ASYMPTOTIC CONDITIONS FOR THE SOLVABILITY OF A FOURTH ORDER BOUNDARY VALUE PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

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ASYMPTOTIC CONDITIONS FOR THE SOLVABILITY OF A FOURTH ORDER BOUNDARY VALUE PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. This paper concerns the existence of solutions of the fourth order periodic boundary value problem

\[- \frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x, u(x)) = e(x), x \in [0, 2\pi],\]
\[u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,\]

under some non-uniform resonance and non-resonance conditions on the asymptotic behavior of \(u^{-1}g(x, u)\) for \(|u| \to \infty\).

1. Introduction. Fourth order boundary value problems arise in the study of the equilibrium of an elastic beam under an external load, (e.g., see [1], [2], [5], [6], [16]) where the existence, uniqueness and iterative methods to construct the solutions have been studied extensively. The author studied in [7] the following fourth order boundary value problems with periodic boundary conditions:

\[
\frac{d^4u}{dx^4} + f(u)u' + g(x, u) = e(x), x \in [0, 2\pi],
\]
\[u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0
\]

and

\[
- \frac{d^4u}{dx^4} + \alpha u' + g(x, u) = e(x), x \in [0, 2\pi],
\]
\[u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,
\]

where \(f : \mathbb{R} \to \mathbb{R}\) is continuous, \(g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}\) satisfies Caratheodory’s conditions, \(e \in L^1[0, 2\pi]\) and \(\alpha \in \mathbb{R}\). The purpose of this paper is to study the analogue of (1.2) when \(\alpha\) is replaced by \(f(u)\); viz. the boundary value problem

\[
- \frac{d^4u}{dx^4} + f(u)u' + g(x, u) = e(x), \quad x \in [0, 2\pi],
\]
\[u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0.
\]

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under more general conditions on the asymptotic behaviour of \( u^{-1}g(x,u) \) relative to the two first eigen-values 0 and 1 of the linear problem

\[
-d^4u \over dx^4 + \lambda u = 0,
\]

\[ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0.\]

Instead of assuming, like in [7], that \( \limsup u^{-1}g(x,u) \leq \beta < 1 \) \((\beta \in \mathbb{R})\) uniformly for a.e. \( x \in [0,2\pi], |u| \to \infty \) we assume in this paper that there exists a function \( \Gamma : [0,2\pi] \to \mathbb{R} \) with \( \Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty \) where \( \Gamma_0(x) \leq 1 \) for a.e. \( x \in [0,2\pi] \) with strict inequality on a subset of \( [0,2\pi] \) of positive measure, \( \Gamma_1 \in L^1[0,2\pi], \Gamma_\infty \in L^\infty[0,2\pi] \) with \( |\Gamma_1|_{L^1} \) and \( |\Gamma_\infty|_{L^\infty} \) sufficiently small such that

\[
\limsup_{|u| \to \infty} u^{-1}g(x,u) \leq \Gamma(x)
\]

uniformly for a.e. \( x \in [0,2\pi] \). Accordingly, the expression \( \limsup_{|u| \to \infty} u^{-1}g(x,u) \) can cross any number of eigenvalues \( n^4 \) of the linear problem (1.4) as far as those crossing take place in subsets \( [0,2\pi] \) of sufficiently small measure.

The methods and results of this paper are motivated by the paper of Gupta-Mawhin ([8] (see also [12], [13]) for the second order boundary value problem with periodic boundary conditions:

\[
\frac{d^4u}{dx^4} + f(u)u' + g(x,u) = e(x), \quad x \in [0,2\pi],
\]

\[ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0.\]

We present in section 2 some lemmas giving a priori inequalities that are needed to apply degree-theoretic arguments to obtain existence of solutions for the problem (1.3). In section 3, non-resonance conditions for the existence of solutions of (1.3) are studied and in section 4 we study (1.3) when it is at resonance. We study in section 5 the boundary value problem (1.2) when \( g \) satisfies asymptotic conditions (1.5) and obtain a theorem which partially extends the theorem of section 4. This requires a rather different lemma, similar to the second order case ([8]), which makes use of an inequality of E. Schmidt [15] for periodic absolutely - continuous functions. The result of section 5 in an improvement over the result of section 4 when \( \Gamma_0 = \Gamma_\infty = 0 \) and \( f \equiv \alpha \); but still is not as sharp as Theorem 2.4 of [7] when applied to the case of a constant \( \Gamma \). But then theorem 3 of section 5 allows \( u^{-1}g(x,u) \) to cross finitely many eigen-values of (1.4).

We note that in addition to using the classical spaces \( C[0,2\pi], C^k[0,2\pi], L^k[0,2\pi] \) and \( L^\infty[0,2\pi] \) of continuous, \( k \)-times continuously differentiable, measurable real-valued functions whose \( k \)-th power of the absolute value is Lebesgue integrable or measurable
functions that are essentially bounded on $[0, 2\pi]$; we shall use the Sobolev-spaces $H^k[0, 2\pi]$, $(k = 2, 3$ or $4)$ defined by

$$H^k[0, 2\pi] = \{ u : [0, 2\pi] \to \mathbb{R} | u^{(j)} \text{ abs. cont. on } [0, 2\pi], \quad j = 0, 1, \ldots, k - 1, u^{(k)} \in L^2[0, 2\pi] \}.$$  

with the inner product defined by

$$(u, v)_{H^k} = \sum_{j=1}^k \frac{1}{2\pi} \int_0^{2\pi} u^{(j)}(x)v^{(j)}(x) dx + \left( \frac{1}{2\pi} \int_0^{2\pi} u(x) dx \right) \left( \frac{1}{2\pi} \int_0^{2\pi} v(x) dx \right),$$  

and the corresponding norm by $| \cdot |_{H^k}$. We also define, for the sake of convenience, the norm in $L^k[0, 2\pi]$ by

$$|u|_{L^k} = \left( \frac{1}{2\pi} \int_0^{2\pi} |u(x)|^k dx \right)^{\frac{1}{k}}.$$  

We also use the Sobolev-space $W^{4, 1}[0, 2\pi]$ defined by $W^{4, 1}[0, 2\pi] = \{ u : [0, 2\pi] \to \mathbb{R} / u', u'', u''' \text{ abs. cont. on } [0, 2\pi] \}$ with norm

$$|u|_{W^{4, 1}} = \sum_{j=0}^4 \int_0^{2\pi} |u^{(j)}(t)| dt.$$  

2. A Priori Inequalities. For $u \in L^1[0, 2\pi]$, let us write

$$\bar{u} = \frac{1}{2\pi} \int_0^{2\pi} u(x) dx, \quad \bar{u}(x) = u(x) - \bar{u},$$

so that $\int_0^{2\pi} \bar{u}(x) dx = 0$. Let $\bar{H}^2[0, 2\pi] = \{ u \in H^2[0, 2\pi] | \bar{u} = 0 \}$.

**Lemma 1.** Let $\Gamma \in L^1[0, 2\pi]$ be such that, for a.e. $x \in [0, 2\pi]$,

$$\Gamma(x) \leq 1,$$

with the strict inequality holding on a subset of $[0, 2\pi]$ of positive measure. Then there exists a $\delta = \delta(\Gamma) > 0$ such that for all $\bar{u} \in \bar{H}^2[0, 2\pi]$ with $\bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0$,

$$B_\Gamma(\bar{u}) = \frac{1}{2\pi} \int_0^{2\pi} [(\bar{u}'') \Gamma(x) \bar{u}^2(x)] dx \geq \delta |\bar{u}|^2_{\bar{H}^2}.$$
Proof. Using (2.2) and Wirtinger’s inequality [3], we see that, for all $\bar{u} \in \bar{H}^2[0,2\pi]$ with $\bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0$,

\begin{equation}
B_\Gamma(\bar{u}) \geq \frac{1}{2\pi} \int_0^{2\pi} ([\bar{u}''(x)]^2 - \bar{u}^2(x)) dx \geq 0,
\end{equation}

and, moreover,

\begin{equation}
B_\Gamma(\bar{u}) = 0.
\end{equation}

if and only if

\begin{equation}
\bar{u}(x) = A \sin(x + \theta),
\end{equation}

for some $A, \theta \in \mathbb{R}$. But then by (2.5), (2.6) we get

\begin{align*}
0 &= B_\Gamma(\bar{u}) = \frac{1}{2\pi} \int_0^{2\pi} [1 - \Gamma(x)] \bar{u}^2(x) dx \\
&= \frac{A^2}{2\pi} \int_0^{2\pi} [1 - \Gamma(x)] \sin^2(x + \theta) dx,
\end{align*}

so that by our assumption (2.2) on $\Gamma$ we have $A = 0$ and hence $\bar{u} = 0$.

Let us next assume that the conclusion of the lemma is false. Then there exists a sequence $\{\bar{u}_n\}$, $\bar{u}_n \in \bar{H}^2[0,2\pi]$ for every $n = 1, 2, 3, \ldots$ such that

\begin{align}
B_\Gamma(\bar{u}_n) &\to 0 \text{ as } n \to \infty, \\
|\bar{u}_n|_{H^2} &= 1, \text{ for every } n = 1, 2, \ldots.
\end{align}

It now follows from (2.7) and the compact imbedding $H^2[0,2\pi] \hookrightarrow C^1[0,2\pi]$ that there exists a $\bar{u} \in \bar{H}^2[0,2\pi]$ such that

\begin{align}
\bar{u}_n &\to \bar{u} \text{ weakly in } H^2[0,2\pi], \\
\bar{u}_n &\to \bar{u} \text{ in } C^1[0,2\pi].
\end{align}

Now (2.8) implies that $\bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0$ and $|\bar{u}|_{H^2} \leq \liminf_{n \to \infty} |\bar{u}_n|_{H^2}$.

Hence we get that

\begin{equation}
0 \leq B_\Gamma(\bar{u}) \leq \liminf_{n \to \infty} B_\Gamma(\bar{u}_n) = 0.
\end{equation}

It, now, follows from (2.9) and the first part of this proof that $\bar{u} = 0$. Also (2.7)-(2.9) imply that

\begin{align*}
\frac{1}{2\pi} \int_0^{2\pi} [\bar{u}_n''(x)]^2 dx &= B_\Gamma(\bar{u}_n) + \frac{1}{2\pi} \int_0^{2\pi} \Gamma(x) \bar{u}_n^2(x) dx
&\to \frac{1}{2\pi} \int_0^{2\pi} \Gamma(x) \bar{u}^2(x) dx = \frac{1}{2\pi} \int_0^{2\pi} [\bar{u}''(x)]^2 dx,
\end{align*}

so that $\bar{u}_n \to \bar{u}$ in $H^2[0,2\pi]$ and $|\bar{u}|_{H^2} = 1$. We have thus arrived at a contradiction.

Hence the lemma is true. □
Lemma 2. Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_{\infty}$ where $\Gamma_{\infty} \in L^\infty[0,2\pi]$, $\Gamma_1 \in L^1[0,2\pi]$ and $\Gamma_0 \in L^1[0,2\pi]$ is such that $\Gamma_0(x) \leq 1$ for a.e. $x \in [0,2\pi]$ with strict inequality holding on a subset of $[0,2\pi]$ of positive measure. Let $\delta(\Gamma_0) > 0$ be as given by Lemma 1. Then for every $\bar{u} \in \tilde{H}^2[0,2\pi]$ with $\bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0$,

\begin{equation}
B_\Gamma(\bar{u}) \geq [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_{\infty}|_{L^\infty}]\bar{u}^2_{H^2} \tag{2.10}
\end{equation}

Proof. We have

\begin{align*}
B_\Gamma(\bar{u}) &= \frac{1}{2\pi} \int_0^{2\pi} \left[ (\bar{u}''(x))^2 - \Gamma_0(x)\bar{u}^2(x) \right] dx \\
&\quad - \frac{1}{2\pi} \int_0^{2\pi} \Gamma_1(x)\bar{u}^2(x) dx - \frac{1}{2\pi} \int_0^{2\pi} \Gamma_{\infty}(x)\bar{u}^2(x) dx.
\end{align*}

Using, now the fact that $H^2[0,2\pi] \subset C^1[0,2\pi]$ and the well-known inequalities (see e.g. [14])

\[ |\bar{u}|_{L^2} \leq |\bar{u}'|_{L^2} \leq |\bar{u}''|_{H^2}, \quad |\bar{u}|_{L^\infty} \leq \frac{\pi}{\sqrt{3}}|\bar{u}'|_{L^2} \leq \frac{\pi}{\sqrt{3}}|\bar{u}|_{H^2} \]

for $\bar{u} \in \tilde{H}^2[0,2\pi]$ with $\bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0$, as well as Lemma 1, we get that

\[ B_\Gamma(\bar{u}) \geq [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_{\infty}|_{L^\infty}]\bar{u}^2_{H^2} \]

Remark 1. The best value for $\delta(0)$ is easily seen to be $\frac{1}{2}$, so that $B_\Gamma(\bar{u}) \geq (\frac{1}{2} - \frac{\pi^2}{3}|\Gamma_1|_{L^1})\bar{u}^2_{H^2}$ for all $\bar{u} \in \tilde{H}^2[0,2\pi]$ with $\bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0$

Lemma 3. Let $\gamma \in L^1[0,2\pi]$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_{\infty}$ be as in lemma 2, $\delta(\Gamma_0)$ be given by lemma 1. Then for all measurable functions $p(x)$ on $[0,2\pi]$ such that $\gamma \leq \tilde{p}$, $p(x) \leq \Gamma(x)$ a.e. on $[0,2\pi]$, all continuous functions $f : \mathbb{R} \to \mathbb{R}$ and all $u \in W^{4,1}[0,2\pi]$ with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, we have

\begin{equation}
\frac{1}{2\pi} \int_0^{2\pi} \left[ \bar{u} - \bar{u}(x) \right] \left[ -\bar{u}(iv)(x) + f(u(x))u'(x) + p(x)u(x) \right] dx \geq \gamma \cdot \bar{u}^2 + \left[ \delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_{\infty}|_{L^\infty} \right] \bar{u}^2_{H^2} \tag{2.11}
\end{equation}

Proof. For $u \in W^{4,1}[0,2\pi]$ with

\[ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \]

we have, on integrating by parts, and using lemma 2 that

\begin{align*}
\frac{1}{2\pi} \int_0^{2\pi} \left[ \bar{u} - \bar{u}(x) \right] \left[ -u(iv)(x) + f(u(x))u'(x) + p(x)u(x) \right] dx &\geq \tilde{p} \cdot \bar{u}^2 + \frac{1}{2\pi} \int_0^{2\pi} \left[ (\bar{u}''(x))^2 - p(x)\bar{u}^2(x) \right] dx \\
&\geq \gamma \cdot \bar{u}^2 + \left[ \delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_{\infty}|_{L^\infty} \right] \bar{u}^2_{H^2}.
\end{align*}
3. Asymptotic conditions for non-resonance. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Caratheodory’s conditions, viz.,

(i) for each $u \in \mathbb{R}$, the function $x \in [0, 2\pi] \to g(x, u) \in \mathbb{R}$ is measurable on $[0, 2\pi]$,
(ii) for a.e. $x \in [0, 2\pi]$, the function $u \in \mathbb{R} \to g(x, u) \in \mathbb{R}$ is continuous on $\mathbb{R}$, and
(iii) for each $r > 0$, there exists a function $\alpha_r(x) \in L^1[0, 2\pi]$ such that $|g(x, u)| \leq \alpha_r(x)$ for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbb{R}$ with $|u| \leq r$.

**Theorem 1.** Let $\gamma \in L^1[0, 2\pi]$ with $\overline{\gamma} > 0$ be given. Also let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ with $\Gamma_1 \in L^1[0, 2\pi]$, $\Gamma_\infty \in L^\infty[0, 2\pi]$, $\Gamma_0$ measurable on $[0, 2\pi]$, $\Gamma_0(x) \leq 1$ with strict inequality holding on a subset of $[0, 2\pi]$ of positive measure, and $\frac{\pi^2}{3} |\Gamma_1|_{L^1} + |\Gamma_\infty|_{L^\infty} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is given by lemma 1. Assume that the inequalities

$$\gamma(x) \leq \liminf_{|u| \to \infty} u^{-1}g(x, u) \leq \limsup_{|u| \to \infty} u^{-1}g(x, u) \leq \Gamma(x),$$

hold uniformly for a.e. $x \in [0, 2\pi]$.

Then for every, given, $e(x) \in L^1[0, 2\pi]$ the boundary value problem

$$-u^{(iv)}(x) + f(u(x))u''(x) + g(x, u(x)) = e(x), x \in [0, 2\pi],$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$$

has at least one solution.

**Proof.** Let $\eta = \frac{1}{2} \min\{\overline{\gamma}, \delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\} > 0$. Then, by (3.1) we can find an $r > 0$ such that for a.e. $x \in [0, 2\pi]$ and every $u \in \mathbb{R}$ with $|u| \geq r$ we have

$$\gamma(x) - \eta \leq u^{-1}g(x, u) \leq \Gamma(x) + \eta.$$

Next, define $\tilde{\gamma} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\gamma}(x, u) = \begin{cases} u^{-1}g(x, u) & |u| \geq r, \\ r^{-1}g(x, r) & 0 < u < r, \\ -r^{-1}g(x, -r) & -r < u < 0 \\ \Gamma(x) & u = 0. \end{cases}$$

Note that $\tilde{\gamma}(x, u)u$ satisfies Caratheodory’s conditions and, from (3.3),

$$\gamma(x) - \eta \leq \tilde{\gamma}(x, u) \leq \Gamma(x) + \eta$$

for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbb{R}$. Now, define $h : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ by

$$h(x, u) = g(x, u) - \tilde{\gamma}(x, u)u,$$
for \( x \in [0, 2\pi], u \in \mathbb{R} \). We then see that

\[
|h(x, u)| \leq \sup_{|u| \leq r} |g(x, u) - \tilde{g}(x, u)u| \leq \alpha(x),
\]

for \( x \in [0, 2\pi], u \in \mathbb{R} \), where \( \alpha(x) \in L^1[0, 2\pi] \) depends on \( \gamma, \Gamma \) and \( \alpha_r \).

Now, the equation in (3.2) is equivalent to the equation

\[
-u^{(iv)}(x) + f(u(x))u'(x) + \tilde{g}(x, u(x))u(x) + h(x, u(x)) = e(x),
\]

to which we shall apply coincidence degree theory [4,9] in a manner similar to the one used in Theorem 1 of [12]. Let \( X = C^1[0, 2\pi], Z = L^1[0, 2\pi], \) dom \( L = \{ u \in W^4,1[0, 2\pi] : u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \} \).

\[
L : \text{dom} \ L \subset X \to Z, u \to -u^{(iv)},
F : X \to Z, u \to f(u(\cdot))u'(\cdot),
G : X \to Z, u \to \tilde{g}(\cdot, u(\cdot))u(\cdot),
H : X \to Z, u \to h(\cdot, u(\cdot)) - e(\cdot),
A : X \to Z, u \to \tilde{g}(\cdot, 0)u(\cdot) = \Gamma(\cdot)u(\cdot).
\]

It is easy to check that \( F, G, H \) and \( A \) are well-defined and \( L \)-compact on bounded subsets of \( X \) and that \( L \) is a linear Fredholm mapping of index zero. (see Lemma 2.1 of [7]). We consider the homotopy \( \Phi : \text{dom} \ L \times [0, 1] \to Z \) defined by

\[
\Phi(u, \lambda) \equiv Lu + \lambda Fu + (1 - \lambda)Au + \lambda Gu + \lambda Hu,
\]

for \( u \in \text{dom} \ L, \lambda \in [0, 1] \). Now, in order to apply Theorem IV.5 of [9] (see also [10], [11]) it suffices to show that the set of possible solutions, of the family of equations

\[
-u^{(iv)}(x) + \lambda f(u(x))u'(x) + [(1 - \lambda)\Gamma(x) + \lambda \tilde{g}(x, u(x))]u(x)
+ \lambda h(x, u(x)) - \lambda e(x) = 0,
\]

\[
u(0) - u(2\pi) = u'(0) - u'(2\pi) - u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi),
\]

is a priori bounded in \( C^1[0, 2\pi] \) independently of \( \lambda \in [0, 1] \). If \( u \) is a solution of (3.6), then multiplying (3.6) by \( \bar{u} - \bar{u} \), integrating over \([0, 2\pi]\) and using (3.4), (3.5) together with Lemma 3 with \( \Gamma_\infty \) replaced by \( \Gamma_\infty + \eta \) and \( \gamma \) by \( \gamma - \eta \), we get

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} (\bar{u} - \bar{u}(x)) \{-u^{(iv)}(x) + \lambda f(u(x))u'(x)
+ [(1 - \lambda)\Gamma(x) + \lambda \tilde{g}(x, u(x))]u(x) + \lambda h(x, u(x)) - \lambda e(x)\} dx
\geq (\gamma - \eta)\bar{u}^2 + [\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta] |\bar{u}|_{H^2}^2
- (|\alpha|_{L^1} + |e|_{L^1}) |\bar{u} - \bar{u}|_{L^\infty}
\geq \frac{1}{2} \bar{v}^2 + \frac{1}{2} [\delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^\infty} - |\Gamma_\infty|] |\bar{u}|_{H^2}^2 - \beta |u|_{H^2}
\geq \eta |u|_{H^2}^2 - \beta |u|_{H^2}
\]
and hence $|u|_{H^2} \leq \beta/\gamma$ which implies that $|u|_{C^1[0,1]} \leq C$ where $C$ is a constant independent of $\lambda \in [0,1]$, in view of the compact imbedding $H^2[0,2\pi] \subset C^1[0,2\pi]$.

This completes the proof of the Theorem. \qed

4. Asymptotic Conditions at Resonance. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $g : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Caratheodory's conditions.

**Theorem 2.** Let $\Gamma \in L^1[0,2\pi]$ be such that

\[
\limsup_{|u| \to \infty} \frac{g(x,u)}{u} \leq \Gamma(x),
\]

uniformly a.e. in $x \in [0,2\pi]$ and $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty[0,2\pi]$, $\Gamma_1 \in L^1[0,2\pi]$ and $\Gamma_0 \in L^1[0,2\pi]$ are such that $\Gamma_0(x) \leq 1$ for a.e. $x \in [0,2\pi]$, with strict inequality holding on a subset of $[0,2\pi]$ of positive measure and $|\Gamma_\infty|_{L^\infty} + \frac{\pi^2}{3} |\Gamma_1|_{L^1} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is given by lemma 1.

Suppose, further, that there exist real numbers $a, A, r$ and $R$ with $a \leq A$ and $r < 0 < R$ such that

\[
g(x,u) \geq A
\]

for a.e. $x \in [0,2\pi]$ and all $u \geq R$, and

\[
g(x,u) \leq a
\]

for a.e. $x \in [0,2\pi]$ and all $u \leq r$.

Then the periodic boundary value problem

\[
-\frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x,u(x)) = \epsilon(x), x \in [0,2\pi],
\]

\[
u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0
\]

has at least one solution for each given $\epsilon \in L^1[0,2\pi]$ with

\[
a \leq \epsilon \leq A.
\]

**Proof.** Define $g_1 : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ by $g_1(x,u) = g(x,u) - \frac{1}{2}(a + A)$ and $\epsilon_1 \in L^1[0,2\pi]$ by $\epsilon_1(x) = \epsilon(x) - \frac{1}{2}(a + A)$, so that for a.e. $x \in [0,2\pi]$ we have by using (4.2), (4.3), (4.5)

\[
g_1(x,u) \geq \frac{1}{2}(A - a) \geq 0 \text{ if } u \geq R,
\]

\[
g_1(x,u) \leq \frac{1}{2}(a - A) \leq 0 \text{ if } u \leq r,
\]
and
\[(4.8) \quad \frac{1}{2}(e - A) \leq \overline{e}_1 \leq \frac{1}{2}(A - a).\]

Now, the equation in (4.4) in clearly equivalent to
\[(4.9) \quad \frac{d^4 u}{dx^4} + f(u(x))u'(x) + g_1(x, u(x)) = e_1(x).\]

Moreover, we have
\[\limsup_{|u| \to \infty} u^{-1}g_1(x, u) \leq \Gamma(x),\]
uniformly a.e. in $x \in [0, 2\pi]$ and for $|u| \geq \max(R, -r)$. a.e. $x \in [0, 2\pi]$, $u^{-1}g_1(x, u) \geq 0$. So that $\Gamma(x) \geq 0$ for a.e. $x \in [0, 2\pi]$.

Let, now $\eta = \frac{1}{3}\delta(G_0) - \frac{\pi^2}{3}\Gamma_1|L^1| - |\Gamma_\infty|L^\infty > 0$. Then, there exists an $r_1 > 0$ such that for a.e. $x \in [0, 2\pi]$ and for all $u \in \mathbb{R}$, $|u| \geq r_1$, we have
\[(4.10) \quad 0 \leq u^{-1}g_1(x, u) \leq \Gamma(x) + \eta.\]

Proceeding as in the proof of Theorem 1 (of Section 3) we can write the equation (4.9) in the equivalent form
\[(4.11) \quad \frac{d^4 u}{dx^4} + f(u(x))u'(x) + \tilde{\gamma}(x, u(x))u(x) + h(x, u(x)) = e_1(x),\]

where $0 \leq \tilde{\gamma}(x, u) \leq \Gamma(x) + \eta$, $|h(x, u)| \leq \alpha(x)$, for a.e. $x \in [0, 2\pi]$, all $u \in \mathbb{R}$ and some $\alpha \in L^1[0, 2\pi]$. Once again, degree arguments will ensure the existence of a solution for (4.4) if the set of all possible solutions of the family of equations
\[(4.12) \quad \frac{d^4 u}{dx^4} + \lambda f(u(x))u'(x) + [(1 - \lambda)(\Gamma(x) + \eta) + \lambda \tilde{\gamma}(x, u(x))u(x)
+ \lambda h(x, u(x)) = \lambda e_1(x), \lambda \in [0, 1],\]

$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, is, a priori, bounded in $C^1[0, 2\pi]$ independently of $\lambda \in [0, 1]$. If, now, $u(x)$ is a possible solution of (4.12) for some $\lambda \in [0, 1]$, then integrating the equation in (4.12) over $[0, 2\pi]$ after multiplying it by $\overline{u} - \hat{u}$, we get on using Lemma 3 with $\gamma = 0$, and $\Gamma_\infty$ replaced by $\Gamma_\infty + \eta$,
\[0 = \frac{1}{2\pi} \int_0^{2\pi} [\overline{u} - \hat{u}(x)]\{ - \frac{d^4 u}{dx^4} + \lambda f(u(x))u'(x) + [(1 - \lambda)(\Gamma(x) + \eta) + \tilde{\gamma}(x, u(x))u(x)
+ \lambda h(x, u(x)) - \lambda e_1(x)]\} dx\]
\[\geq \delta(G_0) - \frac{\pi^2}{3}\Gamma_1|L^1| - |\Gamma_\infty|L^\infty - \eta|\overline{u}|^2_{H^2} - (|\alpha|_{L^1} + |e_1|_{L^1})|\overline{u} - \hat{u}|_{L^\infty}\]
\[\geq \eta|\overline{u}|^2_{H^2} - \beta(|\overline{u}| + |\overline{u}|_{H^2}),\]
for some constant $\beta$, independent of $\lambda \in [0, 1]$. Hence,

\begin{equation}
|\tilde{u}|_{H^2}^2 \leq (\beta / \eta) (|\tilde{u}| + |\tilde{u}|_{H^2}).
\end{equation}

Next, we get on integrating the equation in (4.12) over $[0, 2\pi]$,

\begin{equation}
\left( 1 - \frac{1}{2\pi} \int_0^{2\pi} e_{12}(x) + \frac{1}{2\pi} \lambda \int_0^{2\pi} [g_1(x, u(x)) - e_1(x)] dx \right) = 0.
\end{equation}

If, now, $u(x) \geq R$ for all $x \in [0, 2\pi]$ then (4.6), (4.8) imply that $(1 - \lambda)(\Gamma + \eta)R \leq 0$, contradicting $\Gamma + \eta \geq \eta > 0$. Similarly $u(x) \leq r$ for all $x \in [0, 2\pi]$ leads to a contradiction. So there must exist a $\tau \in [0, 2\pi]$ such that

\[ r < u(\tau) < R. \]

It is then easy to see from $u(x) = u(\tau) + \int_{\tau}^x u'(s)ds$ that

\begin{equation}
|\tilde{u}| \leq \max(R, -r) + |\tilde{u}|_{H^2}
\end{equation}

(4.13) and (4.15) now imply that

\[ |\tilde{u}|_{H^2}^2 \leq (2\beta / \eta)|\tilde{u}|_{H^2} + (\beta / \eta) \cdot \max(R, -r), \]

so that there exists a constant $\rho$, independent of $\lambda \in [0, 1]$ such that

\begin{equation}
|\tilde{u}|_{H^2} \leq \rho.
\end{equation}

Finally (4.15) and (4.16) imply that there is a constant $C$, independent of $\lambda \in [0, 1]$ such that

\[ |u|_{H^2} \leq C \]

which implies that $|u|_{C^1} \leq C_1$, for some constant $C_1$, independent of $\lambda \in [0, 1]$.

This completes the proof of Theorem 2. \[ \]

**Remark 2.** If we take $f(u) \equiv \alpha, \alpha \in \mathbb{R}$ and $\Gamma(x) = \beta < 1$, (i.e. $\Gamma_0 = \beta, \Gamma_1 = \Gamma_\infty = 0$) in theorem 2 above we get Theorem 2.4 of [7] as a corollary to theorem 2.

5. An inequality for a linear fourth order operator with periodic boundary conditions. We obtain a partial extension of Theorem 2 of section 4 when $f$ is a constant function and $\Gamma_0 = \Gamma_\infty = 0$. We need the following lemma which gives an inequality for a linear fourth order operator with periodic boundary conditions.
Lemma 4. Let \( \alpha \in \mathbb{R}, \ e \in L^1[0, 2\pi], \ \Gamma \in L^1[0, 2\pi] \) with \( \bar{\Gamma} \geq 0 \). Then every possible solution \( u(x) \) of the problem

\[
-\frac{d^4 u}{dx^4} + \alpha u'(x) + p(x)u(x) = e(x), \quad x \in [0, 2\pi],
\]

\( u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0, \)

with \( p \in L^1[0, 2\pi] \) such that

\[
\overline{p} \leq \bar{\Gamma}, \quad 0 \leq p(x)
\]

for a.e. \( x \in [0, 2\pi] \) satisfies the inequality

\[
(1 - \frac{\pi^2}{4} \bar{\Gamma}) \frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 \leq 2|e|_{L^1} \frac{d^4 u}{dx^4} + \alpha u'|_{L^1}^2 + \overline{\Gamma} |e|_{L^1} + 3|e|_{L^1}^2
\]

Proof. Let \( p \in L^1[0, 2\pi] \) be as above and \( u(x) \) be a solution of (5.1). Then, on multiplying the equation in (5.1) by \( \frac{u(x)}{2\pi} \) and integrating over \([0, 2\pi]\) we get

\[
-\frac{1}{2\pi} \int_0^{2\pi} (u''(x))^2 + \frac{1}{2\pi} \int_0^{2\pi} p(x)u^2(x)dx = \frac{1}{2\pi} \int_0^{2\pi} e(x)u(x)dx.
\]

Since, now \( \overline{p} \leq \bar{\Gamma} \) we have, by using Schwarz’s inequality

\[
(\frac{1}{2\pi} \int_0^{2\pi} |p(x)u(x)|dx)^2 \leq (\frac{1}{2\pi} \int_0^{2\pi} p(x)dx)(\frac{1}{2\pi} \int_0^{2\pi} p(x)u^2dx)
\]

\[
\leq \bar{\Gamma}(\frac{1}{2\pi} \int_0^{2\pi} p(x)u^2(x)dx),
\]

and hence, using the equation in (5.1),

\[
(\frac{1}{2\pi} \int_0^{2\pi} |e(x) + \frac{d^4 u}{dx^4} - \alpha u'|dx)^2 \leq \bar{\Gamma}(\frac{1}{2\pi} \int_0^{2\pi} p(x)u^2(x)dx)
\]

We next apply an inequality of E. Schmidt [15] (see also [8]) to \( u''' - \alpha \ddot{u} \) to get

\[
\frac{1}{2\pi} \int_0^{2\pi} [u''' - \alpha \ddot{u}]^2 dx = \frac{1}{2\pi} \int_0^{2\pi} (u''')^2 dx + \frac{\alpha^2}{2\pi} \int_0^{2\pi} \ddot{u}^2 dx
\]

\[
\leq \frac{\pi^4}{4}(\frac{1}{2\pi} \int_0^{2\pi} |\frac{d^4 u}{dx^4} - \alpha u'|dx)^2.
\]
Now, we get from (5.4), (5.6) and (5.7) that

\[ \Gamma^{-1} \left( \frac{1}{2\pi} \int_0^{2\pi} |e(x) + \frac{d^4 u}{dx^4} - \alpha u'| dx \right)^2 + \frac{1}{2\pi} \int_0^{2\pi} (u''')^2 dx + \frac{\alpha^2}{2\pi} \int_0^{2\pi} \tilde{u}^2 dx \]

\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \tilde{p}(x) u^2 dx + \frac{\pi^2}{4} \left( \frac{1}{2\pi} \int_0^{2\pi} |\frac{d^4 u}{dx^4} - \alpha u'| dx \right)^2 \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} (u'')^2 dx + \frac{\pi^2}{4} \left( \frac{d^4 u}{dx^4} - \alpha u \right)_{L^2}^2. \]

Hence,

\[ - \frac{\pi^2}{4} \left( \frac{d^4 u}{dx^4} - \alpha u \right)_{L^2}^2 + \Gamma^{-1} |e(x) + \frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 \leq \frac{1}{2\pi} \int_0^{2\pi} (u'')^2 dx - \frac{\pi^2}{2\pi} \int_0^{2\pi} \tilde{u}^2 dx + \frac{1}{2\pi} \int_0^{2\pi} e(x)u(x) dx \leq |e|_{L^1} \cdot |u|_{L^\infty}, \]

in view of Wintinger’s inequality $|u''|_{L^2} \leq |u'''|_{L^2}$. Finally, then

\[ (1 - \frac{\pi^2}{4} \Gamma) |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 = |\frac{d^4 u}{dx^4} - \alpha u' + e - e|_{L^1}^2 - \frac{\pi^2}{4} \Gamma |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 \]

\[ \leq |e + \frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 + 2|e|_{L^1} |e + \frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 \]

\[ + |e|_{L^1}^2 - \frac{\pi^2}{4} \Gamma |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}^2 \]

\[ \leq 2|e|_{L^1} |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1} + \Gamma |e|_{L^1} \cdot |u|_{L^\infty} + 3|e|_{L^1}^2. \]

Hence the lemma. []

**Theorem 3.** Let $\alpha \in \mathbb{R}$ be given and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Carathéodory’s conditions. Assume that there exists $\Gamma \in L^1[0, 2\pi]$ such that

\[ \limsup_{|u| \to \infty} u^{-1} g(x, u) \leq \Gamma(x) \]

uniformly a.e. on $[0, 2\pi]$ and that $\Gamma < \frac{\pi^2}{4}$. Suppose, further that there exist real numbers $a, A, r, R$ with $a \leq A$ and $r < 0 < R$ such that for a.e. $x \in [0, 2\pi]$, $g(x, u) \geq A$ when $u \geq R$ and $g(x, u) \leq a$ when $u \leq r$. Then the periodic boundary value problem

\[ - \frac{d^4 u}{dx^4} + \alpha u' + g(x, u(x)) = e(x), \quad x \in [0, 2\pi], \]

\[ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \]

(5.8)
has at least one solution for each given \( e \in L^1[0, 2\pi] \) with \( a \leq \bar{e} \leq A \).

Proof. We first define \( g_1 \) and \( e_1 \) as in the proof of Theorem 2 (section 4) so that the equation in (5.8) can be written as

\[
(5.9) \quad -\frac{d^4 u}{dx^4} + \alpha u' + g_1(x, u(x)) = e_1(x),
\]

with \( g_1(x, u) \geq 0 \) when \( u \geq R \) and \( g_1(x, u) \leq 0 \) when \( u \leq r \) for a.e. \( x \in [0, 2\pi] \) and \( \limsup_{|u| \to \infty} u^{-1}g_1(x, u) \leq \Gamma(x) \) uniformly for a.e. \( x \in [0, 2\pi] \). Consequently for a.e. \( x \in [0, 2\pi], \Gamma(x) \geq 0 \). Let \( \eta = \frac{1}{2}[\frac{4}{\pi^2} - \bar{\Gamma}] > 0 \) so that \( \bar{\Gamma} + \eta < \frac{4}{\pi^2} \) and let \( r_1 > 0 \) be such that

\[
(5.11) \quad 0 \leq u^{-1}g_1(x, u) \leq \Gamma(x) + \eta
\]

for a.e. \( x \in [0, 2\pi], |u| \geq r_1 \). Proceeding as in the proof of Theorem 1 (Section 3) we can write (5.9) in the form

\[
(5.12) \quad -\frac{d^4 u}{dx^4} + \alpha u' + \tilde{\gamma}(x, u(x))u(x) + h(x, u(x)) = e_1(x),
\]

where \( 0 \leq \tilde{\gamma}(x, u) \leq \Gamma(x) + \eta, |h(x, u)| \leq \beta(x) \) for a.e. \( x \in [0, 2\pi] \) and all \( u \in \mathbb{R} \) and some \( \beta \in L^1[0, 2\pi] \). The same degree arguments will imply the existence of a solution for (5.8) if the set of possible solutions of the family of equations

\[
(5.13) \quad -\frac{d^4 u}{dx^4} + \alpha u'(x) + [(1 - \lambda)(\Gamma(x) + \eta) + \lambda \tilde{\gamma}(x, u(x))]u(x)
\]

\[
= -\lambda h(x, u(x)) + \lambda e_1(x),
\]

\( u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \), is, a priori, bounded in \( C[0, 2\pi] \) independently of \( \lambda \in [0, 1] \). Let \( u(x) \) be a solution of (5.13) for some \( \lambda \in [0, 1] \).

Since now,

\[
0 \leq (1 - \lambda)(\Gamma(x) + \eta) + \lambda \tilde{\gamma}(x_1u(x)) \leq \Gamma(x) + \eta
\]

for a.e. \( x \in [0, 2\pi] \), with \( \bar{\Gamma} + \eta < \frac{4}{\pi^2} \), and since

\[
|e_1 - h(\cdot, u(\cdot))|_{L^1} \leq |e_1|_{L^1} + |\beta|_{L^1},
\]

it follows from Lemma 4 that

\[
(5.14) \quad [1 - \frac{\pi^2}{4}(\bar{\Gamma} + \eta)]\frac{d^4 u}{dx^4} - \alpha u'|^2_{L^1} \leq 2(|e_1|_{L^1} + |\beta|_{L^1})\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}
\]

\[
+ (\bar{\Gamma} + \eta)(|e_1|_{L^1} + |\beta|_{L^1})|u|_{L^\infty}
\]

\[
+ 3(|e_1|_{L^1} + |\beta|_{L^1})^2
\]

13
Also, we see as in the proof of Theorem 2 (section 4) that there exists a \( r \in [0, 2\pi] \) such that

\[
(5.15) \quad r < u(\tau) < R
\]

Next, we use lemma 2.1 of [7] to deduce the existence of constants \( \delta = \delta_1(\alpha) > 0, \delta_2 = \delta_2(\alpha) > 0 \) such that

\[
(5.16) \quad |\dddot{u}|_{L^\infty} \leq \delta_1 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1} = \delta_1 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}
\]

\[
(5.17) \quad |\dddot{u}'|_{L^\infty} \leq \delta_2 |\frac{d^4 \dddot{u}}{dx^4} - \alpha \dddot{u}'| = \delta_2 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}
\]

for every \( u \in C^3[0, 2\pi] \) with \( u''' \) absolutely continuous and satisfying the periodic boundary conditions in (5.13). Using, next, (5.16) in (5.14) we get

\[
(5.11) \quad [1 - \frac{x^2}{4} (\bar{\Gamma} + \eta)] |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1} \leq (|e_1|_{L^1} + |\beta|_{L^1})(2 + \delta_1 (\bar{\Gamma} + \eta)) |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}
\]

Also, it follows from (5.15), (5.17) that

\[
|u(x)| = |u(\tau) + \int_{\tau}^{x} u'(s)ds| < \max(-r, R) + 2\pi |u'|_{L^\infty}
\]

\[
\leq \max(-r, R) + 2\pi \delta_2 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}
\]

so that

\[
(5.19) \quad |\bar{u}| \leq \max(-r, R) + 2\pi \delta_2 |\frac{d^4 u}{dx^4} - \alpha u'|_{L^1}.
\]

Finally, it follows from (5.16), (5.18), (5.19) that there exist a constant \( \rho \), independent of \( \lambda \in [0, 1] \) such that

\[
|u|_{L^\infty} \leq \rho.
\]

This completes the proof of Theorem 3.

**Remark 3.** In the case when \( \Gamma_0 = \Gamma_\infty = 0 \) and \( f \equiv \alpha \) in Theorem 2, we see that Theorem 3 improves the condition on \( \Gamma \) from \( \bar{\Gamma} < \frac{3}{2\pi^2} \) into \( \bar{\Gamma} < \frac{2}{\pi^2} \). (Note that \( \delta(0) = \frac{1}{2} \) in lemma 1). In this sense Theorem 3 is an extension of Theorem 2. However, if \( \Gamma \) is a constant, then Theorem 3 is not as sharp as Theorem 2.
REFERENCES


