AN INVERSE OBSTACLE PROBLEM: A UNIQUENESS THEOREM FOR SPHERES

By

Changmei Liu

IMA Preprint Series # 1324
July 1995

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
An Inverse Obstacle Problem: A Uniqueness Theorem for Spheres

Changmei Liu*

IMA, University of Minnesota
514 Vincent Hall
206 Church Street S.E.
Minneapolis, MN 55455
May, 1995

Abstract

In the scattering of time-harmonic acoustic or electromagnetic waves, whether an impenetrable sound-soft obstacle $\Omega$ can be completely determined by the scattering amplitude (or the far field pattern) $A_{\Omega}(\xi, k)$ given for $|\xi|^2 = |k|^2$ at fixed wave number $|k|$ and fixed incident plane wave direction $k$ is still a question. In this paper, we show that any sphere in $\mathbb{R}^n (n \geq 3)$ can be uniquely determined by

*Research supported by the NSF through IMA.
its scattering amplitude $A_{\Omega}(\cdot, k)$ given at two linearly independent incident directions $\hat{k}_1$ and $\hat{k}_2$ with one fixed wave number $|k|$. We also show that two spheres in $\mathbb{R}^n (n \geq 2)$ with same scattering amplitude $A_{\Omega}(\cdot, k)$ at only one fixed $k \in \mathbb{R}^n$ must coincide.

1 Introduction

Let $\Omega$ be a bounded simply connected domain (also called an obstacle) in $\mathbb{R}^n (n \geq 2)$. In the inverse scattering theory, one of basic problems is to identify the scattering obstacle $\Omega$ if it is an impenetrable obstacle in the scattering of the time-harmonic acoustic or electromagnetic waves. Considering the case of acoustic waves, we assume the incident field is given by the time-harmonic acoustic plane wave

$$u^i(x, t) = e^{i(k \cdot x - \omega t)}$$

where $|k| = \omega / c_0$ is the wave number, $\omega$ the frequency, $c_0$ the speed of sound and $\hat{k} \in S^{n-1}$ the direction of propagation, where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. In the scattering by an impenetrable obstacle $\Omega$, one of the direct scattering problems is to find the total field $\phi$ such that

$$(\Delta + k^2)\phi = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \bar{\Omega} \quad (1.1)$$

$$\phi(x) = e^{ik \cdot x} + \phi^s(x) \quad (1.2)$$

$$\phi = 0 \quad \text{on} \quad \partial \Omega \quad (1.3)$$

$$\lim_{r \to \infty} r^{n-1} \left[ \phi^s(r \hat{x}) - i \phi^s(r \hat{x}) \right] = 0 \quad \text{uniformly for} \quad \hat{x} \in S^{n-1} \quad (1.4)$$
where $\phi^e_r = \frac{\partial \phi^e}{\partial r}$. The boundary condition (1.3) corresponds to a sound-soft obstacle and the condition (1.4) is called the Sommerfeld radiation condition or Outgoing condition (see [1]). If $\Omega$ is a bounded domain with Lipschitz boundary it is known that solution to (1.1)–(1.4) uniquely exists (see [23, 19, 20]) and has the asymptotics for large $|x|$: \[
abla \phi(x, k) = e^{ik\cdot x} + c_n \frac{|k|^{n-3} e^{ik|\mathbf{x}|}}{|x|^{n-1}} A_{\Omega}(|k|, k) + o\left(\frac{1}{|x|^{n-1}}\right) \quad (1.5)\]
where $c_n$ is a constant just depending on the dimension $n$, $\hat{x} = x/|x|$ and $A_{\Omega}$ is the scattering amplitude (or the far field pattern) of $\phi$. It is well known that $A_{\Omega}$ has a representation: \[A_{\Omega}(\xi, k) = \int_{\partial\Omega} e^{-i\xi \cdot y} \frac{\partial \phi}{\partial n}(y, k) d\sigma(y), \quad |\xi|^2 = |k|^2 \quad (1.6)\]
where $\nu(y)$ is the unit outward normal to $\partial\Omega$ at $y$.

One of questions which has been asked in the inverse scattering for decades is: can an obstacle be uniquely determined from the knowledge of the scattering amplitude? The first result in this direction was due to Schiffer (see [5]) who showed that for problem (1.1) – (1.4) the scattering amplitude $A_{\Omega}(\xi, k)$ given for all $\hat{k} \in S^{n-1}$ and $|k|$ on an interval at fixed one direction $\hat{k}$ uniquely determines the scattering obstacle $\Omega$. Closely related to the uniqueness theorem for the inverse obstacle problem is Karp's theorem [12], which states for problem (1.1) – (1.4) that if $A_{\Omega}(\xi, k) = A_{\Omega}(Q\xi, Qk)$ for all rotations $Q$ and all $\xi, k \in S^{n-1}$ then $\Omega$ is a ball centered at origin. Based on Schiffer's idea, an improved uniqueness theorem obtained by Colton and Kress [6] says that if $\Omega_1$ and $\Omega_2$ are two sound-soft scatters such that the scattering amplitudes coincide for an infinite number of incident plane waves with distinct
directions and one fixed wave number, then $\Omega_1 = \Omega_2$. Since then the inverse obstacle problem has been reduced to: can $A_\Omega(\xi, k)$ given for all $\hat{\xi} \in S^{n-1}$ with one fixed incident direction $\hat{k}$ and one fixed wave number $|k|$ completely determine the obstacle?

To answer this question and reconstruct the obstacle, versatile methods have been developed and a number of results were obtained also [5, 6, 11, 2, 13, 22, 17, 21, 24, 14, 15, 16, 7, 8, 9]. In paper [18], we obtained a scattering theory analogue of a Polya's theorem and some uniqueness results. For example, two polyhedra must coincide if they correspond to same scattering amplitude at fixed $k$. A consequence of the scattering theory analogue is that at most finitely many distinct Lipschitz domains correspond to same scattering amplitude at fixed $k$ in $\mathbb{R}^n$. Based on this fact, we are able to prove the following uniqueness theorem for spheres in $\mathbb{R}^n (n \geq 3)$ by looking at the explicit series solution scattered by a sphere and examining the corresponding scattering amplitude under certain rotations.

**Theorem 1.1** Suppose that $\Omega_1$ is a ball and $\Omega_2$ is a Lipschitz domain in $\mathbb{R}^n (n \geq 3)$. If $A_{\Omega_1}(\xi, k_j) = A_{\Omega_2}(\xi, k_j)$ on $|\xi|^2 = |k_j|^2$ at two linearly independent incident directions $\hat{k}_j$, $j = 1, 2$ with one wave number $|k|$, then $\Omega_2$ is a ball also and $\Omega_1 = \Omega_2$.

For circles in $\mathbb{R}^2$, we are not successful in obtaining such a uniqueness result by using same argument. However, it is not hard to obtain a uniqueness among circles even if the scattering amplitude is just given at one incident direction and one wave number. This result is contained in the following theorem.
Theorem 1.2 Suppose that $\Omega_1$ is a ball and $\Omega_2$ is a Lipschitz domain in $\mathbb{R}^n$. Assume $A_{\Omega_1}(\xi, k) = A_{\Omega_2}(\xi, k)$ on $|\xi|^2 = |k|^2$ at one $k \in \mathbb{R}^n$. Then the $k$-core (see the definition 2.1 in section 2) of $\Omega_2$ is the center of the ball $\Omega_1$ and $\partial \Omega_2$ is analytic except a compact set $K$ with finitely many connected components and zero surface measure on $\partial \Omega_2$.

Especially, $\Omega_2 = \Omega_1$ if $\Omega_2$ is a ball also; i.e., any two spheres with same scattering amplitude $A_\Omega(\cdot, k)$ at one incident direction must coincide.

The proof for Theorem 1.1 is given by Theorem 3.1 in Section 3 and Theorem 4.1 in Section 4. Theorem 1.2 is the combination of Theorem 3.1, Theorem 3.2 and Theorem 3.3 in section 3 where the proof for each theorem is given also.

Acknowledgements

It is a pleasure to thank Professor Adrian Nachman for his guiding me to this interesting problem and giving some useful ideas. I also like to thank Professor Michael Taylor for his crucial hint in the proof of Theorem 4.1.

2 Some Preliminaries

For the convenience to prove our theorems in Section 1, we introduce some notations, definitions, the scattering theory analogue of a Polya's theorem and a consequence which can be found in paper [18].
Definition 2.1 Given an exterior solution $\phi$ of the Helmholtz equation $(\Delta + k^2)\phi = 0$ for $|x| > R$, define a convex set $D \subset \{|x| \leq R\}$ which we'll call the $k$-core of $\Omega$ as follows:

For each $\eta \in S^{n-1}$, define

$$j(\eta) = \inf\{a : \phi \text{ solves } (\Delta + k^2)\phi = 0 \text{ in } x \cdot \eta > a\}$$  \hspace{1cm} (2.1)

and

$$D = \cap_{\eta \in S^{n-1}}\{x \cdot \eta \leq j(\eta)\}.$$  \hspace{1cm} (2.2)

Obviously $D$ is a closed convex set and $h_D(\eta) = j(\eta)$ where $h_D(\eta)$ is the supporting function on $D$. $D$ is also the smallest convex set such that $\phi$ extends to $\mathbb{R}^n \setminus D$.

Let $A_\Omega(\xi, k)$ be the scattering amplitude corresponding to $\phi(x, k)$ in (1.1)--(1.4). Then $A_\Omega$ can be extended to an entire function of $\zeta \in \mathbb{C}^n$ by the expression (1.6). It is easy to see that

$$|A_\Omega(\zeta, k)| \leq C e^{h_\Omega(\zeta_I)}$$  \hspace{1cm} (2.3)

where $h_\Omega(\zeta_I) = \sup_{x \in \Omega} x \cdot \zeta_I = \sup_{x \in \partial \Omega} x \cdot \zeta_I$.

Definition 2.2 we define the indicator function $h_\Lambda$ of a scattering amplitude $A_\Omega$ by

$$h_\Lambda(\zeta_I) = \sup_{\omega^\perp} \lim_{r \to \infty} \frac{\log |A_\Omega(\zeta, k)|}{r}$$  \hspace{1cm} (2.4)

where $\zeta \in \mathbb{C}^n$ and $\omega^\perp \in S^{n-1}$ satisfy

$$\omega^\perp \cdot \zeta_I = 0 \text{ and } \zeta = (\sqrt{r^2 + k^2})\omega^\perp + ir\zeta_I$$

(note $\zeta^2 = k^2$).
Note that the indicator function $h_A$ of $A_\Omega$, defined by (2.4), involves only the values of $A_\Omega$ on $\zeta^2 = k^2$. Extend $h_A$ to all $\eta \in \mathbb{R}^n$ by

$$h_A(\eta) = |\eta|h_A(\eta).$$

(2.5)

The relation between the indicator function of $A_\Omega$ and the $k$-core of $\Omega$ (we call a scattering theory analogue of a Polya's theorem) is given by following theorem [18].

**Theorem 2.1** Let $\phi$ be the exterior solution to (1.1)-(1.4) and $A_\Omega$ the corresponding scattering amplitude. Then the indicator function $h_A$ of $A_\Omega$ is equal to the support function of the $k$-core of $\Omega$. Therefore, the $k$-core of $\Omega$ can be uniquely determined from $A_\Omega$.

A consequence of above theorem is as follows [18]:

**Corollary 2.2** At most finite distinct Lipschitz obstacles correspond to same $A_\Omega(\cdot, k)$ at fixed $k \in \mathbb{R}^n$ and all $k$-cores coincide.

### 3 The Uniqueness at One Incident Direction

In this section, we prove Theorem 1.2. The uniqueness statement in Theorem 1.2 is given by following theorem.

**Theorem 3.1** Any two spheres in $\mathbb{R}^n$ corresponding same scattering amplitude $A_\Omega(\cdot, k)$ must coincide.

**Proof:** From the reflection principle across the boundary of a ball [3] we see that the exterior solution $\phi$ to (1.1)-(1.4) for a ball can be extended to the
whole space $\mathbb{R}^n$ except the center of the ball. Then by analytic continuation we obtain that two balls must be concentric if they correspond to same scattering amplitude at one $k$.

Next we prove that these two concentric balls must be same. It is known that the incident plane wave $e^{ik \cdot x}$ has a series form (see [10] for example):

$$e^{ik \cdot x} \sim (2\pi)^{n/2} |x|^{\frac{n-2}{2}} \sum_{l=0}^{\infty} \sum_{m=1}^{N} \frac{i^l \tilde{Y}^*_{lm}(\hat{k})J_{\frac{n-2}{2}+l}(|k||x|)}{H_{\frac{n-2}{2}+l}^{(1)}(|k||x|)} Y_{lm}(\hat{x})$$  \hspace{1cm} (3.1)

where $N = \frac{(l+n-2)!(l+n-2)}{l!(n-2)!}$ is the dimension of the space of spherical harmonics of degree $l$, $Y_{lm}(\hat{x})$, $m = 1, \cdots, N$, is an orthonormal basis for this space, $J_{\frac{n-2}{2}+l}$ is the bessel function of the first kind of order $\frac{n-2}{2} + l$ and * is the complex conjugate. The scattering amplitude $A_\Omega(\cdot, k)$ for the sphere of radius $R$ is given by

$$A_\Omega(\xi, k) = \frac{2^{n+1} \pi^n e^{-i(n+1)\pi/4}}{|k|^{n-2}} \sum_{l=0}^{\infty} \sum_{m=1}^{N} \frac{J_{\frac{n-2}{2}+l}(|k|R)}{H_{\frac{n-2}{2}+l}^{(1)}(|k|R)} Y^*_{lm}(\hat{k})Y_{lm}(\hat{\xi})$$  \hspace{1cm} (3.2)

where $H_{\frac{n-2}{2}+l}^{(1)}$ is the first kind of Hankel function of order $\frac{n-2}{2} + l$ (it is known that this series formula is coincident with formula (1.6)). If there are two spheres with radii $a < b$ corresponding to same scattering amplitude $A_\Omega(\cdot, k)$, then from (3.2) we see that $A_\Omega(\cdot, k)$ has two series representations. Thus

$$\sum_{l=0}^{\infty} \sum_{m=1}^{N} \frac{J_{\frac{n-2}{2}+l}(|k|a)}{H_{\frac{n-2}{2}+l}^{(1)}(|k|a)} Y^*_{lm}(\hat{k})Y_{lm}(\hat{\xi}) = \sum_{l=0}^{\infty} \sum_{m=1}^{N} \frac{J_{\frac{n-2}{2}+l}(|k|b)}{H_{\frac{n-2}{2}+l}^{(1)}(|k|b)} Y^*_{lm}(\hat{k})Y_{lm}(\hat{\xi}).$$

Since $\{Y^*_{lm}(\hat{k})Y_{lm}(\hat{\xi})\}$ are orthonormal in $L^2(S^{n-1} \times S^{n-1})$ we have

$$\frac{J_{\frac{n-2}{2}+l}(|k|a)}{H_{\frac{n-2}{2}+l}^{(1)}(|k|a)} = \frac{J_{\frac{n-2}{2}+l}(|k|b)}{H_{\frac{n-2}{2}+l}^{(1)}(|k|b)}, \quad l = 0, 1, 2, \cdots$$  \hspace{1cm} (3.3)
Formula (3.3) is equivalent to
\[
\frac{J_{\frac{n-2}{2}+l}(|k|a)}{J_{\frac{n-2}{2}+l}(|k|b)} = \frac{H^{(1)}_{\frac{n-2}{2}+l}(|k|a)}{H^{(1)}_{\frac{n-2}{2}+l}(|k|b)}, \quad l = 0, 1, 2, \ldots
\] (3.4)

However, note that for each fixed \( x \in \mathbb{R} \)
\[
J_{\frac{n-2}{2}+l}(x) \sim \frac{(x/2)^{\frac{n-2}{2}+l}}{\Gamma\left(\frac{n-2}{2} + l + 1\right)}, \quad l \to \infty
\]
and
\[
H^{(1)}_{\frac{n-2}{2}+l}(x) \sim -\frac{i}{\pi} \Gamma\left(\frac{n-2}{2} + l\right)(2/x)^{\frac{n-2}{2}+l}, \quad l \to \infty.
\]

So
\[
\lim_{l \to \infty} \frac{J_{\frac{n-2}{2}+l}(|k|a)}{J_{\frac{n-2}{2}+l}(|k|b)} = 0
\]
and
\[
\lim_{l \to \infty} \frac{H^{(1)}_{\frac{n-2}{2}+l}(|k|a)}{H^{(1)}_{\frac{n-2}{2}+l}(|k|b)} = \infty.
\]

This is a contradiction. Thus \( a = b \) and the theorem is proved.

### Theorem 3.2
Suppose that \( \Omega_1 \) is a ball and \( \Omega_2 \) is a Lipschitz domain in \( \mathbb{R}^n \).

Assume \( A_{\Omega_1}(\xi, k) = A_{\Omega_2}(\xi, k) \) on \( |\xi|^2 = |k|^2 \) at one \( k \in \mathbb{R}^n \). Then the \( k \)-core of \( \Omega_2 \) is the center of the ball \( \Omega_1 \) and is not on \( \partial \Omega_2 \). Also, \( \partial \Omega_2 \) is analytic at \( x \) where \( \nabla \phi(x) \neq 0 \).

**Proof:** From the proof of Theorem 3.1 we see that the exterior solution \( \phi^s \) has a series representation:
\[
\phi^s(x, k) = c |x|^{-(n-2)/2} \sum_{l=0}^{\infty} \sum_{m=1}^{N} \frac{J_{\frac{n-2}{2}+l}(|k|R)}{H^{(1)}_{\frac{n-2}{2}+l}(|k|R)} \frac{H^{(1)}_{\frac{n-2}{2}+l}(|k| |x|)}{H^{(1)}_{\frac{n-2}{2}+l}(|k| R)} Y^*_{lm}(\hat{k}) Y_{lm}(\hat{x})
\] (3.5)
where \( c = \sqrt{2\pi} \frac{2^{n+1} \pi^{n+1}}{|k|^{n-2}} \) and \( R \) is the radius of the ball. So the exterior solution to (1.1) - (1.4) \( \phi \) can be extended to the whole space \( \mathbb{R}^n \) except the center of the ball. It is clear that the center (the \( k \)-core of \( \Omega_2 \)) is either inside of \( \Omega_2 \) or on the boundary of \( \Omega_2 \). It is also easy to see that the center cannot be on \( \partial \Omega_2 \). In fact, from series form (3.5) we know that both \( \phi^s \) and \( \frac{\partial \phi^s}{\partial \nu} \) are not \( L^2 \) integrable on any cone neighborhood with the cone vertex at the center of the ball \( \Omega_1 \). However, in paper [18] (Theorem 4.5) we showed that \( \phi^s \) is in \( L^2((\Omega_2)')_\rho \) where \( (\Omega_2)'_\rho \) is the set \( \{ x \in \mathbb{R}^n \setminus \Omega_2 : |x| < \rho \} \). This is a contradiction. So the center of the ball must be inside of \( \Omega_2 \).

The analyticity of \( \partial \Omega_2 \) at \( x \) when \( \nabla \phi(x) \neq 0 \) follows from the analyticity of the exterior solution \( \phi \) and the implicit function theorem.

**Theorem 3.3** If \( \Omega_1, \Omega_2, A_{\Omega_1} \) and \( A_{\Omega_2} \) are as in theorem 3.2, then the boundary of \( \Omega_2 \) is analytic except a compact set \( K \) which has finitely many connected components and zero surface measure on \( \partial \Omega_2 \).

**Proof:** If \( \partial \Omega_2 \) is smooth, we can use Holmgren's Uniqueness theorem (see Theorem 6.12 in [5]) to give an easier proof. But for a general Lipschitz domain, we need to achieve the result without using the Holmgren's Uniqueness theorem.

From Theorem 3.2 we know that the center of the ball \( \Omega_1 \) is inside \( \Omega_2 \) and \( \partial \Omega_2 \) is analytic at \( x \) where \( \nabla \phi(x) \neq 0 \). Since the exterior solution to (1.1) - (1.4) \( \phi \) is analytic near \( \partial \Omega_2 \), the set

\[
K_1 = \{ x \in \partial \Omega_2 : \nabla \phi(x) = 0 \}
\]  

(3.6)

is locally connected and closed. Then the compactness of \( \partial \Omega_2 \) implies that \( K_1 \) is compact. By Corollary 27.11 [25] \( K_1 \) has only finitely many connected
components $K_1^1, K_1^2, \ldots, K_1^{n_1}$. If none of $K_1^i$ has interior points, we are done. In fact, $K_1$ having finite closed components implies that $\partial \Omega_2 \setminus K_1$ has finite open components $U_1, U_2, \ldots, U_m$. Then, denoting the surface measure by $\mu$, we have
\[
\mu(\partial \Omega_2) = \mu(K_1) + \mu(\partial \Omega_2 \setminus K_1)
\]
and
\[
\mu(\partial \Omega_2 \setminus K_1) = \mu(\bigcup_{j=1}^{m} U_j) = \sum_{j=1}^{m} \mu(U_j)
\]
\[
= \sum_{j=1}^{m} \mu(\bar{U}_j) \geq \mu(\bigcup_{j=1}^{m} \bar{U}_j)
\]
\[
= \mu(\bigcup_{j=1}^{m} U_j) = \mu(\partial \Omega_2)
\]

since $K_1$ has no interior points which implies that $\partial \Omega_2 \setminus K_1$ is densely open in $\partial \Omega_2$. Then $\mu(K_1) = 0$.

If some of $\{K_1^i\}$ have interior points, say $K_1^1$, then $\nabla \phi(x) \equiv 0$ on $K_1^1$. Let
\[
K_2 = \{ x \in K_1^1 : \frac{\partial^\beta \phi(x)}{\partial x^\beta} = 0, \ |\beta| = 2 \}.
\]  
(3.7)

Since $K_1^1$ is connected and compact, $K_2$ is compact and locally connected. Again by Corollary 27.11 [25] $K_2$ has finitely many connected components $K_2^1, K_2^2, \ldots, K_2^{n_2}$. If some of $\{K_2^i\}$ have interior points, say $K_2^1$, then repeating above process we get $K_3^1$. Repeating this process again and again we obtain a sequence of sets:
\[
K_1^1 \supset K_2^1 \supset K_3^1 \supset \cdots
\]  
(3.8)

and each $K_j^1$ is connected compact set with interior points and $\frac{\partial^\beta \phi(x)}{\partial x^\beta} \equiv 0$ on $K_j^1$ for all $|\beta| \leq j$. 

11
Claim: This process will stop at some finite integer $M$. If not, from the construction of $K^1_j$ we know that there is a point $x_0 \in \cap_{j=1}^{\infty} K^1_j$ such that

$$\frac{\partial^\beta \phi(x_0)}{\partial x^\beta} = 0$$

for all multi-index $\beta$.

However this fact and the analyticity of $\phi$ imply that $\phi \equiv 0$ in a neighborhood of $x_0$. Thus $\phi$ is identically zero which is a contradiction.

If $\frac{\partial^\beta \phi(x)}{\partial x^\beta} \equiv 0$ in a neighborhood of some point $y$ and $\frac{\partial^{\beta+1} \phi(y)}{\partial x^{\beta+1}} \neq 0$ for some multi-index $\beta$, then the analyticity of $\phi$ implies that $\partial \Omega_2$ is analytic at $y$.

Since each $K_j$ has only finite components and for each connected component with interior points above process stops in finite steps, there are only finitely many closed connected subsets of $\partial \Omega_2$ without interior points such that on each such a subset, $\frac{\partial^\beta \phi(x)}{\partial x^\beta} \equiv 0$, $|\beta| \leq N$ for some integer $N$. Therefore the measure of these subsets are zero. Clearly all other points on $\partial \Omega_2$ are analytic.

Remark: The fact that there are only finitely many closed connected subsets of $\partial \Omega_2$ without interior points such that on each such a subset, $\frac{\partial^\beta \phi(x)}{\partial x^\beta} \equiv 0$, $|\beta| \leq N$ for some integer $N$ proved above maybe is known before. But the author is not aware of any reference. So we give the argument here for the convenience.

4 The Uniqueness at Two Incident Directions

In this section, we give the proof for Theorem 1.1. Since Theorem 3.1 we only need to prove that any domain corresponding to the same scattering
amplitude as that of a sphere at two linearly independent incident directions must be a sphere also. We finish this by the following theorem.

**Theorem 4.1** If a domain $\Omega$ in $\mathbb{R}^n (n \geq 3)$ corresponds to the same scattering amplitude as that of a sphere at two linearly independent incident directions, then this domain must be a sphere.

**Proof:** Let $\hat{k}_1$ and $\hat{k}_1$ denote two linearly independent directions and $S^2_\alpha$ denote a sphere with radius $a$. According to the assumptions in the theorem,

$$A_\Omega(\xi, |k|\hat{k}_j) = A_{S^2_\alpha}(\xi, |k|\hat{k}_j), \quad \text{on} \quad |\xi|^2 = |k|^2, \quad j = 1, 2 \quad (4.1)$$

Note that the identity

$$A_\Omega(\xi, |k|\hat{k}_j) = A_{Q\Omega}(Q\xi, |k|Q\hat{k}_j). \quad (4.2)$$

holds for any rotation $Q$ in $\mathbb{R}^n$ where $Q\Omega$ means the transform of $\Omega$ under rotation $Q$. Also note that $Q(S^2_{\alpha}) = S^2_{\alpha}$ for any rotation $Q$. For fixed $\hat{k}_1$, let $\text{Rot}$ be the set of those rotations in $\mathbb{R}^n$ such that $Q\hat{k}_1 = \hat{k}_1$ for $Q$ in $\text{Rot}$ ($\text{Rot}$ is empty in $\mathbb{R}^2$, this is why the proof is broken down for $n = 2$). Note that if $Q$ is in $\text{Rot}$ so is $Q^{-1}$ and $Q^{-1} = Q^*$ where $Q^{-1}$ is the inverse of $Q$ and $Q^*$ is the transpose of $Q$. Let $Q$ in $\text{Rot}$, then

$$A_\Omega(\xi, |k|\hat{k}_1) = A_{S^2_\alpha}(\xi, |k|\hat{k}_1) = A_{S^2_\alpha}(Q\xi, |k|Q\hat{k}_1)$$

$$= A_{S^2_\alpha}(Q\xi, |k|\hat{k}_1) = A_\Omega(Q\xi, |k|\hat{k}_1)$$

$$= A_{Q\cdot\Omega}(\xi, |k|Q^*\hat{k}_1) = A_{Q\cdot\Omega}(\xi, |k|\hat{k}_1)$$

Thus $A_\Omega(\xi, |k|\hat{k}_1) = A_{Q\cdot\Omega}(\xi, |k|\hat{k}_1)$ for all rotations $Q$ in $\text{Rot}$. This implies that $\Omega$ is symmetric w.r.t. the vector $\hat{k}_1$ for otherwise we can find an infinite
sequence \( \{Q_j\}_{j=1}^{\infty} \) such that \( Q_j^*\Omega \neq Q_i^*\Omega \) for \( j \neq l \). This is contradictory to Corollary 2.2.

Replacing \( \hat{k}_1 \) by \( \hat{k}_2 \) we obtain that \( \Omega \) is symmetric w.r.t. the vector \( \hat{k}_2 \) also. It is known that a geometric obstacle symmetric to two linearly independent vectors \( \hat{k}_1 \) and \( \hat{k}_2 \) must be a sphere. Therefore the proof is completed.

References


<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1229</td>
<td>Zhangxin Chen</td>
<td>Large-scale averaging analysis of multiphase flow in fractured reservoirs</td>
</tr>
<tr>
<td>1230</td>
<td>Bruce Hajek &amp; Babu Narayanan</td>
<td>Multigraphs with the most edge covers</td>
</tr>
<tr>
<td>1231</td>
<td>K.B. Athreya</td>
<td>Entropy maximization</td>
</tr>
<tr>
<td>1232</td>
<td>F.I. Karpelevich &amp; Yu.M. Suhov</td>
<td>Functional equations in the problem of boundedness of stochastic branching dynamics</td>
</tr>
<tr>
<td>1233</td>
<td>E. Dibenedetto &amp; V. Vespri</td>
<td>On the singular equation $\beta(u)_t = \Delta u$</td>
</tr>
<tr>
<td>1234</td>
<td>M.Ya. Kelbert &amp; Yu.M. Suhov</td>
<td>The Markov branching random walk and systems of reaction-diffusion (Kolmogorov-Petrovskii-Piskunov) equations</td>
</tr>
<tr>
<td>1235</td>
<td>M. Hildebrand</td>
<td>Random walks on random regular simple graphs</td>
</tr>
<tr>
<td>1236</td>
<td>W.S. Don &amp; A. Solomonoff</td>
<td>Accuracy enhancement for higher derivatives using Chebyshev collocation and a mapping technique</td>
</tr>
<tr>
<td>1237</td>
<td>D. Gurarie</td>
<td>Symmetries and conservation laws of two-dimensional hydrodynamics</td>
</tr>
<tr>
<td>1238</td>
<td>Z. Chen</td>
<td>Finite element methods for the black oil model in petroleum reservoirs</td>
</tr>
<tr>
<td>1239</td>
<td>G. Bao &amp; A. Friedman</td>
<td>Inverse problems for scattering by periodic structure</td>
</tr>
<tr>
<td>1240</td>
<td>G. Bao</td>
<td>Some inverse problems in partial differential equations</td>
</tr>
<tr>
<td>1241</td>
<td>G. Bao</td>
<td>Diffractive optics in periodic structures: The TM polarization</td>
</tr>
<tr>
<td>1242</td>
<td>C.C. Lim &amp; D.A. Schmidt</td>
<td>On noneven digraphs and symplectic pairs</td>
</tr>
<tr>
<td>1243</td>
<td>H.M. Soner, S.E. Shreve &amp; J. Cvitanić</td>
<td>There is no nontrivial hedging portfolio for option pricing with transaction costs</td>
</tr>
<tr>
<td>1244</td>
<td>D.L. Russell &amp; B-Y. Zhang</td>
<td>Exact controllability and stabilizability of the Korteweg-de Vries equation</td>
</tr>
<tr>
<td>1245</td>
<td>B. Morton, D. Enns &amp; B-Y. Zhang</td>
<td>Stability of dynamic inversion control laws applied to nonlinear aircraft pitch-axis models</td>
</tr>
<tr>
<td>1246</td>
<td>S. Hansen &amp; G. Weiss</td>
<td>New results on the operator Carleson measure criterion</td>
</tr>
<tr>
<td>1247</td>
<td>V.A. Malyshev &amp; F.M. Spieksma</td>
<td>Intrinsic convergence rate of countable Markov chains</td>
</tr>
<tr>
<td>1248</td>
<td>G. Bao, D.C. Dobson &amp; J.A. Cox</td>
<td>Mathematical studies in rigorous grating theory</td>
</tr>
<tr>
<td>1249</td>
<td>G. Bao &amp; W.W. Symes</td>
<td>On the sensitivity of solutions of hyperbolic equations to the coefficients</td>
</tr>
<tr>
<td>1250</td>
<td>D.A. Huntley &amp; S.H. Davis</td>
<td>Oscillatory and cellular mode coupling in rapid directional solidification</td>
</tr>
<tr>
<td>1251</td>
<td>M.J. Donahue, L. Gurvits, C. Darken &amp; E. Sonntag</td>
<td>Rates of convex approximation in non-Hilbert spaces</td>
</tr>
<tr>
<td>1252</td>
<td>A. Friedman &amp; B. Hu</td>
<td>A Stefan problem for multi-dimensional reaction diffusion systems</td>
</tr>
<tr>
<td>1253</td>
<td>J.L. Bona &amp; B-Y. Zhang</td>
<td>The initial-value problem for the forced Korteweg-de Vries equation</td>
</tr>
<tr>
<td>1254</td>
<td>A. Friedman &amp; R. Gulliver, Organizers</td>
<td>Mathematical modeling for instructors</td>
</tr>
<tr>
<td>1255</td>
<td>S. Kichenassamy</td>
<td>The prolongation formula for tensor fields</td>
</tr>
<tr>
<td>1256</td>
<td>S. Kichenassamy</td>
<td>Fuchsian equations in Sobolev spaces and blow-up</td>
</tr>
<tr>
<td>1257</td>
<td>H.S. Dumas, L. Dumas, &amp; F. Golse</td>
<td>On the mean free path for a periodic array of spherical obstacles</td>
</tr>
<tr>
<td>1258</td>
<td>C. Liu</td>
<td>Global estimates for solutions of partial differential equations</td>
</tr>
<tr>
<td>1259</td>
<td>C. Liu</td>
<td>Exponentially growing solutions for inverse problems in PDE</td>
</tr>
<tr>
<td>1260</td>
<td>Mary Ann Horn &amp; I. Lasiecka</td>
<td>Nonlinear boundary stabilization of parallely connected Kirchhoff plates</td>
</tr>
<tr>
<td>1261</td>
<td>B. Cockburn &amp; H. Gau</td>
<td>A posteriori error estimates for general numerical methods for scalar conservation laws</td>
</tr>
<tr>
<td>1262</td>
<td>B. Cockburn &amp; P-A. Gremaud</td>
<td>A priori error estimates for numerical methods for scalar conservation laws. Part I: The general approach</td>
</tr>
<tr>
<td>1263</td>
<td>R. Spigler &amp; M. Vianello</td>
<td>Convergence analysis of the semi-implicit euler method for abstract evolution equations</td>
</tr>
<tr>
<td>1264</td>
<td>R. Spigler &amp; M. Vianello</td>
<td>WKB-type approximation for second-order differential equations in $C^*$-algebras</td>
</tr>
<tr>
<td>1265</td>
<td>M. Menshikov &amp; R.J. Williams</td>
<td>Passage-time moments for continuous non-negative stochastic processes and applications</td>
</tr>
<tr>
<td>1266</td>
<td>C. Mazza</td>
<td>On the storage capacity of nonlinear neural networks</td>
</tr>
<tr>
<td>1267</td>
<td>Z. Chen, R.E. Ewing &amp; R. Lazarov</td>
<td>Domain decomposition algorithms for mixed methods for second order elliptic problems</td>
</tr>
<tr>
<td>1268</td>
<td>Z. Chen, M. Espedal &amp; R.E. Ewing</td>
<td>Finite element analysis of multiphase flow in groundwater hydrology</td>
</tr>
<tr>
<td>1270</td>
<td>S. Kichenassamy &amp; G.K. Srinivasan</td>
<td>The structure of WTC expansions and applications</td>
</tr>
<tr>
<td>1271</td>
<td>A. Zinger</td>
<td>Positiveness of Wigner quasi-probability density and characterization of Gaussian distribution</td>
</tr>
<tr>
<td>1272</td>
<td>V. Malkin &amp; G. Papanicolaou</td>
<td>On self-focusing of short laser pulses</td>
</tr>
<tr>
<td>1273</td>
<td>J.N. Kutz &amp; W.L. Kath</td>
<td>Stability of pulses in nonlinear optical fibers using phase-sensitive amplifiers</td>
</tr>
<tr>
<td>1274</td>
<td>S.K. Patch</td>
<td>Recursive recovery of a family of Markov transition probabilities from boundary value data</td>
</tr>
<tr>
<td>1275</td>
<td>C. Liu</td>
<td>The completeness of plane waves</td>
</tr>
<tr>
<td>1276</td>
<td>Z. Chen &amp; R.E. Ewing</td>
<td>Stability and convergence of a finite element method for reactive transport in ground water</td>
</tr>
</tbody>
</table>
1277 Z. Chen & Do Y. Kwak, The analysis of multigrid algorithms for nonconforming and mixed methods for second order elliptic problems
1278 Z. Chen, Expanded mixed finite element methods for quasilinear second order elliptic problems II
1279 M.A. Horn & W. Littman, Boundary control of a Schrödinger equation with nonconstant principal part
1281 S. Maliassov, Substructuring preconditioning for finite element approximations of second order elliptic problems. II. Mixed method for an elliptic operator with scalar tensor
1282 V. Jakšić & C.-A. Pillet, On model for quantum friction II. Fermi's golden rule and dynamics at positive temperatures
1283 V. M. Malkin, Kolmogorov and nonstationary spectra of optical turbulence
1284 E.G. Kalnins, V.B. Kuznetsov & W. Miller, Jr., Separation of variables and the XXZ Gaudin magnet
1285 E.G. Kalnins & W. Miller, Jr., A note on tensor products of $q$-algebra representations and orthogonal polynomials
1286 E.G. Kalnins & W. Miller, Jr., $q$-algebra representations of the Euclidean, pseudo-Euclidean and oscillator algebras, and their tensor products
1287 L.A. Pastur, Spectral and probabilistic aspects of matrix models
1288 K. Kastella, Discrimination gain to optimize detection and classification
1289 L.A. Peletier & W.C. Troy, Spatial patterns described by the Extended Fisher-Kolmogorov (EFK) equation: Periodic solutions
1290 A. Friedman & Y. Liu, Propagation of cracks in elastic media
1291 A. Friedman & C. Huang, Averaged motion of charged particles in a curved strip
1292 G. R. Sell, Global attractors for the 3D Navier-Stokes equations
1293 C. Liu, A uniqueness result for a general class of inverse problems
1294 H.O. Kreiss, Numerical solution of problems with different time scales II
1295 B. Cockburn, G. Gripenberg, S-O. Londen, On convergence to entropy solutions of a single conservation law
1296 S-H. Yu, On stability of discrete shock profiles for conservative finite difference scheme
1297 H. Behncke & P. Rejto, A limiting absorption principle for separated Dirac operators with Wigner Von Neumann type potentials
1298 R. Lipton B. Vernescu, Composites with imperfect interface
1299 E. Casas, Pontryagin’s principle for state-constrained boundary control problems of semilinear parabolic equations
1300 G.R. Sell, References on dynamical systems
1301 J. Zhang, Swelling and dissolution of polymer: A free boundary problem
1302 J. Zhang, A nonlinear nonlocal multi-dimensional conservation law
1303 M.E. Taylor, Estimates for approximate solutions to acoustic inverse scattering problems
1304 J. Kim & D. Sheen, A priori estimates for elliptic boundary value problems with nonlinear boundary conditions
1305 B. Engquist & E. Luo, New coarse grid operators for highly oscillatory coefficient elliptic problems
1306 A. Boutet de Monvel & I. Egorova, On the almost periodicity of solutions of the nonlinear Schrödinger equation with the cantor type spectrum
1307 A. Boutet de Monvel & V. Georgescu, Boundary values of the resolvent of a self-adjoint operator: Higher order estimates
1308 S.K. Patch, Diffuse tomography modulo Graßmann and Laplace
1309 A. Friedman & J.J.L. Velázquez, Liouville type theorems for fourth order elliptic equations in a half plane
1310 T. Aktosun, M. Klaus & C. van der Mee, Recovery of discontinuities in a nonhomogeneous medium
1311 V. Bondarevsky, On the global regularity problem for 3-dimensional Navier-Stokes equations
1312 M. Cheney & D. Isaacson, Inverse problems for a perturbed dissipative half-space
1313 B. Cockburn, D.A. Jones & E.S. Titi, Determining degrees of freedom for nonlinear dissipative equations
1314 B. Engquist & E. Luo, Convergence of a multigrid method for elliptic equations with highly oscillatory coefficients
1315 L. Pastur & M. Shcherbina, Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles
1316 V. Jakšić, S. Molchanov & L. Pastur, On the propagation properties of surface waves
1317 J. Nečas, M. Ružička & V. Šverák, On self-similar solutions of the Navier-Stokes equations
1318 S. Stojanovic, Remarks on $W^{2,p}$-solutions of bilateral obstacle problems
1319 E. Luo & H-O. Kreiss, Pseudospectral vs. Finite difference methods for initial value problems with discontinuous coefficients
1320 V.E. Kriukov, Soliton’s rebuilding in one-dimensional Schrödinger model with polynomial nonlinearity
1321 J.M. Harrison & R.J. Williams, A multiclass closed queueing network with unconventional heavy traffic behavior
1322 M.E. Taylor, Microlocal analysis on Morrey spaces
1323 C. Huang, Homogenization of biharmonic equations in domains perforated with tiny holes
1324 C. Liu, An inverse obstacle problem: A uniqueness theorem for spheres