DEGENERATE HAMILTON–JACOBI–BELLMAN EQUATIONS IN A BOUNDED DOMAIN

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IMA Preprint Series # 549
August 1989
DEGENERATE HAMILTON–JACOBI–BELLMAN EQUATIONS
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The Dirichlet problem for a class of degenerate Hamilton–Jacobi–Bellman equations in a bounded convex domain is considered. It is proved that there exists a unique viscosity solution and that it belongs to $W^{1,\infty}(\Omega) \cap W^{2,p}_{\text{loc}}(\Omega)$ for any $p < \infty$. The class of equations includes, for example,

$$\max \left\{ \sum_{i=1}^{k} a_i \frac{\partial^2 u}{\partial x_i^2}, \sum_{j=1}^{n} b_j \frac{\partial^2 u}{\partial x_j^2} \right\} + f(x) = 0$$

where $a_i, b_j$ are positive constants and $l \leq k + 1$.

§1. Introduction. The Dirichlet problem for the Hamilton–Jacobi–Bellman equation (HJB) in a bounded domain $\Omega$ in $\mathbb{R}^n$ has the form

$$(1.1) \quad \sup_{\omega \in A} [A(\omega)u(x) + f(x, \omega)] = 0 \quad \text{in} \quad \Omega,$$

$$(1.2) \quad u = g \quad \text{on} \quad \partial \Omega$$

where

$$A(\omega) = \sum_{i,j=1}^{n} a_{ij}(x, \omega) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, \omega) \frac{\partial}{\partial x_i} + c(x, \omega)$$

is a family of elliptic operators with parameter $\omega$ varying in a closed subset $A$ of a euclidean space $\mathbb{R}^n$. In case the $A(\omega)$ are uniformly elliptic, with respect to both $x \in \overline{\Omega}$ and $\omega \in A$, there exists a unique solution of (1.1), (1.2) which belongs to $C^{2+\alpha}(\Omega)$ (under some regularity conditions on the coefficients of $A(\omega), f, \partial \Omega$ and $g$); see [13] [5] [3] [9]. When some of the elliptic operators $A(\omega)$ are degenerate the known results are not as complete.

In case $\Omega = \mathbb{R}^n$, $W^{2,p}$ and $W^{2,\infty}$ regularity results have been obtained by Krylov [8] and Lions [10] [12] under some restrictions on the coefficients of $A(\omega)$. For a bounded domain $\Omega, W^{2,\infty}$ regularity was established by Lions [11] provided $A(\omega)$ is uniformly elliptic for all $x \in \partial \Omega, \omega \in A$, and $c(x, \omega) \geq \lambda_0 > 0$, $\lambda_0$ sufficiently large.

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Krylov [9; Chap. 8] has developed a general method for proving $W^{2,\infty}$ regularity for (1.1) in case of degenerate $A(\omega)$. His method requires the construction of a barrier function satisfying very complicated elliptic differential inequality. He has carried out the construction only in case $\Omega$ is a ball (Theorem 8.1 in [9]), with $A(\omega)$ any degenerate elliptic operators with constant coefficients $A(\omega) = \sum a_{ij}(\omega) \frac{\partial^2}{\partial x_i \partial x_j}$, satisfying:

$$0 < c \leq \sum_{i=1}^{n} a_{ii}(\omega) \leq C < \infty, \quad \sup_{\omega} \Sigma a_{ij}(\omega) \xi_i \xi_j > 0 \quad \forall \xi \neq 0.$$

Recently Mandelbaum, Shepp and Vanderbei [16] considered the special case

(1.3) \[ \max \left( \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2} \right) = 0 \quad \text{in} \ \Omega, \]

(1.4) \[ u = g \quad \text{on} \ \partial \Omega \]

where $\Omega$ is a rectangle $0 < x_1 < a$, $0 < x_2 < b$. The problem arises as a stochastic optimization problem whereby one is allowed to switch from one dimensional Brownian motion to another without penalty, and the payoff is the value of $g(x(\tau))$ at the exit time $\tau$ from $\Omega$. Assuming that $g$ is either linear or strictly concave along each side of $\partial \Omega$, they determined the optimal switching strategy and, as by-product, concluded that $u$ is piecewise in $C^2$.

The present work is aimed at generalizing the regularity result of Mandelbaum, Shepp and Vanderbei to the case where $\Omega$ is any 2-dimensional convex domain; when $\Omega$ is a ball, $W^{2,\infty}$ regularity was already established by Krylov [9; Th., 8.1] as mentioned above.

Some of our results actually apply also to

(1.5) \[ \max \left( \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_n^2} \right) = 0 \quad \text{in} \ \Omega, \]

where $\Omega$ is a convex domain in $\mathbb{R}^n$, as well as to more general HJB equations.

In §2 we prove that there exists a unique viscosity solution of (1.5), (1.4) and it belongs to $W^{1,\infty}(\Omega)$ (\Omega need to be convex).

In §3 we establish boundary estimates on mixed normal–tangential derivatives of the solution $u$ to (1.5), (1.4); here we use the method of Krylov [9; §8.6], and our main effort is in establishing a barrier function.

Using the boundary estimates derived in §3, we establish in §4 $L^\infty(\Omega)$ bound on the derivatives $\frac{\partial^2 u}{\partial x_i^2}$; in particular, $u \in W^{2,p}_{\text{loc}}(\Omega)$ for all $p < \infty$. It is at this stage that the restriction $n = 2$ is imposed.

Finally, extensions of our results to more general HJB equations are briefly given in §5.
§2. Existence and uniqueness. In this and in the next section we consider the
Dirichlet problem (1.5), (1.4) with \( \Omega \) a bounded domain and \( g \in C(\partial \Omega) \). In order to state
a uniqueness result, we recall the concept of a viscosity solution.

Denote by \( S(n) \) the set of all symmetric \( n \times n \) matrices. Consider the partial differential operator

\[
F(D^2 u, Du, u) = 0
\]

where \( u = u(x) \) and \( F \) is a continuous function on \( S(n) \times \mathbb{R}^n \times \mathbb{R} \). We say that \( F \) is
degenerate elliptic if

\[
F(M, p, t) \geq F(N, p, t) \quad \forall \ M \succ N, \quad p \in \mathbb{R}^n, t \in \mathbb{R}
\]

where \( M, N \in S(n) \) and \( M \succ N \) means that \( M - N \) is positive semi-definite. We say that
\( F(M, p, t) \) is uniformly decreasing in \( t \) if there exists a positive constant \( \delta > 0 \) such that

\[
F(M, p, t) - F(M, p, s) \leq -\delta (t - s) \quad \forall \ t > s, M \in S(n), p \in \mathbb{R}^n.
\]

Let \( w \in C(\Omega) \) where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). The superdifferential \( D^+ w(x) \) \((x \in \Omega)\) is defined as the set

\[
D^+ w(x) = \{(p, M) \in \mathbb{R}^n \times S(n); w(x + z) \leq w(x) + p \cdot z + \frac{1}{2} (Mz, z) + o(|z|^2)\}.
\]

Similarly the subdifferential \( D^- w(x) \) is defined by reversing the last inequality.

A function \( w \in C(\Omega) \) is called a viscosity supersolution of (13.21) in \( \Omega \) if

\[
F(M, p, w(x)) \leq 0 \quad \forall (p, M) \in D^- w(x), \quad x \in \Omega;
\]

it is a viscosity subsolution if

\[
F(M, p, w(x)) \geq 0 \quad \forall (p, M) \in D^+ w(x), x \in \Omega.
\]

The following result is due to Jensen [7].

**Theorem 2.1.** Suppose \( F(M, p, t) \) is degenerate elliptic and uniformly decreasing in \( t \). If \( u \) is a viscosity supersolution and \( v \) is a viscosity subsolution in \( \Omega \), such that \( u \in C(\overline{\Omega}), v \in C(\overline{\Omega}) \) and \( u \geq v \) on \( \partial \Omega \), then \( u \geq v \) in \( \Omega \).

If \( u \) is both a viscosity supersolution and a viscosity subsolution then it is called a viscosity solution of (2.1). Theorem 2.1 implies:
THEOREM 2.2. If \( F(M, p, t) \) is as in Theorem 2.1, then for any given function \( g \in C(\partial \Omega) \) there exists at most one viscosity solution (in \( C(\overline{\Omega}) \)) of (2.1) in \( \Omega \), satisfying (1.4).

Using Theorem 2.1 we shall prove:

THEOREM 2.3. There exists at most one viscosity solution of (1.5), (1.4).

Proof. For any small \( \delta > 0 \) consider the operator

\[
\tilde{L}_\delta u = Lu - \delta u
\]

where

\[
Lu = \max \left\{ \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_n^2} \right\}.
\]

\( \tilde{L}_\delta \) is nonlinear degenerate elliptic operator, strictly decreasing in the lowest order coefficient. Hence Theorem 2.1 is applicable to \( \tilde{L}_\delta \). If \( u, v \) are viscosity supersolution and subsolution, respectively, for \( L \) in \( \Omega \), with \( u \geq v \) on \( \partial \Omega \), then, for any \( \epsilon > 0 \), the same is true of

\[
\bar{u} \equiv u - \frac{\epsilon}{2} |x|^2 + C \epsilon, \quad \bar{v} \equiv v + \frac{\epsilon}{2} |x|^2
\]

with respect to \( \tilde{L}_\delta \), provided \( \delta \) is sufficiently small (depending on \( \epsilon \)), and \( C \) is sufficiently large positive constant independent of \( \delta, \epsilon \). Applying Theorem 2.1 we conclude that \( \bar{u} \geq \bar{v} \) in \( \Omega \). Letting \( \epsilon \to 0 \) we get \( u \geq v \) in \( \Omega \). Thus Theorem 2.1 extends to \( L \), and this implies the assertion of Theorem 2.3.

It was proved by Lions [15] that if \( u \in W^{2,p}(\Omega), p > n \), then \( u \) satisfies (1.5) a.e. if and only if it is a viscosity solution. Hence:

COROLLARY 2.4. There exists at most one solution of (1.5), (1.4) in \( C(\overline{\Omega}) \cap W^{2,p}_\text{loc}(\Omega) \), \( p > n \).

This result follows also, more simply, from the maximum principle of Bony [1] (see also [2]).

A function \( g \) is called marginally concave if \( \partial^2 g / \partial x_i^2 \leq 0 \) \( \forall i \).

A domain \( \Omega \) is said to satisfy the exterior ball condition if for any \( x_0 \in \partial \Omega \) there exists a ball \( B : |x - x^*| < \rho^* \) (\( \rho^* \) independent of \( x_0 \)) such that

\[
B \cap \Omega = \phi, \quad \overline{B} \cap \overline{\Omega} = \{x_0\}.
\]

We now state an existence result.
Theorem 2.5. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying the exterior ball condition, and let $g \in C^2(\overline{\Omega})$ be a marginally concave function. Then there exists a viscosity solution $u$ of (1.5), (1.4), and $u \in W^{1,\infty}(\Omega)$.

Proof. Set $\beta(r_1, \ldots, r_n) = \max\{r_1, \ldots, r_n\}$. We introduce mollifiers

$$\beta_\varepsilon(r) = (\beta * \rho_\varepsilon)(r), \quad r = (r_1, \ldots, r_n),$$

where

$$\rho_\varepsilon(x) = \varepsilon^{-n} \rho\left(\frac{x}{\varepsilon}\right), \quad \rho \in C_0^\infty\{|x| < 1\}, \quad \int \rho(x) dx = 1.$$ 

Then

$$0 \leq \frac{\partial \beta_\varepsilon}{\partial r_i} \leq 1, \quad \left(\frac{\partial^2 \beta_\varepsilon}{\partial r_i \partial r_j}\right) \text{ is nonnegative definite,}$$

$$\sup_r |\beta_\varepsilon(r) - \beta(r)| \leq \eta_0(\varepsilon) \to 0 \quad \text{if} \quad \varepsilon \to 0.$$ 

Consider the Dirichlet problem

\begin{align*}
L_\varepsilon u &\equiv \varepsilon \Delta u + \beta_\varepsilon(u_{x_1x_1}, \ldots, u_{x_nx_n}) - \beta_\varepsilon(0) = 0 \quad \text{in} \quad \Omega, \\
u &= g \quad \text{on} \quad \partial \Omega.
\end{align*}

By [6; Theorem 17.17] this problem has unique solution in $C^2(\Omega) \cap C(\overline{\Omega})$; we designate it by $u_\varepsilon$.

For any $x_0 \in \partial \Omega$ take

$$\eta(x) = e^{-\lambda \rho_\ast} - e^{-\lambda|x-x^\ast|}$$

where $x^\ast$, $\rho_\ast$ are defined above (in the "exterior ball condition"). Set

$$w(x) = g(x) - C\eta(x).$$

We compute that

$$\beta(w_{x_1x_1}, \ldots, w_{x_nx_n}) > 1$$

if $\lambda R^2/n \geq 2$ and $C$ is sufficiently large ($R$ is the diameter of $\Omega$). Hence

$$L_\varepsilon w > 0 \quad \text{in} \quad \Omega$$

if $\varepsilon$ is sufficiently small. Since $w \leq u_\varepsilon$ on $\partial \Omega$, we get, by comparison, $w \leq u_\varepsilon$ in $\Omega$.

Next, since $g$ is marginally concave,

$$L_\varepsilon g = \varepsilon \Delta g + \sum_j \beta_\varepsilon(r_j)(\theta g_{x_1x_1}, \ldots, \theta g_{x_nx_n})g_{x_jx_j} \leq 0 \quad \text{in} \quad \Omega$$
(for some \( \theta \in (0, 1) \)), so that, by comparison, \( u_\varepsilon \leq g \) in \( \Omega \).

From the estimates

\[
g - C \eta \leq u_\varepsilon \leq g \quad \text{in} \quad \Omega
\]

with equalities at \( x = x_0 \) it follows that

\[
|\nabla u_\varepsilon(x_0)| \leq C, \quad \forall \ x_0 \in \partial \Omega.
\]

Differentiating (2.4) in any direction \( \xi \) we get

\[
\epsilon \Delta u_{\varepsilon, \xi} + \Sigma \beta_{\varepsilon, r_j}(u_{\varepsilon, \xi}) x_j x_j = 0.
\]

Applying the maximum principle to \( u_{\varepsilon, \xi} \) and using (2.6), we find that

\[
|\nabla u_\varepsilon| \leq C \quad \text{in} \quad \Omega.
\]

Now let \( \varepsilon \to 0 \). The limiting function \( u = \lim u_\varepsilon \) is the viscosity solution (by [14]).

We shall adopt the following definition:

A domain \( \Omega \) with \( C^2 \) boundary is strictly convex if there exists a function \( \psi \) in \( C^2(\overline{\Omega}) \) satisfying:

\[
\begin{align*}
\left( \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} \right) & \quad \text{is negative definite} \quad \forall \ x \in \overline{\Omega}, \\
\psi(x) & > 0 \quad \text{in} \quad \Omega, \\
\psi(x) & = 0, \ \nabla \psi(x) \neq 0 \quad \forall \ x \in \partial \Omega.
\end{align*}
\]

If \( g \) is any \( C^2(\overline{\Omega}) \) function, then \( g + A \psi \) is concave if \( A \) is sufficiently large whereas \( u = g + A \psi \) on \( \partial \Omega \); consequently, by Theorem 2.5:

**Theorem 2.6.** Let \( \Omega \) be a \( C^2 \) strictly convex domain and let \( g \) be any \( C^2(\overline{\Omega}) \) function. Then there exists a unique viscosity solution \( u \) of (1.5), (1.4), and \( u \in W^{1, \infty}(\overline{\Omega}) \).

**§3. Normal–tangential boundary estimates.** For any \( \mu > 0 \), denote by \( \Omega_{\mu} \) the \( \mu \) \( \Omega \)-neighborhood of \( \partial \Omega \). In the sequel we assume that \( \Omega \) is \( C^2 \) strictly convex, and

\[
\partial \Omega \in C^4, \quad g \in C^4(\overline{\Omega}_{\mu_0}) \quad \text{for some} \quad \mu_0 > 0.
\]

In the definition of \( \beta_\varepsilon \) above, we choose \( \rho(x) \) as in [9; pp. 331–2 and p. 276]. Then, in addition to all the above properties of \( \beta_\varepsilon \), there also holds:

\[
\beta_\varepsilon(r) = \max_{\omega \in \mathcal{B}} \left\{ \sum_{i=1}^{n} a_i^\varepsilon(\omega) r_i + a_0^\varepsilon(\omega) \right\},
\]

where \( \mathcal{B} \) is a closed set in \( \mathbb{R}^n \),

\[
0 \leq a_i^\varepsilon(\omega) \leq 1 \quad \text{if} \quad 1 \leq i \leq n, \quad \sum_{i=1}^{n} a_i^\varepsilon(\omega) = 1, \quad |a_0^\varepsilon(\omega)| \leq \varepsilon;
\]
it should be emphasized that for each $r$,

\begin{equation}
\text{(3.3)}
\text{the maximum in (3.2) is attained.}
\end{equation}

**Lemma 3.1.** The following estimate holds for all sufficiently small $\epsilon$:

\begin{equation}
\text{(3.4)}
\left| \frac{\partial^2 u_r(x)}{\partial \tau \partial \nu} \right| \leq C \quad \forall \ x \in \partial \Omega
\end{equation}

where $\partial/\partial \tau, \partial/\partial \nu$ are tangential and normal derivatives at $x$, respectively, and $C$ is a constant independent of $x, \epsilon$.

For technical clarity we shall first give a formal proof in case $\epsilon = 0$, i.e., we shall derive the estimate

\begin{equation}
\text{(3.5)}
\left| \frac{\partial^2 u(x)}{\partial \tau \partial \nu} \right| \leq C \quad \forall \ x \in \partial \Omega,
\end{equation}

assuming that $u$, the solution of (1.5), (1.4), is in $C^4(\overline{\Omega})$; the constant $C$ should not depend of course on the assumed smoothness of $u$. The rigorous proof of (3.4) will then be derived by minor changes.

Set

\begin{equation}
\text{(3.6)}
\overline{u} = u - g.
\end{equation}

Then

\begin{equation}
\text{(3.7)}
\max_{1 \leq i \leq n} \{ \overline{u}_{x_i x_i} + g_{x_i x_i} \} = 0 \quad \text{in} \ \overline{\Omega},
\end{equation}

\begin{equation}
\overline{u} = 0 \quad \text{on} \ \partial \Omega.
\end{equation}

We next introduce a function $v(x)$ in $\Omega$ by

\begin{equation}
\text{(3.8)}
\overline{u}(x) = \psi(x)v(x).
\end{equation}

Then

\begin{equation}
\text{(3.9)}
\max_{1 \leq i \leq n} \{ \psi v_{x_i x_i} + 2\psi x_i v_{x_i} + \psi_{x_i x_i} v + g_{x_i x_i} \} = 0 \quad \text{in} \ \Omega,
\end{equation}

and

\begin{equation}
\text{(3.10)}
v(x) = \frac{\partial \overline{u}(x)/\partial \nu}{\partial \psi(x)/\partial \nu} \quad \text{on} \ \partial \Omega
\end{equation}
since \( \overline{u}(x) = \overline{\psi}(x) = 0 \) on \( \partial \Omega \). The estimate (3.5) is equivalent to the estimate

\[
(3.11) \quad \left| \frac{\partial v(x)}{\partial r} \right| \leq C \quad \forall \ x \in \partial \Omega.
\]

Notice that any solution of (3.9) which is in \( C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega) \) \( (p > n) \) gives rise, via (3.8), to a solution of (3.7), or to a viscosity solution \( u \) of (1.5), (1.4) where \( u = \overline{u} + g \). By uniqueness of the viscosity solution \( u \) it thus follows that \( v \) is uniquely determined.

We shall now proceed (as in Krylov [9; Chap. 8]) to “unfold” \( v \): We introduce a function \( \tilde{w}(x,r) \ (x \in \overline{\Omega}, r \geq 0) \) by \( \tilde{w}(x,r) = v(x) \) and then introduce a function

\[
w(x, x_{n+1}, \ldots, x_{n+4}) \ (x \in \overline{\Omega}, (x_{n+1}, \ldots, x_{n+4}) \in \mathbb{R}^4) \text{ by } w(x, x_{n+1}, \ldots, x_{n+4}) = \tilde{w}(x,r)
\]

where

\[
r = \left( \sum_{\nu = n+1}^{n+4} x_{\nu}^2 \right)^{1/2}.
\]

In particular, we have

\[
v(x) = \tilde{w}(x, \sqrt{\psi(x)}) = w(x, x_{n+1}, \ldots, x_{n+4}) \quad \text{on } S
\]

where \( S \) is the manifold in \( \mathbb{R}^{n+4} \):

\[
S : \sum_{\nu = n+1}^{n+4} x_{\nu}^2 = \psi(x) \quad (S \subset \mathbb{R}^{n+4}).
\]

We shall now consider the function \( w \) on \( S \) only.

Expressing \( v_{x_i} \) in terms of \( \tilde{w}_{x_i}, \tilde{w}_r \), and \( v_{x_i x_i} \) in terms of \( \tilde{w}_{x_i x_i}, \tilde{w}_{x_i r}, \tilde{w}_{r r}, \tilde{w}_{x_i}, \tilde{w}_r \) and then using the relations

\[
\sum_{\nu = n+1}^{n+4} x_{\nu} w_{x_{\nu}} = r \tilde{w}_r, \quad \sum_{\nu = n+1}^{n+4} x_{\nu} w_{x_i x_{\nu}} = r w_{x_i r},
\]

\[
\sum_{r = n+1}^{n+4} w_{x_r x_r} = \tilde{w}_{r r} + \frac{3}{r} \tilde{w}_r,
\]

we find [9; pp. 377–379] that

\[
\psi v_{x_i x_i} + 2\psi v_{x_i} + \psi v_{x_i x_i} v + g_{x_i x_i} = L_i w
\]
where

\[ L_i w = \left( \sum_{\nu=n+1}^{n+4} x_{\nu}^2 \right) w_{x_i x_i} + \psi_{x_i} \sum_{\nu=n+1}^{n+4} x_{\nu} w_{x_i x_{\nu}} \]

\[ + \frac{1}{4} \left( \psi_{x_i} \right)^2 \sum_{\nu=n+1}^{n+4} w_{x_{\nu} x_{\nu}} + 2 \psi_{x_i} w_{x_i} \]

\[ + \frac{1}{2} \psi_{x_i x_i} \sum_{\nu=n+1}^{n+4} x_{\nu} w_{x_{\nu}} + \psi_{x_i x_i} w. \]

(3.14)

Thus,

(3.15) \[ \max_{1 \leq i \leq n} \{ L_i w + g_{x_i x_i} \} = 0 \quad \text{on } S. \]

Instead of working with \( v \) in \( \Omega \) we shall now work with \( w \) on the manifold \( S \).

Next, as in [9; p. 314] we introduce the operators \( \partial (\xi) \):

(3.16) \[ \partial (\xi) w = \sum_{j=1}^{n+4} \xi_j w_{x_j} + \xi_{n+5} w \quad (\xi = (\xi_1, \ldots, \xi_{n+5})). \]

We then have [9; p. 315, Lemma 1 (c)]

(3.17) \[ \partial (\xi) \tilde{L}_i w = \tilde{L}_i (\partial (\xi) w) \quad \text{for any } w = w(X) \text{ in } C^3, \]

where \( X = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+4}) \) and

(3.18) \[ \tilde{L}_i w(x, \xi) \]

\[ = \left\{ \left( \sum_{\nu=n+1}^{n+4} x_{\nu}^2 \right) w_{x_i x_i} + \psi_{x_i} \sum_{\nu=n+1}^{n+4} x_{\nu} w_{x_i x_{\nu}} + \frac{1}{4} \left( \psi_{x_i} \right)^2 \sum_{\nu=n+1}^{n+4} w_{x_{\nu} x_{\nu}} \right\} \]

\[ + \left( \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 \right) w_{\xi_i \xi_i} + \left( \sum_{j=1}^{n} \psi_{x_i x_j} \xi_j \right) \cdot \left( \sum_{\nu=n+1}^{n+4} \xi_{\nu} w_{\xi_i \xi_{\nu}} \right) \]

\[ + \frac{1}{4} \left( \sum_{j=1}^{n} \psi_{x_i x_j} \xi_j \right)^2 \sum_{\nu=n+1}^{n+4} w_{\xi_\nu \xi_\nu} \]

\[ + 2 \left( \sum_{\nu=n+1}^{n+4} x_{\nu} \xi_{\nu} \right) w_{x_i \xi_i} + \left( \sum_{j=1}^{n} \psi_{x_i x_j} \xi_j \right) \sum_{\nu=n+1}^{n+4} x_{\nu} w_{x_i \xi_{\nu}}. \]
\[ + \sum_{\nu=n+1}^{n+4} (\psi_{x_i \xi_\nu})w_{x_i x_\xi} + \frac{1}{2} \psi_{x_i} \left( \sum_{j=1}^{n} \psi_{x_i x_j \xi_j} \right) \cdot \left( \sum_{\nu=n+1}^{n+4} w_{x_\nu \xi_\nu} \right) \]

\[ + \left\{ 2\psi_{x_i}w_{x_i} + \frac{1}{2} \psi_{x_i x_i} \sum_{\nu=n+1}^{n+4} x_\nu w_{x_\nu} + 2 \left( \sum_{j=1}^{n} \psi_{x_i x_j \xi_j} \right) w_{x_i} \right\} \]

\[ + \sum_{\nu=n+1}^{n+4} \left[ \frac{1}{2} \sum_{j=1}^{n} (\psi_{x_i x_i x_j \xi_j} x_\nu + \frac{1}{2} \psi_{x_i x_i \xi_\nu} \right] w_{x_i} \]

\[ + \left( \sum_{j=1}^{n} \psi_{x_i x_i x_j \xi_j} \right) w_{x_{n+5}} \} + \psi_{x_i x_i} w; \]

observe that

(3.19) \[ \tilde{L}_i w = L_i w \quad \text{if} \quad w = w(x_1, \ldots, x_{n+4}) \equiv w(X). \]

To verify (3.18) we note that we can write

\[ L_i = \sum_{l,k=1}^{n+4} a_{l_k} \frac{\partial^2}{\partial x_l \partial x_k} + \sum_{k=1}^{n+4} b_{l_k} \frac{\partial}{\partial x_k} + c^i \]

where

\[ a_{l_k} = \frac{1}{2} \sum_{m=1}^{4} \sigma_{l,m} \sigma_{k,m} \]

and

\[ \sigma_{l,m} = \sqrt{2} \quad x_{n+m}, \]

\[ \sigma_{n+m,n+m} = \frac{\sqrt{2}}{2} \quad \psi_{x_i}, \quad (1 \leq m \leq 4), \]

\[ \sigma_{l,m} = 0 \quad \text{for all other pairs} \quad (l, m). \]

Using the recipe in [9; p. 314] with \( \pi = 0 \), formula (3.18) can now easily be checked.

We introduce the manifolds

\[ \Sigma_1 = \left\{ (X, \xi); \quad \psi(x) - \sum_{\nu=n+1}^{n+4} x_\nu^2 = 0 \right\}, \]

\[ \Sigma_2 = \left\{ (X, \xi); \quad \sum_{j=1}^{n} \psi_{x_j}(x) \xi_j - 2 \sum_{\nu=n+1}^{n+4} x_\nu \xi_\nu = 0 \right\}. \]
For any function \( u(X) \) define

\[
u(\xi)(X) = \sum_{j=1}^{n+4} \xi_j \frac{\partial}{\partial x_j} u(X).
\]

Observe that if \( (X, \xi) \in \Sigma_1 \cap \Sigma_2 \) then

\[
(3.20) \quad u(\xi)(X) \text{ is a tangential derivative on the manifold } S.
\]

By [9; p. 411, Lemma 6 (b)]

\[
(3.21) \quad \tilde{L}_i \left( \psi(x) - \sum_{\nu=n+1}^{n+4} x_\nu^2 \right) = \tilde{L}_i \left( \psi(x) - \sum_{\nu=n+1}^{n+4} x_\nu^2 \right)^2 = 0 \text{ on } \Sigma_1,
\]

and by [9; p. 384, Theorem 7 (c)]

\[
(3.22) \quad \tilde{L}_i \left( \sum_{j=1}^{n} \psi x_j \xi_j - 2 \sum_{\nu=n+1}^{n+4} x_\nu \xi_\nu \right)
\]

\[
= \tilde{L}_i \left( \sum_{j=1}^{n} \psi x_j \xi_j - 2 \sum_{\nu=n+1}^{n+4} x_\nu \xi_\nu \right)^2 = 0 \text{ on } \Sigma_1 \cap \Sigma_2.
\]

These relations imply (by [9; p. 381, Lemma 2]) the following maximum principle on the manifold \( \Sigma_1 \cap \Sigma_2 \):

\[
(3.23) \quad \text{if } w(X, \xi) \text{ is in } C^2 \text{ in a neighborhood of } (\Sigma_1 \cap \Sigma_2) \text{ and, restricted to } \Sigma_1 \cap \Sigma_2, \text{ attains local maximum at } (X_0, \xi_0), \text{ then } \tilde{L}_i w(X_0, \xi_0) \leq 0.
\]

Let \( d(x) = \text{dist}(x, \partial \Omega), \ x \in \Omega \). From [6; (14.101)] with \( \psi = d(x) \) it follows that \( d \in C^4 \) in some \( \Omega \)-neighborhood of \( \partial \Omega \). We shall construct a barrier \( \phi(X, \xi) \) in

\[
Q_\mu \equiv \Sigma_1 \cap \Sigma_2 \cap \{ d < \mu \},
\]

for some small \( \mu > 0 \).

Set

\[
(3.24) \quad h = d - d^2 \quad \text{in} \quad \Omega_{\mu_0}, \quad \mu_0 \quad \text{small}.
\]
One can check that

\[(3.25) \quad \sum_{i,j=1}^{n} h_{x_i x_j} \zeta_i \zeta_j \leq -\delta |\zeta|^2 \quad (\delta > 0).\]

We shall make a special choice of \(\psi\):

\[(3.26) \quad \psi = h - Kd^2\]

where \(K\) is a positive constant to be determined later on. The barrier function will be

\[(3.27) \quad \phi(X, \xi) = -\lambda \sum_{j,k=1}^{n} h_{x_j x_k}(x) \xi_j \xi_k + 2\lambda \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 + \xi_{n+5}^2 + \lambda\]

where \(\lambda\) is a positive constant to be determined.

**Lemma 3.2.** For suitable choice of the positive constants \(K, \lambda\) and \(\mu\) there exists a positive constant \(M\) such that

\[(3.28) \quad \tilde{L}_i \phi(X, \xi) \leq -M \left( \sum_{j=1}^{n+5} \xi_j^2 + 1 \right) \quad \text{in} \quad Q_{\mu}, \quad \forall \ i\]

Let us first assume that the lemma is true and complete the proof of (3.5), and then of (3.4).

From Lemma 3.2 we get

\[(3.29) \quad \tilde{L}_i (A\phi) + \partial(\xi) g_{x_i x_i} \leq -1 \quad \text{in} \quad Q_{\mu}, \quad \forall \ i\]

for some positive constant \(A\).

Consider the function

\[W = \partial(\xi)v - A\phi \quad \text{in} \quad Q_{\mu}\]

which can be extended as a \(C^2\) function to a neighborhood of \(Q_{\mu}\) in the \((X, \xi)\)-space. We claim that

\[(3.30) \quad W \leq 0 \quad \text{in} \quad Q_{\mu}.\]

Indeed, since \(W \to -\infty\) as \(|\xi| \to \infty\), the maximum of \(W\) must occur either (i) on \(\{\psi(x) = \mu\}\), in which case \(W \leq 0\) if \(A\) is chosen large enough since \(|\nabla v| \leq C\) on \(|\psi(x)| = \mu\), or (ii) at an interior point \((X_0, \xi_0)\) of \(Q_{\mu}\). Thus, in order to complete the proof of (3.30), it suffices to rule out case (ii).
Suppose (ii) holds. Then by the maximum principle (3.23),

\[ \tilde{L}_i \partial(\xi)v - \tilde{L}_i (A\phi) \leq 0 \quad \text{at} \quad (X_0, \xi_0), \]

so that, by (3.29), (3.17),

\[ \partial(\xi)(\tilde{L}_i v + g_{x ix_i}) \leq -1 \quad \text{at} \quad (X_0, \xi_0), \quad \forall \ i. \]  

(3.31)

Since

\[ (\tilde{L}_i v + g_{x ix_i})(x, \xi) \leq 0 \quad \forall \ (X, \xi) \in Q_\mu, \quad 1 \leq i \leq n \]

and equality holds at \((X_0, \xi_0)\) for some \(i = i_0\) (by (3.19) and the HJB equation for \(v\)), we have

\[ \partial(\xi)(\tilde{L}_{i_0} v + g_{x_{i_0}x_{i_0}}) = 0 \quad \text{at} \quad (X_0, \xi_0), \]  

(3.32)

which is a contradiction to (3.31).

Having proved that \(W \leq 0\) in \(Q_\mu\), we note that (in view of (3.20)) this inequality translates into (3.11). Thus the proof of (3.11) is complete.

We are now ready to prove the assertion (3.4), or equivalently, the estimate

\[ \left| \frac{\partial v_\epsilon(x)}{\partial r} \right| \leq C \]

where \(\psi(x)v_\epsilon(x) = u_\epsilon(x) - g(x)\). Recall that \(u_\epsilon\) is in \(C^4(\Omega_{\mu_0}) \cap C^3(\overline{\Omega}_{\mu_0})\). Instead of working with the operators \(\tilde{L}_j \equiv u_{x_jx_j} \quad (1 \leq j \leq n)\) we work with the operators (cf. (3.2))

\[ \tilde{\mathcal{L}}_\omega \equiv \sum_{j=1}^{n} a_\omega^{\epsilon}(\omega) \frac{\partial^2}{\partial x_j^2} + a_\omega^0(\omega), \quad \omega \in \mathcal{B}. \]

The operators \(L_i\) and \(\tilde{L}_i\) have to be suitably modified; the necessary changes in the proof of Lemma 3.2 (given below) are rather obvious; the barrier function is the same. We can now proceed as before to establish that \(W \leq 0\). Thus, if \(W\) takes positive maximum at \((X_0, \xi_0) \in Q_\mu\) then (3.31) holds with \(\tilde{L}_i\) replaced by \(\tilde{\mathcal{L}}_\omega\), \(\forall \ \omega \in \mathcal{B}\). Also, for at least one \(\omega = \omega_0\) equality holds in (3.32) (with \(i = \omega_0\)), which leads to a contradiction as before.
Proof of Lemma 3.2. Substituting $\phi$ from (3.27) into (3.18) we find that

\begin{align*}
(3.33) \quad \tilde{L}_i \phi & = \left( \sum_{\nu=n+1}^{n+4} x_{i\nu}^2 \right) \left( -\lambda \sum_{k,j=1}^{n} h_{x_k x_j x_i x_i} \xi_k \xi_j \right) \\
& + \left( \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 \right) \left( -2\lambda h_{x_i x_i} \right) \\
& + \frac{1}{4} \left( \sum_{j=1}^{n} \psi_{x_i x_j} \xi_j \right)^2 \cdot 16\lambda \\
& + 2 \left( \sum_{\nu=n+1}^{n+4} x_{i\nu} \xi_{\nu} \right) \left( -\lambda \sum_{j=1}^{n} 2h_{x_i x_j x_i} \xi_j \right) \\
& + 2\psi_{x_i x_i} \left( -\lambda \sum_{k,j=1}^{n} h_{x_k x_j x_i x_i} \xi_k \xi_j \right) \\
& + 2 \left( \sum_{j=1}^{n} \psi_{x_i x_j} \xi_j \right) \left( -\lambda \sum_{j=1}^{n} 2h_{x_i x_j x_i} \xi_j \right) \\
& + \frac{1}{2} \left( \sum_{j=1}^{n} \psi_{x_i x_j x_j} \xi_j \right) \left( \sum_{\nu=n+1}^{n+4} x_{\nu} \cdot 4\lambda \xi_{\nu} \right) \\
& + \frac{1}{2} \psi_{x_i x_i} \sum_{\nu=n+1}^{n+4} \xi_{\nu} \cdot 4\lambda \xi_{\nu} \\
& + \left( \sum_{j=1}^{n} \psi_{x_i x_i x_j} \xi_j \right) 2\xi_{n+5} + \psi_{x_i x_i} \phi \equiv \sum_{k=1}^{10} J_k.
\end{align*}

We compute:

\[ J_1 + J_4 + J_7 \leq C\lambda(K + 1)\sqrt{\mu} \left( \sum_{j=1}^{n} \xi_j^2 + \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 \right) \]

in $Q_{\mu}$.

Since

\begin{align*}
(3.34) \quad \psi_{x_i x_j} = h_{x_i x_j} - 2Kd_{x_i}d_{x_j} - 2Kdd_{x_i x_j},
\end{align*}

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we also have

\[ J_1 + J_8 = -2K \left( \sum_{\nu=n+1}^{n+4} \xi_{x_\nu}^2 \right) \cdot (d_{x_i}^2 + dd_{x_i; x_j}) \cdot 2\lambda \]

\[ \leq C\lambda K d \left( \sum_{\nu=n+1}^{n+4} \xi_{x_\nu}^2 \right); \]

the negative term involving \(d_{x_i}^2\) was dropped.

Next, by (3.34),

\[ J_3 + J_6 = -4\lambda \left( \sum_{j=1}^{n} \psi_{x_i; x_j} x_j \right) \cdot 2K \left[ \sum_{j=1}^{n} \left( d_{x_i} + dd_{x_i; x_j} \right) \right] \]

\[ = -8K\lambda \left( \sum_{j=1}^{n} \psi_{x_i; x_j} x_j \right) \left\{ d_{x_i} \left[ \sum_{j=1}^{n} \left( \psi_{x_j} + 2(K+1)dd_{x_j} \right) x_j \right] \right. \]

\[ + d \sum_{j=1}^{n} dd_{x_i; x_j} x_j \}

since

(3.35)

\[ \psi_{x_j} = d_{x_j} - 2(K+1)dd_{x_j} . \]

Recalling that \( \sum_{j=1}^{n} \psi_{x_j} x_j = 2 \sum_{\nu=n+1}^{n+4} x_\nu \xi_{x_\nu} \) on \( \Sigma_2 \), we get

\[ J_3 + J_6 = -8K\lambda \left( \sum_{j=1}^{n} \psi_{x_i; x_j} x_j \right) \left\{ d_{x_i} \left[ 2 \sum_{\nu=n+1}^{n+4} x_\nu \xi_{x_\nu} + 2(K+1)d \sum_{j=1}^{n} dd_{x_j} x_j \right] \right. \]

\[ + d \sum_{j=1}^{n} dd_{x_i; x_j} x_j \}

\[ \leq (CK\lambda \sqrt{\mu} + C(K^2 + 1)\lambda d) \left( \sum_{j=1}^{n} \xi_{x_j}^2 + \sum_{\nu=n+1}^{n+4} \xi_{x_\nu}^2 \right). \]

Next,

\[ J_5 \leq C\lambda |\psi_{x_i}| \sum_{j=1}^{n} \xi_{x_j}^2 \leq \left( \frac{\delta}{2} \right)^2 \lambda \sum_{j=1}^{n} \xi_{x_j}^2 + C\lambda |\psi_{x_i}|^2 \sum_{j=1}^{n} \xi_{x_j}^2 \]

\[ \leq \left( \frac{\delta}{2} \right)^2 \lambda \sum_{j=1}^{n} \xi_{x_j}^2 + C\lambda d_{x_i}^2 \sum_{j=1}^{n} \xi_{x_j}^2 + C\lambda (K^2 + 1)d^2 \sum_{j=1}^{n} \xi_{x_j}^2 \]
by (3.35). Finally,

$$J_9 \leq \frac{\delta}{2} \xi_{n+5}^2 + C\delta \sum_{j=1}^{n} \xi_j^2 \quad (\delta \text{ as in (3.25)})$$.

Combining the estimates on the $J_k$ and using the inequality

$$\psi_{x;ix_i} \leq -\delta - 2Kd^2_{x_i} - 2Kdd_{x;ix_i}$$

which follows from (3.34), (3.25), we get

$$\tilde{L}_i \phi \leq \left( \sum_{j=1}^{n} \xi_j^2 + \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 \right) \left\{ C\lambda(K + 1)\sqrt{\mu} + C\lambda Kd + CK \lambda \sqrt{\mu} 
+ C(K^2 + 1)\lambda d + C\delta + C\lambda(K^2 + 1)d^2 \right\} + \frac{\delta}{2} \xi_{n+5}^2$$

$$+ \left( \frac{\delta}{2} \right)^2 \lambda \sum_{j=1}^{n} \xi_j^2 + C\lambda d_{x_i}^2 \sum_{j=1}^{n} \xi_j^2
- (\delta + 2Kd_{x_i}^2 + 2Kdd_{x;ix_i}) \phi.$$

Substituting $\phi$ from (3.27) into the last term and using (3.25), we get

$$\tilde{L}_i \phi \leq \left( \sum_{j=1}^{n} \xi_j^2 + \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 \right) \left\{ C\lambda(K + 1)\sqrt{\mu} + C(K^2 + 1)\lambda d + C\delta \right\}$$

$$+ \frac{\delta}{2} \xi_{n+5}^2 + \left( \frac{\delta}{2} \right)^2 \lambda \sum_{j=1}^{n} \xi_j^2 + C^* \lambda d_{x_i}^2 \sum_{j=1}^{n} \xi_j^2 + 2Kdd_{x;ix_i} \xi_{n+5}^2 + 2K\lambda dd_{x;ix_i} - B$$

where

$$B = \delta^2 \lambda \sum_{j=1}^{n} \xi_j^2 + 2K\delta \lambda d_{x_i}^2 \sum_{j=1}^{n} \xi_j^2 + 2\delta \lambda \sum_{\nu=n+1}^{n+4} \xi_{\nu}^2 + \delta \xi_{n+5}^2 + \delta \lambda;$$

all the terms in $B$ come from the last summand in (3.36); some terms coming from the last summand are (implicitly) included elsewhere on the right-hand side of (3.37), and $\mu$ was already assumed to be small (depending on $K, \delta$); $C^*$, like $C$, is a constant depending on $h$ but independent of $K, \delta, \mu$.

We now fix $K$ so that $2K\delta \geq C^*$. Next, choosing $\lambda$ large and then $\mu$ small, the inequality (3.28) follows.
Remark 3.1. The above method of estimating \( \partial^2 u_\epsilon / \partial \tau \partial \nu \) is due to Krylov [9; p. 421, Lemma 8]. The novelty of our result is in implementing the method by actually constructing a barrier (by means of the special choices of both \( \psi \) and \( \phi \)).

Remark 3.2. If the HJB equation satisfies a “weakly nondegenerate” condition [9; p. 417] then \( \partial^2 u_\epsilon / \partial \nu^2 \) can be estimated from the estimates on \( \partial^2 u / \partial \tau \partial \nu \). This then leads to uniform \( C^2 \) estimates for \( u_\epsilon \) in \( \Omega \) and therefore also for the limit \( u \). The nondegeneracy condition [9; p. 417, (4)] must then hold also for \( u \). In our case however this nondegeneracy condition is not valid for \( u \), which means that the HJB equation is not “weakly nondegenerate” in Krylov’s sense. Because of this degeneracy, we are unable to establish \( W^{2,\infty} \) estimates on \( u_\epsilon \) and \( u \), except in case \( n = 2 \), as will be done in §4.

§4. \( u_{x_1 x_1} \) are locally bounded \( (n = 2) \). Denote by \( \{a_i < x_i < b_i\} \) the projection of \( \Omega \) on the \( x_i \)-axis and let

\[
\Omega_{i,\mu} = \Omega \cap \{a_i + \mu < x_i < b_i - \mu\}, \quad \tilde{\Omega}_\mu = \bigcap_{i=1}^{n} \Omega_{i,\mu}.
\]

Theorem 4.1. If \( n = 2 \) then \( u_{x_1 x_1} \) and \( u_{x_2 x_2} \) belong to \( L^\infty(\tilde{\Omega}_\mu) \), for any \( \mu > 0 \).

Proof. The proof is based on the method of Evans and Friedman [4]. Differentiating (2.4) twice in any direction and using the fact that \( \beta_\epsilon \) is concave, we get

\[
\begin{align*}
(4.1) \quad & \epsilon \Delta u_\epsilon,\xi + \sum_{i=1}^{n} \beta_{\epsilon, r_i}(u_\epsilon)_{x_i x_i} = 0, \\
(4.2) \quad & \epsilon \Delta u_\epsilon,\xi \xi + \sum_{i=1}^{n} \beta_{\epsilon, r_i}(u_\epsilon)_{x_i x_i} \geq 0.
\end{align*}
\]

We first estimate \( u_{\epsilon, x_1 x_1} \) from below:

\[
(4.3) \quad u_{\epsilon, x_1 x_1} \geq -C \quad \text{in} \quad \Omega_{1,2\delta}
\]

for any \( \delta > 0 \). To do this we introduce a cut-off function \( \zeta \), \( \zeta = \eta^2 \) where \( \eta = \eta(x_1) \) satisfies:

\[
\eta \in C^\infty_0 \{a_1 + \delta < x_1 < b_1 - \delta\}, \ 0 \leq \eta \leq 1, \ \eta(x_1) = 1 \quad \text{if} \quad a_i + 2\delta < x_1 < b - 2\delta.
\]

Consider the function

\[
(4.4) \quad w = \zeta(x_1)(u_{\epsilon, x_1 x_1})^2 + c^* |\nabla u_\epsilon|^2.
\]
The maximum of $w$ in $\Omega$ is attained at some point $y_0$ in $\Omega$. If $y_0 \in \Omega$ and $u_{\epsilon,x_1x_1}(y_0) \geq -1$ then (4.3) follows. Thus, if $y_0 \in \Omega$ then without loss of generality we may assume that

$$u_{\epsilon,x_1x_1} < -1$$

at $y_0$. But then also

$$u_{\epsilon,x_2x_2} > u_{\epsilon,x_1x_1} + \frac{1}{2}$$

at $y_0$; for otherwise

$$\max(u_{\epsilon,x_1x_1}, u_{\epsilon,x_2x_2}) < -\frac{1}{2}$$

and, since

$$|\beta_\epsilon(r_1, r_2) - \max(r_1, r_2)| \leq \eta_0(\epsilon) \to 0 \text{ if } \epsilon \to 0,$$

we get

$$L_\epsilon u_\epsilon = \epsilon \Delta u_\epsilon + \beta_\epsilon(u_{\epsilon,x_1x_1}, u_{\epsilon,x_2x_2}) - \beta_\epsilon(0) < 0$$

at $y_0$, if $\epsilon$ is small enough, which is a contradiction.

By continuity, (4.5) and (4.6) hold in some neighborhood $N_0$ of $y_0$. This implies that $\beta_\epsilon(u_{\epsilon,x_1x_1}, u_{\epsilon,x_2x_2})$ is independent of $u_{\epsilon,x_1x_1}$ for $x \in N_0$, so that in $N_0$

$$\beta_\epsilon,x_1x_1(u_{\epsilon,x_1x_1}, u_{\epsilon,x_2x_2}) = 0 \text{ in } N_0.$$ (4.7)

We next compute $w_{x_i}$ and $w_{x_1x_1}$ directly from (4.4) and find (writing $u_\epsilon = u$) that in $N_0$

$$-\epsilon \Delta w - \sum_{i=1}^{n} \beta_\epsilon,x_1x_1 w_{x_1x_1} \leq 2\Omega u_{x_1x_1} \left\{-\epsilon \Delta u_{x_1x_1} - \sum_{i=1}^{n} \beta_\epsilon,x_1x_1(u_{x_1x_1})_{x_1x_1} \right\}$$

$$-2 \sum_{i=1}^{n} (\epsilon + \beta_\epsilon,x_1x_1)(2\Omega u_{x_1x_1}^2_{x_1x_1})$$

$$-\sum_{i=1}^{n} (\epsilon + \beta_\epsilon,x_1x_1)(\epsilon'' \delta_{x_1x_1} u_{x_1x_1}^2 + 4\delta_{x_1x_1} u_{x_1x_1} u_{x_1x_1})$$

$$-\sum_{i=1}^{n} (\epsilon + \beta_\epsilon,x_1x_1) \left\{2c^* \sum_{k=1}^{n} u_{x_k} u_{x_kx_1x_1} \right\}$$

$$-\sum_{i=1}^{n} (\epsilon + \beta_\epsilon,x_1x_1) \left\{2c^* \sum_{k=1}^{n} u_{x_kx_1x_1}^2 \right\} \equiv \sum_{j=1}^{5} J_j.$$
We proceed to evaluate the right-hand side in $N_0$. By (4.2) and (4.5), $J_1 \leq 0$ in $N_0$. Since $\beta_{\epsilon, r_i} \geq 0$,

$$J_2 \leq -2\epsilon \zeta \sum_{i=1}^{n} u_{x_1 x_1 x_i}^2.$$ 

By (4.7),

$$J_3 = \{-\epsilon'' u_{x_1 x_1}^2 + 4\epsilon' u_{x_1 x_1} u_{x_1 x_1 x_1}\}$$

$$\leq C\epsilon u_{x_1 x_1}^2 + 2\epsilon u_{x_1 x_1 x_1}^2 + 2\epsilon\frac{(\zeta' u_{x_1 x_1})^2}{\zeta}$$

$$\leq \epsilon \left(C + \frac{2(\zeta')^2}{\zeta}\right) u_{x_1 x_1}^2 + 2\epsilon\zeta u_{x_1 x_1 x_1}^2.$$ 

Next, $J_4 = 0$ by (4.1), and

$$J_5 \leq -\epsilon \cdot 2c^* \sum_{i,k=1}^{n} u_{x_k x_i}^2.$$ 

Combining these estimates we obtain

$$-\epsilon \Delta w - \sum_{i=1}^{n} \beta_{\epsilon, r_i} w_{x_i x_i} \leq \epsilon \left(C + \frac{2(\zeta')^2}{\zeta} - 2c^*\right) u_{x_1 x_1}^2 < 0 \quad \text{in } N_0$$

provided $c^*$ is sufficiently large (note that $(\zeta')^2/\zeta = (2\eta y')^2/\eta^2 \leq C$). This is a contradiction to the maximum principle. Hence $w$ cannot take maximum at an interior point $y_0$.

We conclude that the maximum is attained at a point $y_0$ on $\partial\Omega$. If $y_0 \in \partial\Omega \setminus \partial\Omega_1,\delta$ then $\zeta(y_0) = 0$ and $w \leq C$ ($C$ independent of $\epsilon$), and therefore (4.3) follows. Thus it remains to consider the case where $y_0 \in \partial\Omega \cap \Omega_1,\delta$.

Introduce orthogonal coordinates $(y_1, \ldots, y_n)$ at $y_0$ with $y_n$ in the normal direction (of course, $n = 2$). Then

$$\frac{\partial y_i}{\partial x_i} = a_{ji}, \quad (a_{ij}) \quad \text{is orthogonal}$$

and

$$a_{nn} = a_{22} \geq c > 0$$

(since $y_0 \in \partial\Omega \cap \partial\Omega_1,\delta$).

We can write

$$u_{\epsilon, x_1 x_1} = u_{\epsilon, y_1 y_n} a_{n_i}^2 + \sum_{(k,j) \neq (n,n)} u_{\epsilon, y_j y_k} a_{ji} a_{ki}$$

$$= u_{\epsilon, y_n y_n} a_{n_i}^2 + O(1), \quad \text{by Lemma 3.1.}$$
Also, as before, we may assume that (4.5), (4.6) hold in $N_0$, an $\Omega$-neighborhood of $y_0$. Since

$$|\varepsilon \Delta u_\varepsilon + \max(u_{\varepsilon,x_1x_1}, u_{\varepsilon,x_2x_2})| \leq \frac{1}{2},$$

it follows from (4.6) that

$$|\varepsilon \Delta u_\varepsilon + u_{\varepsilon,x_2x_2}| < \frac{1}{2}. \tag{4.10}$$

Substituting from (4.9) into (4.10) and using (4.8), we conclude that

$$|u_{\varepsilon,y_2y_2}| \leq C \quad \text{at } y_0 \quad (\text{using } |u_{\varepsilon,y_1y_1}| \leq C \text{ at } y_0) \tag{4.11}$$

and then also

$$|u_{\varepsilon,x_1x_1}| \leq C \quad \text{at } y_0.$$

Consequently $w(y_0) \leq C$, and this completes the proof of (4.3).

Similarly,

$$u_{\varepsilon,x_2x_2} \geq -C \quad \text{in } \Omega_{2,2\delta}.$$

Since finally,

$$\varepsilon \Delta u_\varepsilon + \max(u_{\varepsilon,x_1x_1}, u_{\varepsilon,x_2x_2}) \leq 1$$

and $\Delta u_\varepsilon \geq -2C$ in $\tilde{\Omega}_{2\delta}$, it follows that

$$u_{\varepsilon,x_1x_1} \leq C \quad \text{in } \tilde{\Omega}_{2\delta},$$

and Theorem 4.1 follows by taking $\varepsilon \to 0$.

**Remark 4.1.** The above proof breaks down if $n \geq 3$ because of the fact that $a_{in} \quad (i = 2, \ldots, n - 1)$ vanish at some points on $\partial \Omega \cap \partial \Omega_{1,\delta}$.

§5. **Extensions.** Theorem 4.1 can be extended to a large class of HJB equations with degenerate elliptic operators having constant coefficients, provided $n = 2$.

We can also extend Theorem 4.1 to some HJB equations with $n \geq 3$. We shall consider here one example:

$$\max(L^1 u, L^2 u) = f \quad \text{in } \Omega, \tag{5.1}$$

$$u = g \quad \text{on } \partial \Omega, \tag{5.2}$$

where

$$L^1 = a_1 \frac{\partial^2}{\partial x_1^2} + \cdots + a_k \frac{\partial^2}{\partial x_k^2}, \quad 1 \leq k \leq n, \tag{5.3}$$

$$L^2 = b_1 \frac{\partial^2}{\partial x_1^2} + \cdots + b_n \frac{\partial^2}{\partial x_n^2}, \quad 1 \leq l \leq k + 1,$$

$$a_j, b_m \geq \alpha > 0.$$
We can assume without loss of generality that \( f = 0 \).

It is readily seen that the results of §§2,3 extend to the present case. In order to extend Theorem 4.1 we denote by \( \Omega_1 \) the projection of \( \Omega \) on the space \( (x_1, \ldots, x_{l-1}) \) and take

\[
 w = \zeta(x_1, \ldots, x_{l-1})((L^1u)\mathbf{c})^2 + c^*|\nabla u|^2
\]

where \( \zeta \) is a cut-off function with compact support in \( \Omega_1 \). The proof of Theorem 4.1 extends up to the point where the maximum of \( w \) is attained at \( y_0 \), where \( y_0 \) lies on the boundary of \( \partial \Omega \) and \( \zeta(y_0) > 0 \). Analogously to (4.10) we have

\[
(5.4) \quad |\epsilon \Delta u_\epsilon + L^2 u_\epsilon| < \frac{1}{2} \quad \text{near} \quad y_0,
\]

and it remains to show that

\[
(5.5) \quad |u_{\epsilon,y_n,\overrightarrow{y}_n}(y_0)| \leq C.
\]

But, at \( y_0 \),

\[
L^2 u_\epsilon = \sum_{i=1}^{n} b_i u_{\epsilon,x_i} x_i = u_{\epsilon,y_n,\overrightarrow{y}_n} \sum_{i=1}^{n} a_{ni}^2 + O(1)
\]

where Lemma 3.1 has been used. Since \( a_{ni} = d_{x_i} \), it remains to show that

\[
(5.6) \quad \sum_{i=1}^{n} d_{x_i}^2(y_0) = 0
\]

is impossible. Indeed, (5.6) implies that \( d_{x_i} = 0 \) \( (l \leq i \leq n) \) and therefore the normal \( \overrightarrow{n} \) at \( y_0 \) is orthogonal to \( \overrightarrow{x}_i, l \leq i \leq n \); i.e., \( \overrightarrow{n} \) is in the span of \( \overrightarrow{x}_1, \ldots, \overrightarrow{x}_{l-1} \), which is a contradiction since dist \((y_0, \partial \Omega_1) > 0 \) and \( \partial \Omega \) is strictly convex.

Using (5.5) we can deduce, as in §4, that \( w \geq -C \), i.e., \( L^1 u_\epsilon \geq -C \) locally in \( \Omega \). Similarly we can prove that \( L^2 u_\epsilon \geq -C \) and then also that \( L^1 u_\epsilon, L^2 u_\epsilon \) are locally bounded in \( \Omega \).

We summarize:

**Theorem 5.1.** Let \( \Omega \) be a bounded strictly convex domain with \( C^4 \) boundary and let \( f \in C^2(\overline{\Omega}), g \in C^4(\overline{\Omega}) \). Then there exists a unique viscosity solution \( u \) of (5.1), (5.2), and \( u \in W^{1,\infty}(\Omega) \cap W^{2,p}_{\text{loc}}(\Omega) \) for any \( p < \infty \); moreover, \( L^1 u \) and \( L^2 u \) belong to \( L^\infty_{\text{loc}}(\Omega) \).

The same result holds if we replace \( L^1 u \) by \( L^1 u - c_i u \) where \( c_i \) is any nonnegative constant.

**Acknowledgement.** This work is partially supported by National Science Foundation Grant DMS–86–12880.
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