G'(\tau) = -\int \nabla y\left(\frac{e^{-|y|^2/2}}{u(\tau,y)}\right) \cdot a_1 \nabla_y u(\tau,y))dy$$

$$= \int e^{-|y|^2/2} y \cdot a_1(\nabla_y \log u(\tau,y))dy + \int e^{-|y|^2/2} \nabla y \log u \cdot a_1(\nabla_y \log u)dy$$

$$= -\frac{1}{2}\int e^{-|y|^2/2} y \cdot a(y)dy + \frac{1}{2}\int e^{-|y|^2/2}(y + \nabla y \log u)\cdot a_1(y + \nabla_y \log u)dy$$

$$+ \frac{1}{2}\int e^{-|y|^2/2} (\nabla_y \log u) \cdot a_1(\nabla_y \log u)dy.$$

Hence

(2.2) $$G'(s) \geq -A + \frac{\lambda}{2}\int e^{-|y|^2/2} |\nabla_y \log u|^2 dy.$$

In particular $G(s) + As$ is nondecreasing on $[\frac{1}{2}, 1]$. Also since

$$\int e^{-|y|^2/2}(\log u(s,y) - G(s))^2 dy \leq c \int e^{-|y|^2/2} |\nabla_y \log u|^2 dy$$

we have

(2.3) $$G'(s) \geq -A + B\int e^{-|y|^2/2} (\log u(s,y) - G(s))^2 dy$$

for constants A and B depending only on λ.

Next observe that $\left(\frac{\log u - G(s)}{u}\right)^2$ is nonincreasing as a function of u in $[e^{2+G(s)}, \infty)$. Also from Theorem 1.9 $\sup_{1/2 \leq s \leq 1} u(s,y) \leq K$, an absolute constant. Combined with (2.3), this implies

(2.4) $$G'(s) \geq -A + B\left(\frac{\log K - G(s)}{K}\right)^2 \int_{u(s,y) \geq e^{2+G(s)}} e^{-|y|^2/2} u(s,y)dy$$

for $s \in [\frac{1}{2}, 1]$. At the same time

$$\int_{u(s,y) \geq e^{2+G(s)}} e^{-|y|^2/2} u(s,y)dy \geq \int e^{-|y|^2/2} u(s,y)dy - (2\pi)^{n/2} e^{2+G(s)}$$

$$\geq e^{-R^2/2} \int_{|y|<R} u(s,y)dy - (2\pi)^{n/2} e^{2+G(s)}$$

$$= e^{-R^2/2}\left[1 - \int_{|y| \geq R} u(s,y)dy\right] - (2\pi)^{n/2} e^{2+G(s)}$$

Note that, by Theorem (1.4) there exists R_λ, depending only on λ such $\sup_{1/2 < s < 1} \int_{|y|>R_\lambda} u(s,y)dy < \frac{1}{2}$.

Applying these last remarks to (2.3) and also remembering that $G(s) + As$ is nondecreasing on $[\frac{1}{2},1]$, we can conclude that there exists δ_λ and M_λ, both depending only on λ, such that

$$G'(s) > \delta_\lambda G(s)^2 \quad \text{for} \quad s \in [\tfrac{1}{2},1] \tag{2.5}$$

provided $G(1) < -M_\lambda$. But if (2.5) holds then $G(1) > -\frac{2}{\delta_\lambda}$. That is we have proved

$$G(1) > -\max(\frac{2}{\delta_\lambda}, M_\lambda).$$

Lemma (2.6). There exists C, depending only on λ such that

$$\Gamma_a(t,x;s,y) > \frac{1}{C(t-s)^{n/2}}$$

for all x and y satisfying $|x - y| < \sqrt{t - s}$.

Proof.

By rescaling, we may take $s = 0$ and $t = 2$. We write

$$\Gamma_a(2,x;0,y) = \int \Gamma_a(1,\xi;0,y)\Gamma_{\tilde{a}}(1,x;0,\xi)d\xi$$

where $\tilde{a} = a(\cdot + 1,\cdot)$. Clearly, this leads to

$$\Gamma_a(2,x;0,y) > \int \Gamma_a(1,\xi;0,y)\Gamma_{\tilde{a}}(1,x;0,\xi)e^{-|\xi|^2/2}d\xi$$

and by Jensen's inequality

$$\log[(2\pi)^{-n/2}\Gamma_a(2,x;0,y)] \geq (2\pi)^{-n/2}[\int e^{-|\xi|^2/2}\log \Gamma_a(1,x;0,\xi)d\xi$$

$$+ \int e^{-|\xi|^2/2}\log \Gamma_a(1,\xi;0,y)d\xi]$$

$$\geq -c_\lambda \text{ be Lemma (2.1).}$$

(Remember $\Gamma_a(1,\xi;0,y) = \Gamma_{a_1}(1,y;0,\xi)$ where $a_1 = a(1 - \cdot,\cdot)$.)

We are now ready to prove the lower bound estimate for the fundamental solution.

Theorem (2.7) (Aronson). There exists C, depending only on λ and n, such that

$$\Gamma_a(t,x;s,y) \geq \frac{1}{C(t-s)^{n/2}} \exp(-C|x-y|^2/(t-s)).$$

Proof.

Again we may assume $s = 0$, $t = 1$, and this time we also assume, as we may, that $y = 0$. That is, we wish to show that $\Gamma_a(1,x;0,0) \geq \frac{1}{C}\exp(-C|x|^2)$. Because of Lemma (2.6), we may also assume $|x| > 1$.

Given $x \in R^n$ with $|x| > 1$, let k be the smallest integer dominating $4|x|^2$ and set $S = \prod_{\ell=1}^{k-1} B(\frac{\ell x}{k}, \frac{1}{2\sqrt{k}})$ $(B(y,r) \equiv \{\xi \in R^n: |\xi - y| < r\})$. Then, for $(\xi_1,\ldots,\xi_{k-1}) \in S$: $|\xi_1| < \frac{1}{\sqrt{k}}$, $\max_{1<\ell<k} |\xi_\ell - \xi_{\ell-1}| < \frac{1}{\sqrt{k}}$, and $|x - \xi_{k-1}| < \frac{1}{\sqrt{k}}$. Hence, by Lemma (2.6):

$$\Gamma(1,x;0,0) = \int \cdots \int \Gamma(1,x;\tfrac{k-1}{k},\xi_{k-1})\Gamma(\tfrac{k-1}{k},\xi_{k-1};\tfrac{k-2}{k},\xi_{k-2})$$

$$\cdots \Gamma(\tfrac{1}{k},\xi_1;0,0)d\xi_1 \cdots d\xi_{k-1}$$

$$> \int_S \cdots \int \Gamma(1,x;\tfrac{k-1}{k},\xi_{k-1})\Gamma(\tfrac{k-1}{k},\xi_{k-1};\tfrac{k-2}{k},\xi_{k-2})$$

$$\cdots \Gamma(\tfrac{1}{k},\ \xi_1;0,0)d\xi_1 \cdots d\xi_{k-1}$$

$$> \left(\frac{k^{n/2}}{C}\right)^{k-1} |S| = \left(\frac{k^{n/2}}{C}\right)^{k-1} (\Omega_n(2k)^{-n/2})^k$$

$$= \frac{C}{k^{n/2}}\left(\frac{\Omega_n}{2^{n/2}C}\right)^k.$$

Clearly the required estimate is immediate from this.

Section 3:

In this section we show how to derive both Nash's continuity result as well as Moser's Harnack principle from (*). Actually, there are a variety of ways in which this can be done. Our choice has been dictated by our desire to show that (*), and nothing more, suggices. The proof here is modelled on the argument given by Krylov ([9]) in (cf. [5] for a similar derivation of the Harnack principle for solutions to certain degenerate equations).

In what follows, $\Gamma_a^{(\xi,R)}(t,x;s,y)$ denotes the fundamental solution ot $Lu = 0$ with zero boundary data on $\partial B(\xi,R)$. That is, if $(s,y) \in (0,\infty) \times B(\xi,r)$ and $u(t,x) = \Gamma_a^{(\xi,R)}(t,x;s,y)$, then $Lu = 0$ in $(s,\infty) \times B(\xi,R)$, $u = 0$ on $[s,\infty) \times \partial B(\xi,R)$, and $u(s,x) = \delta(x - y)$.

(5.1) Lemma: For each $\delta,\gamma \in (0,1)$ there is an $\varepsilon = \varepsilon(n,\lambda,\delta,\gamma) > 0$ such that

$$\Gamma_a^{(\xi,R)}(t,x;s,y) > \frac{\varepsilon}{|B(\xi,\delta R)|}$$

for all $x,y \in B(\xi,\delta R)$ and $s < t$ satisfying $\gamma R^2 \leq t - s \leq R^2$.

Proof: By rescaling and translation, we may and will assume that $\xi = 0$ and $R = 1$. For convenience, we use $\hat{\Gamma}$ to denote $\Gamma_a^{(0,1)}$. Clearly we need only treat the case when $s = 0$.

Note that

$$\hat{\Gamma}(t,x;0,y) = \Gamma(t,x;0,y) - \int_{[0,t)\times\partial B(0,1)} \Gamma(t,x;r,\xi)\mu_{0,y}(d\tau \times d\xi)$$

where $\mu_{0,y}$ is a non-negative measure with total mass less than or equal to one. Hence, by (*):

$$\hat{\Gamma}(t,x;0,y) \geq \frac{1}{Ct^{n/2}} \exp[-C|y - x|^2/t] - C \sup_{0<\tau<t} \frac{1}{\tau^{n/2}} \exp[-(1 - \delta)^2/C\tau]$$

for $x \in B(0,\delta)$, $t > 0$, and $y \in B(0,1)$. In particular, there is an $r \in (0,1-\delta)$ depending only on C and δ, such that

$$\hat{\Gamma}(t,x;0,y) \geq \frac{1}{2Ct^{n/2}} \exp[-C|y - x|^2/t]$$

for all $x \in B(0,\delta)$, $t \in (0,r^2]$, and y with $|y - x| < r$.

Finally, we use the reproducing property of $\hat{\Gamma}$ to conclude from the above that

$$\hat{\Gamma}(t,x;0,y) \geq \frac{\alpha}{t^{n/2}} \exp[-K|y - x|^2/t]$$

for some $\alpha > 0$ and $K < \infty$, depending only on r and C, for all $t \in (0,1]$ and $x,y \in B(0,\delta)$ (cf. the argument used in Section 2 to pass from the lower bound of $\Gamma(t,x;s,y)$ for $|y - x|^2/(t - s)$ small to the general result.) Obviously, our estimate follows immediately from here.

In the following we use the notation $\underline{Osc(u;s,\xi,R)}$ to denote $\sup\{|u(t,x) - u(t',x')|: s - R^2 < t,t' < s \text{ and } x,x' \in B(\xi,R)\}$.

(5.2) <u>Lemma</u>. For each $\delta \in (0,1)$ there is a $\rho = \rho(n,\lambda,\delta) \in (0,1)$ such that for all $(s,\xi) \in R^1 \times R^N$ and $R > 0$:

$$Osc(u;s,\xi,\delta R) \leq \rho\, Osc(u;s,\xi,R)$$

whenever $u \in C^\infty([s - R^2,s] \times \overline{B}(\xi,R))$ satisfies $Lu = 0$ in $(s - R^2,s) \times B(\xi,R)$.

<u>Proof</u>: Let $m(r)$ and $M(r)$ denote, respectively, the minimum and maximum values of u on $[s - r^2,s] \times \overline{B(\xi,r)}$.

Set $S = \{x \in B(\xi,\delta R): u(s - R^2,x) \geq (M(R) + m(R))/2\}$, and assume that $|S|/|B(\xi,\delta R)| \geq \frac{1}{2}$. Then, for $(t,x) \in [s - \delta^2 R^2, s] \times B(\xi,\delta R)$:

$$u(t,x) - m(R) \geq \int (u(s - R^2,y) - m(R))\, \Gamma_a^{(\xi,R)}(t,x;s - R^2,y)dy$$

$$\geq \frac{M(R) - m(R)}{2} \int_S \Gamma_a^{(\xi,R)}(t,x;s - R^2,y)dy$$

$$\geq \varepsilon(M(R) - m(R))/4 ;$$

and so $m(\delta R) \geq \varepsilon M(R)/4 + (1 - \varepsilon/4)m(R)$. Hence

$$M(\delta R) - m(\delta R) \leq M(R) - m(\delta R) \leq (1 - \varepsilon/4)(M(R) - m(R)).$$

In other words, we can take $\rho = 1 - \frac{\varepsilon}{4}$.

(5.3) Theorem (Nash): For each $\delta \in (0,1)$ there exist $C = C(u,\lambda,\delta) < \infty$ and $\beta = \beta(n,\lambda,\delta) \in (0,1)$ such that for all $(s,\xi) \in R^1 \times R^n$ and $R > 0$:

$$|u(t,x) - u(t',x')| \leq C\|u\|_{C_b([s - R^2,R^2] \times \overline{B}(\xi,R))} \left(\frac{|t - t'|^{1/2} \vee |x - x'|}{R} \right)^{\beta}$$

for $(t,x), (t',x') \in [s - (1 - \delta^2)R^2, s] \times \overline{B}(\xi,(1 - \delta)R)$ whenever $u \in C^\infty([s - R^2,s] \times \overline{B}(\xi,R))$ satisfies $Lu = 0$ in $(s - R^2,s) \times B(\xi,R)$.

Proof: Let $(t,x), (t',x') \in [s - (1 - \delta^2)R^2, s] \times \overline{B}(\xi,(1 - \delta)R)$ with $t' \leq t$ be given, and set $\ell = (t - t')^{1/2} \vee |x - x'|$. If $\ell \geq \delta R$, then there is nothing to do. If $\ell < \delta R$, choose $k \in Z^+$ so that $\delta^{k+1} \leq \ell/R \leq \delta^k$. Then $[t - (\ell\delta^{-k+1})^2,t] \times B(x,\delta^{-k+1}\ell)$ $[s - R^2,s] \times \overline{B}(\xi,R)$ and $(t,x') \in [t - \ell,t] \times B(x,\ell)$. Hence:

$$|u(t,x) - u(t',x')| \leq \mathrm{Osc}(u;t,x,\ell) \leq \rho^{k-1}\, \mathrm{Osc}(u;t,x,\delta^{-k+1}\ell)$$

$$\leq 2\|u\|_{C_b([s - R^2,s] \times \overline{B}(\xi,R))}\, \rho^{k-1}.$$

Finally, define β by $\rho = \delta^{\beta}$. Then

$$|u(t,x) - u(t',x')| \leq 2\rho^{-2}\|u\|_{C_b([s - R^2,s] \times \overline{B}(\xi,R))}(\delta^{k+1})^{\beta}$$

$$\leq 2\rho^{-2}\|u\|_{C_b([s - R^2,s] \times \overline{B}(\xi,R)}\ell^{\beta}.$$

We can now prove the following statement of the Harnack principle for the operator L. Although our statement is not precisely the one given by Moser, it can be used to easily prove his Theorem (2) in [7].

(5.4) Theorem: Let $0 < \alpha < \beta < 1$ and $\gamma \in (0,1)$ be given. Then there is an $M = M(n,\lambda,\alpha,\beta,\gamma) < \infty$ such that for all $(s,x) \in R^1 \times R^N$, all $R > 0$, and all non-negative $u \in C^{\infty}([s - R^2,s] \times \overline{B}(x,R))$ satisfying $Lu = 0$, one has that

$$u(t,y) \leq Mu(s,x)$$

for all $(t,y) \in [s - \beta R^2, s - \alpha R^2] \times \overline{B}(x,\delta R)$.

Proof: By translation and rescalng we may and will assume that $(s,x) = (0,0)$ and $R = 1$. Also, we assume that $u(0,0) = 1$.

From Lemma (5.1) we know that there is an $\varepsilon = \varepsilon(n,\lambda,\alpha) > 0$ such that for all $r \in [-1,\alpha]$ and $\lambda > 0$:

$$1 = u(0,0) \geq \int \Gamma_a^{(0,1)}(0,0;r,\eta)u(r,\eta)d\eta$$

$$\geq \varepsilon\lambda|S(r,\lambda)|$$

where $S(r,\lambda) \equiv \{\eta \in B(0, \frac{1+\delta}{2}): u(r,\eta) \geq \lambda\}$.

Next, let $\rho = \rho(n,\lambda,1/2)$ be the constant in Lemma (5.2) and set $\sigma = (1 - \rho)/2$ and $K = (1 + 1/\rho)/2$. Also define $r(\lambda) = (2/\Omega_n\varepsilon\sigma\lambda)^{1/n}$ for $\lambda > 0$, where $\Omega_n = |B(0,1)|$. Now suppose that $(t,y) \in (-1,-\alpha) \times B(0,\frac{1+\delta}{2})$

and $\lambda > 0$ have the property that $u(t,y) \geq \lambda$ and $[t - 4r(\lambda)^2, t] \times B(y, 2r(\lambda)) \subset [-1,\alpha] \times \overline{B}(0, \frac{1+\delta}{2})$. Since for $r \in [-1,\alpha]$ $|S(r,\lambda\sigma)| \leq 1/\varepsilon\sigma\lambda$ and $|B(y,r(\lambda))| = 2/\varepsilon\sigma\lambda$, there exists an $\eta \in B(y,r(\lambda))$ such that $u(r,\eta) < \sigma\lambda$. Hence, $Osc(u;t,y,r(\lambda)) \geq u(t,y) - u(r,\eta) > (1-\sigma)\lambda$; and so, by Lemma (5.2), $Osc(u;t,y,2r(\lambda)) > \frac{1}{\rho}(1-\sigma)\lambda = K\lambda$. In particular, there exists a $(t',y') \in [t - 4r(\lambda)^2, t] \times B(y, 2r(\lambda))$ such that $u(t',y') \geq K\lambda$.

Finally, define M by the relation

$$r(M) = \left(\frac{(1-\beta) \wedge (1-\delta)}{2}\right)(1 - 1/K^{1/n});$$

and suppose that there were a $(t,y) \in [-\beta,-\alpha] \times B(0,\delta)$ such that $u(t,y) \geq M$. Then, by the preceding paragraph, we could inductively find (t_m,y_m), $m \geq 0$, so that $(t_0,y_0) = (t,y)$, $(t_{m+1},y_{m+1}) \in [t_m - 4r(K^m M), t_m] \times \overline{B}(y_m, r(K^m M)) \subset (-1,-\alpha) \times B(0, \frac{1+\delta}{2})$, and $u(t_m,y_m) \geq K^m M$. But this would mean that u is unbounded in $[-\beta,-\alpha] \times \overline{B}(0, \frac{1+\delta}{2})$, and so no such (t,y) exists.

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