INVERSION OF THE BLOCH TRANSFORM IN MAGNETIC RESONANCE IMAGING USING ASYMMETRIC TWO-COMPONENT INVERSE SCATTERING

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IMA Preprint Series # 532
June 1989
Inversion of the Bloch transform in magnetic resonance imaging using asymmetric two-component inverse scattering

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March 1989

Abstract

In magnetic resonance imaging, the relation between the radio-frequency modulation of the magnetic field and the desired final magnetization state is called the Bloch transform. Selective excitation then amounts to inverting this transform, which is highly nonlinear. Previous attempts to formulate this problem as an inverse scattering problem have restricted attention to solutions using reflectionless potentials. This paper uses fast numerical algorithms for inverse scattering problems to obtain a much larger set of solutions. Numerical examples are included.

ICP Classification Numbers: 8760D, 3325, 0380
1. Introduction

In magnetic resonance imaging (MRI), a magnetic field is applied to align proton spins. Another magnetic field, transverse to the first field, is then modulated using radio-frequency (RF) modulation in such a way that the axes of the proton spins in a selected region are rotated or flipped, relative to those in the rest of the object being imaged. The superposition of the proton spins results in a net magnetization. When the modulation stops, the proton spin axes return to the original aligned direction, radiating at the Larmor frequency; the time constant of this relaxation gives information about the composition of the object in the selected region. For more details on MRI see [1].

To achieve selective excitation (of proton spins) in the object being imaged, the transverse magnetic field must be modulated in such a way that only the proton spins in a thin slice of the object are flipped; the spins in the rest of the object must be unaltered when the modulation stops. Then the radiation and relaxation time information are known to apply solely to the selected thin slice. Typically, the spins are to be flipped 90 or 180 degrees in a thin slice, and not flipped elsewhere; in any case the desired final magnetization state is known.

The relation between the RF modulation pulse and the magnetization state resulting from it is a complicated relation called the Bloch transform [2]. The problem of determining what RF modulation to use to achieve a desired magnetization is thus inversion of the Bloch transform.

Several approaches have been used here. The most straightforward is to linearize the Bloch transform into a Fourier transform [3]. This is the so-called “small tip angle” approximation. An interesting interpretation of this approximation, in the context of the scattering formulation used in this paper, is given in Section 4.4 below. Other approaches have been taken in [4]-[6]; the optimal control problem formulation in [6] is notable.

The most promising approach has been the inverse scattering formulation due to A.
Grunbaum and his co-workers, notably A. Hasenfeld. The inverse Bloch transform can be transformed into a Schrödinger equation inverse potential problem by stereographically projecting the magnetization [7]-[10]. The inverse scattering transform maps solutions of the Schrödinger equation associated with reflectionless potentials to solutions of the Korteweg-de Vries equation [11]. This allows the body of knowledge on the latter equation to be applied to this problem.

A recent paper by Grunbaum [12] transforms the inverse Bloch transform into a two-component wave system inverse scattering problem of Zakharov-Shabat type (see [11]). This is much more promising, since fast algorithms for solving inverse scattering problems for asymmetric two-component systems [13] can now be directly applied to this problem.

This paper applies the fast algorithms of [13] to the two-component wave system inverse scattering formulation of the inverse Bloch transform. The result is a numerical procedure for computing the RF magnetic field modulation needed to achieve a desired final magnetization state. By considering non-reflectionless potentials, a much greater range of solutions is made available.

The paper is organized as follows. In Section 2 we review the Bloch transform, and the reformulation of the inverse Bloch transform as a two-component wave system inverse scattering problem [12]. In Section 3 we review the pertinent results of [13] on solving this inverse scattering problem. In Section 4 we apply the results of Section 3 to the problem posed in Section 2. Section 5 presents some numerical examples of the new procedure. Section 6 concludes by summarizing the results and noting directions for future research.

2. The Bloch transform and inverse scattering

2.1 The Bloch transform

Let \( M(x, y, z, t) \) be the magnetization due to proton spins aligning locally with an imposed magnetic field \( B(x, y, z, t) \). Since the RF magnetic field modulation is short in duration (about 3 ms), the decay terms in the Bloch equations can be neglected, resulting
in
\[
\frac{d}{dt} M(x,y,z,t) = \gamma B(x,y,z,t) \times M(x,y,z,t)
\] (2.1)

where \( \gamma \) is the gyromagnetic ratio of the nucleus.

The magnetic field has three components: a strong (about 10,000 Gauss) constant component in the \( z \)-direction; a component in the \( z \)-direction linearly varying with \( z \) (about 1 Gauss/cm); and a time-varying RF-modulated component in the \( x-y \) plane. Thus
\[
B(x,y,z,t) = B_1(t)(\cos\omega_1(t)i + \sin\omega_1(t)j) + (B_o + Gz)k
\] (2.2)

Here \( B_1(t) \) is the amplitude modulation and
\[
\phi(t) = \omega_1(t) - \omega_L t; \omega_L = \gamma B_o
\] (2.3)

is the phase modulation of the transverse magnetic field in the \( x-y \) plane. Note that \( \omega_L \) is the Larmor frequency of precession of the proton spins at \( z = 0 \).

Following [1] we transform to coordinates rotating at \( \omega_L \):
\[
N(x,y,z,t) = \begin{bmatrix}
\cos \omega_L t & -\sin \omega_L t & 0 \\
\sin \omega_L t & \cos \omega_L t & 0 \\
0 & 0 & 1
\end{bmatrix} M(x,y,z,t)
\] (2.4)

Using (2.2)-(2.4), the Bloch equations (2.1) become
\[
\frac{dN}{dt} = \begin{bmatrix}
0 & \Delta \omega & -\gamma B_1 \sin \phi \\
-\Delta \omega & 0 & \gamma B_1 \cos \phi \\
\gamma B_1 \sin \phi & -\gamma B_1 \cos \phi & 0
\end{bmatrix} N(x,y,\Delta \omega,t)
\] (2.5)

where spatial position along the \( z \)-axis has been replaced with
\[
\Delta \omega = \gamma Gz.
\] (2.6)

In (2.5) \( \Delta \omega = \gamma Gz \) is the difference between the local Larmor frequency at \( z \) and the Larmor frequency \( \omega_L \) at \( z = 0 \). Note that the gradient in the magnetic field along the \( z \)-axis has produced spatial encoding; position along the \( z \)-axis has been replaced with the
offset $\Delta \omega$ in resonant frequency from $\omega_L$. Since $B_0 \approx 10,000$ G and $G \approx 1$ G/cm, $\Delta \omega$ is an offset in kHz from an $\omega_L$ in the MHz range.

Equation (2.5) describes how the rotating magnetization vector $N(x, y, z, t)$ evolves in time for a given modulation $\{B_1(t), \phi(t)\}$. The final magnetization state $N(x, y, z, T)$ is found by integrating (2.5) from $t = 0$ to $t = T$, where $T$ is the length of the RF modulation pulse. Hence (2.5) implements the Bloch transform [2]

$$B\{B_1(t), \phi(t), 0 \leq t \leq T\} \rightarrow N(x, y, z, T)$$ (2.7)

In the sequel, we confine attention to the initial magnetization state

$$M(x, y, z, 0) = N(x, y, z, 0) = [0, 0, -1]^T$$ (2.8)

(i.e., all proton spins are aligned in the $-z$ direction), and to final magnetization states with no $z$ and $y$ variation. This means that all $x$ and $y$ dependencies can be dropped in the sequel, leaving dependencies on offset frequency $\Delta \omega$ (in lieu of $z$) and time $t$.

2.2 Transformation to a two-component wave system

Here we follow [12] and transform (2.5) to a two-component wave system of Zakharov-Shabat type. This is a special case of the asymmetric two-component system treated in [13].

Consider the density matrix $P(t, \Delta \omega)$ defined from $N(t, \Delta \omega) = [N_x, N_y, N_z]^T$ by

$$P(t, \Delta \omega) = \frac{i}{2} \begin{bmatrix} -N_x & N_z - iN_y \\ N_x + iN_y & N_z \end{bmatrix}$$

$$= \frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} N_z + \frac{i}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} N_y + \frac{i}{2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} N_x$$ (2.9)

The set of matrices are the Pauli spin matrices; they show why we pick this particular $P(t, \Delta \omega)$.

Since the system matrix in (2.5) is antisymmetric, $||N(t, \Delta \omega)||$ does not vary with time $t$. Hence the eigenvalues $\pm (i/2) \sqrt{||N(t, \Delta \omega)||}$ of $P(t, \Delta \omega)$ are also independent of $t$. This
implies that the time evolution of $P(t, \Delta \omega)$ can be described by a unitary transformation

$$
P(t, \Delta \omega) = U(t, \Delta \omega) P(0, \Delta \omega) U^H(t, \Delta \omega)$$  \hspace{1cm} (2.10a)

$$
U(t, \Delta \omega) = \begin{bmatrix}
\alpha(t, \Delta \omega) & \beta(t, \Delta \omega) \\
-\beta^*(t, \Delta \omega) & \alpha^*(t, \Delta \omega)
\end{bmatrix}
$$  \hspace{1cm} (2.10b)

$$
|\alpha(t, \Delta \omega)|^2 + |\beta(t, \Delta \omega)|^2 = 1
$$  \hspace{1cm} (2.10c)

At this point we diverge from [12]. Define the matrix $D = (dU/dt)U^{-1}$, so that

$$
\frac{dU}{dt}(t, \Delta \omega) = DU(t, \Delta \omega) = \begin{bmatrix}
d(t, \Delta \omega) & e(t, \Delta \omega) \\
e^*(t, \Delta \omega) & d^*(t, \Delta \omega)
\end{bmatrix} U(t, \Delta \omega).
$$  \hspace{1cm} (2.11)

The symmetries in $D$ can be seen by exchanging rows and columns in (2.11) and using (2.10b). Differentiating (2.10a) with respect to $t$, inserting (2.11) and then (2.9), and finally comparing to (2.5) yields

$$
\frac{d}{dt} \begin{bmatrix}
\beta(t, \Delta \omega) \\
\alpha^*(t, \Delta \omega)
\end{bmatrix} = \begin{bmatrix}
\frac{i\Delta \omega}{2} & r(t) \\
-r^*(t) & -\frac{i\Delta \omega}{2}
\end{bmatrix} \begin{bmatrix}
\beta(t, \Delta \omega) \\
\alpha^*(t, \Delta \omega)
\end{bmatrix}
$$  \hspace{1cm} (2.12a)

$$
r(t) = (i/2)\gamma B_1(t) e^{i\phi(t)}
$$  \hspace{1cm} (2.12b)

Equations (2.12) are an asymmetric two-component wave system of Zakharov-Shabat type [11]. They are a special case of the asymmetric two-component wave system considered in [13] (specifically, $s(z) = -r^*(z)$ in (2.1) of [13]). Note that in (2.12) time $t$ takes the role of spatial position, and resonance offset $\Delta \omega$ takes the role of wavenumber.

2.3 Comments

It is not surprising that an asymmetric two-component wave system arises here. The Schrodinger equation formulation of the Bloch transform used in [7]-[10] has a complex potential with its imaginary part proportional to energy, when phase modulation is allowed [12]. Such potentials are associated with absorbing media [14], and absorbing media are known to be readily handled using asymmetric two-component wave systems [13].

It should also be noted that the inverse scattering formulation used here differs from that used in [15]. Although the two-component system (2.12) can be put into the form of
either a Riccati equation or a Schrodinger equation [13], these equations are different from those appearing in [13]. To see this, note that the Riccati equation associated with (2.12) propagates

\[ R(t, \Delta \omega) = \frac{\beta(t, \Delta \omega)}{\alpha^*(t, \Delta \omega)} = \frac{2\alpha(t, \Delta \omega) \beta(t, \Delta \omega)}{2|\alpha(t, \Delta \omega)|^2} = \frac{N_z(t, \Delta \omega) - iN_y(t, \Delta \omega)}{N_z(t, \Delta \omega) - 1} \tag{2.13} \]

where the last equality follows quickly from (2.9) and (2.10a). However, in [15] the different quantity

\[ \frac{N_z(t, \Delta \omega) + iN_y(t, \Delta \omega)}{N_z(t, \Delta \omega) - 1} \]

was used, and the result of [15] applies only to this formulation.

3. Solution using asymmetric Schur algorithm

In this section we briefly review the results of [13] on asymmetric two-component wave systems that pertain to the Zakharov-Shabat system (2.12). For more details and references, see [13].

3.1 The Zakharov-Shabat wave scattering system

The Zakharov-Shabat wave system

\[ \frac{d}{dz} \begin{bmatrix} D(z, k) \\ U(z, k) \end{bmatrix} = \begin{bmatrix} -ik & -r(z) \\ r^*(z) & ik \end{bmatrix} \begin{bmatrix} D(z, k) \\ U(z, k) \end{bmatrix} \tag{3.1} \]

is a special case of the asymmetric two-component wave system treated in [13]. Here \( D(z, k) \) and \( U(z, k) \) are downgoing and upgoing waves, respectively, \( r(z) \) is the reflectivity function, \( z \) is depth (increasing downward), and \( k \) may be either frequency or wavenumber. The system (3.1) describes the scattering medium illustrated in Figure 1. This medium varies smoothly for \( 0 < z < L \), and it is homogeneous (i.e., \( r(z) = 0 \)) for \( z < 0 \) and \( z > L \).

In the time domain (3.1) becomes the pair of equations

\[ \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right) \bar{D}(z, t) = -r(z)\bar{U}(z, t) \tag{3.2a} \]
\[
\left( \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right) \hat{U}(z, t) = r^*(z) \hat{D}(z, t)
\]  

(3.2b)

where \( \hat{D}(z, t) = \int_{-\infty}^{\infty} D(z, k)e^{ikt}dk \) is the inverse Fourier transform of \( D(z, k) \), and similarly for \( \hat{U}(z, t) \). \( D(z, k) \) and \( U(z, k) \) are considered to be waves since (3.2) describes quantities that propagate in increasing and decreasing depth \( z \) as \( t \) increases. The reflectivity function \( r(z) \) describes how much of each wave is reflected into the other wave at each \( z \). If \( r(z) = 0 \) then the medium is locally homogeneous, and no scattering occurs.

### 3.2 Inverse scattering problem

Now initialize (3.1) with

\[
D(z, k) = e^{-ikz}; \quad U(z, k) = R(k)e^{ikz}, \quad z \leq 0
\]  

(3.3a)

\[
D(z, k) = T(k)e^{-ikz}; \quad U(z, k) = 0, \quad z > L
\]  

(3.3b)

This is the same as initializing (3.2) with

\[
\hat{D}(z, t) = \delta(t - z); \quad \hat{U}(z, t) = \hat{R}(t + z), \quad z \leq 0.
\]  

(3.4a)

\[
\hat{D}(z, t) = \hat{T}(t - z); \quad \hat{U}(z, t) = 0, \quad z > L
\]  

(3.4b)

These equations describe an inverse scattering experiment that consists of probing the medium with an impulsive plane wave \( \delta(t - z) \), incident from above and propagating downward, and getting back a reflection response \( \hat{R}(t) \) that is causal and a transmission response \( \hat{T}(t) \) at the far end of the medium. This is illustrated in Figure 2.

Since \( \hat{R}(t) \) is causal it is clear that \( \hat{D}(z, t) \) and \( \hat{U}(z, t) \) have the forms

\[
\hat{D}(z, t) = \delta(t - z) + \hat{D}(z, t)1(t - z)
\]  

(3.5a)

\[
\hat{U}(z, t) = \hat{U}(z, t)1(t - z)
\]  

(3.5b)

where \( \hat{D}(z, t) \) and \( \hat{U}(z, t) \) are the smooth parts of \( \hat{D}(z, t) \) and \( \hat{U}(z, t) \) (both of which jump at \( t = z \)), and where \( 1(-) \) is the unit step or Heaviside function. Equations (3.5) are simply a statement of causality.
Inserting (3.5) into (3.2) and using a propagation of singularities argument (this amounts to equating coefficients of \( \delta(t - z) \)) yields

\[
\left( \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right) \tilde{D}(z, t) = -r(z) \tilde{U}(z, t) \quad (3.6a)
\]

\[
\left( \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right) \tilde{U}(z, t) = r^*(z) \tilde{D}(z, t) \quad (3.6b)
\]

\[
r^*(z) = -2\tilde{U}(z, z^+) \quad (3.6c)
\]

The derivation of (3.6) from (3.2) is analogous to the derivation of transport equations for a system of partial differential equations.

3.3 The asymmetric Schur algorithm

Discretizing depth \( z \) and time \( t \) into integer multiples of a small constant \( \Delta \) and using forward differences, (3.6) discretizes into

\[
\begin{bmatrix}
\tilde{D}(z + \Delta, t + \Delta) \\
\tilde{U}(z + \Delta, t - \Delta)
\end{bmatrix}
= \begin{bmatrix}
1 & -r(z)\Delta \\
r^*(z)\Delta & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{D}(z, t) \\
\tilde{U}(z, t)
\end{bmatrix} \quad (3.7a)
\]

\[
r^*(z)\Delta = -\tilde{U}(z, z) / \tilde{D}(z, z) \quad (3.7b)
\]

\[
D(0, t) = 0; \quad U(0, t) = \tilde{R}(t) \quad (3.7c)
\]

Equations (3.7) constitute a layer-recursive procedure for reconstructing \( r(z) \) from \( \tilde{R}(t) \). Note the transmission response \( \tilde{T}(t) \) is not needed.

3.4 Comments

1. Equations (3.7) differ from the well-known Schur algorithm only in that the reflectivity functions \( r(z) \) and \( -r^*(z) \) are unequal. This is why a simple forward discretization is used—to identify this inverse scattering problem solution with a well-known, numerically stable algorithm. Indeed, by inserting an additional factor of \( 1/\sqrt{1 + |r(z)\Delta|^2} \) the transformation at each step becomes a multiplication by a unitary matrix (note this additional factor cancels out in (3.7b)).
2. In physical inverse scattering problems, wave systems like (3.1) describe absorbing media [13]:
   a. Example: Acoustic media with constant density and wave speed, but varying absorption (Maxwell model);
   b. Example: Electromagnetic media with constant permittivity and permeability, but varying conductivity.

3. In fact, (3.7) is a special case of the asymmetric Schur algorithm. Even though the reflectivity functions are unequal, we can obviously compute one from the other. Hence reflection data from one end only of the scattering medium is sufficient to reconstruct it, even though the medium is lossy.

4. These types of absorbing media can also be formulated as Schrodinger equation inverse potential problems [14]. The potential turns out to be complex, with an imaginary part linearly proportional to wavenumber \( k \). Again the reflection coefficient \( R(k) \) constitutes sufficient data; however, solution of an integral equation replaces (3.7) [14].

5. The *Born approximation* is commonly used in inverse scattering problems. This is a single-scattering assumption in which only direct or primary scattering events are considered; multiple scattering is neglected. Here this amounts to neglecting the coupling in (3.7a), which simplifies to

\[
-r^*(z) = \tilde{U}(z, z) = \tilde{U}(0, 2z) = \tilde{R}(2z)
\]

(3.8)

i.e., each value of \( \tilde{R}(t) \) is assumed to come directly from a reflection of the impulse \( \delta(t-z) \) from \( -r^*(z) \). This will be used to interpret linearization of the Bloch transform in Section 4.4.

6. (3.7b) follows immediately from (3.7a) by setting \( t = z \) and noting that \( \tilde{U}(z+\Delta, z-\Delta) \) is zero by causality. Note that a factor of two in (3.6c) disappears in the discretization.

4. Solution of the inverse Bloch transform problem
Comparing (2.12) and (3.1), we see that \( \alpha^*(t, \Delta \omega) \) and \( \beta(t, \Delta \omega) \) correspond to \( D(z, k) \) and \( U(z, k) \), respectively, and that \( t \) and \( \Delta \omega \) in (2.12) correspond to \( z \) and \( k \) in (3.1). It is clear that the asymmetric Schur algorithm can be used to compute \( r(t) \) in (2.12), provided that a scattering interpretation can be attached to (2.12) and a reflection response characterizing the desired final magnetization state can be produced.

4.1 Scattering interpretation

The initial magnetization state is given by (2.8). Inserting (2.8) into (2.9) and (2.10), and repeating (2.10c), gives

\[
N_x(t, \Delta \omega) = |\beta(t, \Delta \omega)|^2 - |\alpha(t, \Delta \omega)|^2
\]

\[ (4.1a) \]

\[
N_x(t, \Delta \omega) - iN_y(t, \Delta \omega) = -2\alpha(t, \Delta \omega)\beta(t, \Delta \omega)
\]

\[ (4.1b) \]

\[
|\alpha(t, \Delta \omega)|^2 + |\beta(t, \Delta \omega)|^2 = 1
\]

\[ (4.1c) \]

Equations (4.1) describe the time evolution of \( N(t, \Delta \omega) \) directly in terms of the quantities \( \alpha(t, \Delta \omega) \) and \( \beta(t, \Delta \omega) \) in (2.12), for the initial magnetization state (2.8).

For \( t = 0 \) and \( N(t, \Delta \omega) \) as in (2.8), the solution to (4.1) is

\[
\alpha(0, \Delta \omega) = 1; \quad \beta(0, \Delta \omega) = 0
\]

\[ (4.2) \]

This is illustrated in Figure 3, from which we see that \( \alpha^*(t, \Delta \omega) \) and \( \beta(t, \Delta \omega) \) are the Jost solutions for a scattering experiment with an impulse incident from the right. Thus, although we have successfully attached a scattering interpretation to (2.12), we also seem to require the (unknown) Jost solutions at \( t = T \). We now show that this is not so.

4.2 Initialization at \( t = T \)

Let the medium represented by (2.12) be probed from the right with an impulse, resulting in a reflection response \( R(T, \Delta \omega) \) and a transmission response \( T(T, \Delta \omega) \) (this is as in Figure 2, but with left and right interchanged). By linearity, we can divide the
responses in Figure 3 by $\alpha^*(t, \Delta \omega)$. Comparing the result with Figure 2 (with left and right interchanged), we have

$$R(T, \Delta \omega) = \frac{\beta(T, \Delta \omega)}{\alpha^*(T, \Delta \omega)} = \frac{|\beta(T, \Delta \omega)|}{|\alpha(T, \Delta \omega)|} \exp \left[ \text{ARG}[\alpha(T, \Delta \omega)] + \text{ARG}[\beta(T, \Delta \omega)] \right] j$$

$$T(T, \Delta \omega) = \frac{1}{\alpha^*(T, \Delta \omega)}$$

(4.3)

This shows that it is NOT necessary to find the Jost solutions at $t = T$; merely specifying $R(T, \Delta \omega)$, rather than $\alpha^*(T, \Delta \omega)$ and $\beta(T, \Delta \omega)$ separately, is sufficient to compute the $r(t)$.

Note that we have assumed $\alpha^*(t, \Delta \omega)$ has no zeros in the lower half of the complex $\Delta \omega$ plane; this is tantamount to assuming there are no bound states. In practice, we will choose reflection responses without bound states, to give results different from [7]-[10].

4.3 Summary of Procedure

Given a desired final magnetization state $N(T, \Delta \omega)$, the Schur algorithm is initialized at $t = T$ as follows:

1. Solve (4.1) for $|\alpha(T, \Delta \omega)|$, $|\beta(T, \Delta \omega)|$, and $\text{ARG}[\alpha(T, \Delta \omega)] + \text{ARG}[\beta(T, \Delta \omega)]$.
2. Compute $R(T, \Delta \omega)$ using (4.3). Note that the three quantities computed above uniquely specify $R(T, \Delta \omega)$.
3. Compute

$$\tilde{R}(\tau) = \mathcal{F}^{-1}_{\Delta \omega \rightarrow \tau} \{R(T, \Delta \omega)\}$$

(4.4)

where $R(T, \Delta \omega)$ is analytically extended into the complex $\Delta \omega$ plane so that $\tilde{R}(\tau)$ is causal. Mathematically, this is similar to an inverse Laplace transform; in practice, it is easy to choose $R(T, \Delta \omega)$ so that this can be avoided (see Section 5).

4. Initialize the Schur algorithm using $\tilde{R}(\tau)$.
5. Run the Schur algorithm, computing $r(t)$. Reverse them in time (replace $r(t)$ with $r(T - t)$).
6. Compute the amplitude modulation $B_1(t)$ and phase modulation $\phi(t)$ achieving the desired $N(T, \Delta\omega)$ using (2.12b).

The only difficulty is that arbitrary selection of $\alpha(T, \Delta\omega)$ and $\beta(T, \Delta\omega)$ can lead to $r(t)$ (and therefore modulations) that are impractically large, or of too long a duration. Hence some judgment must be exercised in selecting the exact shape of the final $M(T, \Delta\omega)$ so that the $r(t)$ leading to it constitute a practical modulation.

Note that this procedure allows tradeoffs to be identified quickly:

1. If the selective excitation region is too narrow, the $r(t)$ will have long duration (infinitely long for the case of an infinitely narrow slice);
2. The larger the angle of flip, the larger $r(t)$, hence the amplitude modulation $B_1(t)$, will be.

4.4 The Born approximation to the Bloch transform

Applying the Born approximation (see Section 3) to this procedure is most illuminating. Applying (3.8) directly, and using (4.1) and (4.3), gives

\[
\frac{(i/2)\gamma B_1(t)e^{-i\phi(t)}}{r^*(t)} = \mathcal{F}^{-1}\{R(T, \Delta\omega)\} = \mathcal{F}^{-1}\left\{\frac{\beta(T, \Delta\omega)}{\alpha^*(T, \Delta\omega)}\right\}
\]

\[
= \mathcal{F}^{-1}\left\{\frac{2\alpha(T, \Delta\omega)\beta(T, \Delta\omega)}{2|\alpha(T, \Delta\omega)|^2}\right\} = \mathcal{F}^{-1}\left\{\frac{N_x(T, \Delta\omega) - iN_y(T, \Delta\omega)}{N_z(T, \Delta\omega) + 1}\right\}
\]

(4.5)

This shows that the inverse Bloch transform can be approximated by an inverse Fourier transform. This is the same result obtained in [2] and [3] by linearizing a series expansion.

However, the Born approximation interpretation lends more physical insight—examination of the $r(t)$ obtained using the Born approximation can be used to decide whether multiple reflections in (2.12) will be significant enough to warrant using the Schur algorithm instead of the Born approximation.

Note that the $r(t)$ are directly proportional to the amplitude modulation $B_1(t)$. Note also that the larger $N_x(T, \Delta\omega) - iN_y(T, \Delta\omega)$ is, the bigger the $r(t)$ will be. Hence for large pulse amplitudes, or large tip angles, the Schur algorithm should be used.
5. Numerical examples

Two numerical examples of the procedure are given. The first example is a 90 degree rotation of proton spins in a thin slice. The second example is a 180 degree rotation of proton spins in a thin slice.

5.1 Example #1: Initialization

The goal here is to achieve a final magnetization state

\[ N(T, \Delta \omega) = \begin{cases} 
[0, 0, -1]^T, & \text{for most } \Delta \omega; \\
[1, 0, 0]^T, & \text{for } \Delta \omega \approx \Delta \omega_0.
\end{cases} \] (5.1)

where \( \Delta \omega_0 \) is the slice selected. This corresponds to a 90 degree rotation from the z-direction to the x-direction. Inserting (5.1) into (4.1) results in

\[ |\alpha(T, \Delta \omega)| = 1; \quad |\beta(T, \Delta \omega)| = 0 \text{ for most } \Delta \omega \]

\[ |\alpha(T, \Delta \omega)| = |\beta(T, \Delta \omega)| = 0.7071 \text{ for } \Delta \omega \approx \Delta \omega_0. \]

\[ ARG[\alpha(T, \Delta \omega)] + ARG[\beta(T, \Delta \omega)] = 0. \] (5.2)

This leads to a reflection response

\[ R(T, \Delta \omega) = \begin{cases} 
0, & \text{for most } \Delta \omega; \\
1, & \text{for } \Delta \omega \approx \Delta \omega_0.
\end{cases} \] (5.3)

\( r(t) \) depends on the exact shape of \( R(T, \Delta \omega) \) near \( \Delta \omega_0 \).

Suppose first that \( R(T, \Delta \omega) \) is constant at one in a region of finite width in \( \Delta \omega \) and centered at the origin (this corresponds to selective excitation of a slice of finite thickness at \( z = 0 \)). Then applying the Born approximation (4.5) to (5.3) immediately gives the sinc function which is generally used to achieve a 90 degree flip [3].

Now we explore another possible shape for \( R(T, \Delta \omega) \) near \( \Delta \omega_0 \). The medium was discretized in \( \Delta \omega \) into 32 slices. The second-order autoregressive process spectrum

\[ \beta(T, \Delta \omega) = 0.01/(0.99 + 1.9801e^{j\Delta \omega} + 0.99e^{2j\Delta \omega}) \] (5.4)
has poles at -0.99 and -1.01, so there is a sharp peak at $\Delta \omega = \pi$. This corresponds to selective excitation in the \textit{exact middle} of the medium. $\alpha(T, \Delta \omega)$ was then computed from (4.1c), and $R(T, \Delta \omega)$ computed using (4.3); this results in a $R(T, \Delta \omega)$ with a smoother peak than (5.3).

\textit{5.2 Example #1: Results}

The results are shown in Figures 4. Figure 4a is the z-component of the final magnetization state $N(T, \Delta \omega)$. Note that it is indeed minus one for most $\Delta \omega$, and zero for a thin slice of $\Delta \omega$ in the middle. The x-component of $N(T, \Delta \omega)$ is zero for most $\Delta \omega$ and one for a thin slice of $\Delta \omega$ near zero, and is not plotted. Note that the excitation is indeed quite selective in $x$. Although we do not show it here, the time evolution of $N(t, \Delta \omega)$ is a gradual reduction in the z-component in the middle, and a corresponding growth in the x-component.

The real part of the reflectivity function $r(t)$ outputted by the Schur algorithm is plotted in Figure 4b (the imaginary part is similar). Note that the envelope starts at zero, increases gradually, and then goes back down to zero. Use of an $R(T, \Delta \omega)$ sharper than that defined from (5.4) results in a longer $r(t)$. Use of an $R(T, \Delta \omega)$ wider than this results in a sinc envelope similar to the results of [3].

The rapid oscillation of $r(t)$ in Figure 4b is due to the selective excitation in the middle of the medium. To see this, consider matters in the Born approximation. Then a shift $\Delta$ in the $\Delta \omega$ domain corresponds to a modulation $e^{i\Delta t}$ in $r(t)$; the bigger the shift, the faster the modulation. Note that selective excitation of any slice $\Delta \omega_o \neq 0$ will require complex $r(t)$, i.e., phase modulation.

\textit{5.3 Example #2: Initialization}

The goal is now to achieve a final magnetization state

\begin{equation}
N(T, \Delta \omega) = \begin{cases} 
[0,0,-1]^T, & \text{for most } \Delta \omega; \\
[0,0,1]^T, & \text{for } \Delta \omega \approx \Delta \omega_o.
\end{cases}
\end{equation}

(5.5)
where $\Delta \omega_o$ is the slice selected. This corresponds to a 180 degree rotation from the z-direction to the x-direction. Inserting (5.5) into (4.1) results in

$$|\alpha(T, \Delta\omega)| = 1; \quad |\beta(T, \Delta\omega)| = 0 \text{ for most } \Delta\omega$$

$$|\alpha(T, \Delta\omega)| = 0; \quad |\beta(T, \Delta\omega)| = 1 \text{ for } \Delta\omega \approx \Delta\omega_o$$

$$\text{ARG}[\alpha(T, \Delta\omega)] + \text{ARG}[\beta(T, \Delta\omega)] = \text{irrelevant.} \quad (5.6)$$

This leads to a reflection response

$$R(T, \Delta\omega) = \begin{cases} 0, & \text{for most } \Delta\omega; \\ \infty, & \text{for } \Delta\omega \approx \Delta\omega_o. \end{cases} \quad (5.7)$$

Obviously $R(T, \Delta\omega)$ in (5.7) cannot be realized exactly. We use the two-sided geometric spectrum

$$\beta(T, \Delta\omega = n\Delta) = 0.99 	imes (0.5)^{|16-n|}, \quad n = 0 \ldots 32 \quad (5.8)$$

(recall $\Delta\omega$ has been discretized into 32 slices, so 16 is in the middle of the medium). $\alpha(T, \Delta\omega)$ is computed from (4.1c) and $R(T, \Delta\omega)$ computed using (4.3), as before. This results in a $R(T, \Delta\omega)$ with a very sharp peak in the middle.

### 5.4 Example #2: Results

The results are shown in Figures 5. Figure 5a is the z-component of the final magnetization state $N(T, \Delta\omega)$. Note that it is indeed minus one for most $\Delta\omega$, and one for a thin slice of $\Delta\omega$ in the middle. The transverse components are all zero, except in the transition region in the middle where $N_x(T, \Delta\omega) \approx 0$. Note that the excitation is indeed quite selective in $z$.

The real part of the reflectivity function $r(t)$ outputted by the Schur algorithm is plotted in Figure 5b (the imaginary part is again similar). Note that the envelope shape is considerably different from Example #1, and the amplitudes are much larger. Applying the Born approximation (4.5) to (5.8) gives roughly the same shape, but there are differences.
The rapid oscillation of $r(t)$ is again due to the selective excitation in the middle of the medium.

6. Conclusion

A new procedure has been proposed for inverting the Bloch transform in magnetic resonance imaging. Following [12], the Bloch transform is recast as a Zakharov-Shabat two-component wave system, in which the reflectivity function $r(t)$ is related to the modulation needed to achieve a desired final magnetization state $M(T, \Delta \omega)$. This is a special case of the inverse scattering problem for asymmetric two-component wave systems treated in [13]. Hence the asymmetric Schur algorithm from [13] can be applied to this problem. Due to the special form of the Zakharov-Shabat system, reflection data from only one side is sufficient to reconstruct $r(t)$, even though the scattering medium represented by the system (NOT the medium be magnetic resonance imaged) is lossy.

There is considerable grounds for further work. The problem of selecting an exact form of $R(T, \Delta \omega)$ that will lead to a "reasonable" $r(t)$ is still not pinned down, although the Born approximation helps—we need Fourier transform pairs that are narrow in both time and frequency. A more interesting problem is considering initial magnetization states $M(x, y, z, 0)$ different from $[0, 0, -1]^T$—what set of initial states can be treated, and what set of final states can be attained from it? The method proposed here will work on any $M(x, y, z, 0)$ invariant in $x$ and $y$ (all we need is $\alpha(0, \Delta \omega) = 1$ and $\beta(0, \Delta \omega) = 0$), but it is not clear what final states correspond to realizable reflection responses.

Acknowledgments

It is a pleasure to acknowledge the hospitality of the Institute for Mathematics and its Applications at the University of Minnesota, where the author learned about this problem. Several conversations with Andy Hasenberg and Alberto Grunbaum formulated the problem. Der-Shan Luo performed the computer simulations. This work was supported in part by the Institute for Mathematics and its Applications, and in part by the National
Science Foundation under grant #MIP-8858082.

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Figure headings

1. An infinitesimal section of the Zakharov-Shabat wave system (3.1).

2. An inverse scattering experiment with probing from the left, resulting in reflection response $R(k)$ and transmission response $T(k)$.

3. The inverse scattering experiment associated with the Bloch transform. $\alpha(T, \Delta \omega)$ and $\beta(T, \Delta \omega)$ are as defined in (2.12).

4a. The z-component of the final magnetization state $N(T, \Delta \omega)$, plotted against slice number (discretized $\Delta \omega$) for example #1.

4b. The real part of the reflectivity function $r(t)$ outputted by the Schur algorithm, for example #1. The envelope represents the amplitude modulation, and the rapid oscillation represents the cosine of the phase modulation.

5a. The z-component of the final magnetization state $N(T, \Delta \omega)$, plotted against slice number (discretized $\Delta \omega$) for example #2.

5b. The real part of the reflectivity function $r(t)$ outputted by the Schur algorithm, for example #2. The envelope represents the amplitude modulation, and the rapid oscillation represents the cosine of the phase modulation.
FIG. 1
FIG. 2

FIG. 3
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