GHOST TORI FOR MONOTONE MAPS

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IMA Preprint Series # 677
July 1990
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Abstract. Certain higher dimensional analogs of twist maps on \( T^n \times \mathbb{R}^n \) come equipped with a discrete variational problem. We study the gradient flow of the action and find invariant sets with the cohomology of \( T^n \). This implies a topological lower bound on the number of periodic orbits of each given prime rotation vector. For the twist map case, one can extract circles that interpolate Aubry-Mather sets from these invariant sets.

Key words. Symplectic, monotone maps, ghost tori, Aubry-Mather sets

0. Introduction. This article is a survey of the author’s work on twist maps of \( S^1 \times \mathbb{R} \) and their higher dimensional symplectic analog on \( T^n \times \mathbb{R}^n \), called monotone maps ([G 1,2,3]). With such a map \( F \) is associated a "discrete lagrangian" on the space \( (\mathbb{R}^n)^2 \) of bi-infinite sequences whose critical points correspond one to one to the orbits of \( F \). This is analogous to the Hamiltonian setting where one finds periodic orbits as critical points of the action on the loops space of the manifold. Our main result is a variation of the Arnold’s conjecture [Ar] which states that certain symplectic maps on a manifold have at least as many fixed points as Morse theory garanties critical points for a real valued function on this manifold.

More precisely, we find at least \( cl(T^n) = n + 1 \) periodic orbits of any given prime rotation vector and \( sb(T^n) = 2^n \) if they are nondegenerate. Since \( T^n \times \mathbb{R}^n \) is not compact, we need some boundary condition at infinity on our maps, e.g. that they be completely integrable there, or not too far from it. Other than that, the maps can be as far as we want from completely integrable. In fact, the result is valid for arbitrary finite compositions of these maps. This is one of the differences between our work and the one of Bernstein and Katok [B-K], whose setting we otherwise very much adopted. Another marked difference with their work is that we drop their convexity assumption, replacing it by a more general nondegeneracy condition. Herman [He] has shown that the dynamics of monotone maps can differ drastically whether one assumes or not this convexity, especially regarding invariant tori (our result covers what he calls the indefinite case).

To a certain extend, we exploit the analogy between this setting and the Hamiltonian one to obtain our result. We build isolating blocks with the same properties as those appearing in the seminal work of Conley and Zehnder [C-Z 1]. One problem arises when we want to count periodic orbits and not only points. One has to take a quotient of the isolating block by the group \( \mathbb{Z}_q \) where \( q \) is the period. In some cases, some topological information is lost in the process.

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To remedy this situation, we also consider monotone maps that continue to completely integrable ones through a curve of monotone maps. This enables us to use Floer's theorem of continuation of normally hyperbolic invariant set that he used in his thesis [Fl 1], [Fl-Z] to prove Arnold's conjecture for surfaces. This theorem implies in our case that the invariant torus of rotation vector \( p/q \), seen as completely critical invariant set for the energy flow in the space of periodic sequences of rotation vector \( p/q \), can be continued in this space as an invariant set for the flow, in the sense of Conley [Co] and that it conserves the cohomology of \( T^n \). For this reason, we call such an invariant set a *ghost torus*.

It is not known to this author whether or not all monotone maps can be continued to a completely integrable one through a time periodic Hamiltonian flow. If this were the case, our result would derive from the analogous one of Josellis [J]. However, we hope to show that the techniques exposed here can be of some use other than for the existence of periodic orbits.

One question which remains open is whether or not one can find orbits of all rotation vectors. Note that this is weaker than asking for quasiperiodic orbits, on which the action of the map would be conjugated to a rigid rotation. In our setting, we conjecture the existence of a ghost torus for all given rotation vector. We think that this is the first step in order to answer positively to the above question. To support this conjecture, we prove it in the case of twist maps. For this, we use the monotonicity of the energy flow associated to twist maps ([An]) to develop the concepts of *ghost circles* and more generally of *sigma-Aubry-Mather sets* (abr. \( \sigma AM \) sets). The \( \sigma AM \) sets are subsets of \( \mathbb{R}^\mathbb{Z} \) that are closed, completely ordered, shift and integer translation invariant (The partial order on \( \mathbb{R}^\mathbb{Z} \) is the usual one, given by the positive cone). In a sense, they are all the potential Aubry-Mather sets for twist maps. They are homeomorphic to lifts of closed invariant sets of circle homeomorphisms. The rotation number is well defined and is continuous on the set of \( \sigma AM \) sets. Aubry-Mather sets can be defined as critical \( \sigma AM \) sets and ghost circles as connected, flow invariant \( \sigma AM \) sets.

Our second result shows that any Aubry-Mather set (as defined above) can be embedded in a ghost circle. A corollary of this is the existence of ghost circles of all given rotation number, i.e. a positive answer to our conjecture.

Finally we indicate how ghost circles might useful to understand the \( \Delta W \) of Mather and how the properties of ghost circles could be used in the theory of transport.

The author would like to thank warmly the numerous participants of the dynamical systems year at the IMA who helped him complete this work.
1. Notation and statement of the main result.

We let $T^n$ be $\mathbb{R}^n/\mathbb{Z}^n$ and consider the space $A^n := T^n \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ with (global) coordinates $(x,y)$. The group $\mathbb{Z}^n$ acts as deck transformations by

$$T_m(x,y) = (x + m, y), \quad m \in \mathbb{Z}^n.$$ 

$A^n$ is endowed with the canonical symplectic form

$$\Omega = \sum_{k=1}^{n} dx_k \wedge dy_k = d\alpha, \quad \text{where} \quad \alpha = \sum_{k=1}^{n} y_k dx_k.$$ 

A diffeomorphism $F$ of $A^n$ is symplectic if

$$F^*\Omega = \Omega,$$

and exact symplectic if $F^*\alpha - \alpha$ is exact, that is if:

$$F^*\alpha - \alpha = dS,$$

where $S$ is a $C^2$ real valued function on $A^n$. Obviously, an exact symplectic map is symplectic.

In the following, we write $F(x,y) = (X,Y)$.

**Definition 1 (Monotone Maps, Generating Functions).**

A diffeomorphism $F$ of $A^n$ is called a monotone map if:

1. $F$ is exact symplectic: $F^*\alpha - \alpha = dS$, $S : A^n \rightarrow \mathbb{R}$
2. $F$ is a lift: $F \circ T_m = T_m \circ F$
3. For each $x_0$, the map $(x_0, y) \rightarrow (x_0, X)$ is a diffeomorphism of $\mathbb{R}^n$ (and hence $(x, y) \rightarrow (x, X)$ is a diffeomorphism of $\mathbb{R}^{2n}$)

The function $S$ of (1.2) is called the generating function of the monotone map, thought of as a function of $(x,X)$.

That $(x,X) \rightarrow (x,y)$ is a diffeomorphism implies the nondegeneracy condition:

$$det \partial_1 \partial_2 S(x,X) \neq 0$$

which generalizes the so-called twist condition ($\partial_1$ (resp. $\partial_2$) means the partial derivative with respect to the first (resp. the second) component). We will see in the next section in what sense $S$ generates $F$.

When $n = 1$, we have:
DEFINITION 2 (Twist Maps).

A diffeomorphism \( f \) of \( A \) is called an area preserving monotone twist map, or, in short (in this paper), twist map if it is monotone and if the scalar \(-\partial_1 \partial_2 S(x, X)\) is strictly positive for all \((x, X)\).

COMMENTS.

(1) Definition 1.3 is slightly different from the one that we used previously [Go 1,2] where we put condition 1.4 map \((x, X) \rightarrow (x, y)\) is a diffeomorphism. The term monotone was introduced by Herman ([He]) in a more general setting: his monotone maps do not necessarily have a generating function. Some authors impose another condition on \(S\), i.e. that \(-\partial_1 \partial_2 S\) be positive definite. They call these maps symplectic twist maps (see e.g. [K-M]). Definition 1.6, on the other hand, is equivalent to the usual one: \(f\) is an area preserving map with zero flux which sends any vertical line \(x = c\) into a graph over the \(x\)-axis satisfying \(f(c, +\infty) = (+\infty, +\infty)\) and \(f(c, -\infty) = (-\infty, -\infty)\).

(2) Note that some authors use "exact symplectic" for maps that are homologous to the identity, i.e., time one maps of (time dependant Hamiltonian flows (what E.Zehnder now calls Hamiltonian maps). Whereas for the case \(n = 1\) (twist maps) it is true that the maps we consider can be suspended by a hamiltonian flow ([Mo1]), it seems to be an open question whether this holds for the monotone maps we defined above.

The type of objects that we are looking for are:

DEFINITION 3 ((p,q)-Periodic Points). Let \((p,q)\) be an element of \(\mathbb{Z}^n \times \mathbb{Z}\). \((x_0, y_0)\) in \(A^n\) is a \((p,q)\)-periodic point \(((p,q)\)-point, in short) if, when we denote by \((x_k, y_k) = F^k(x_0, y_0)\), the following holds:

\[
x_{k+q} = x_k + p
\]

The orbit of a \((p,q)\)-point is called a \((p,q)\)-orbit. \((p,q)\) is called relatively prime if \(q\) is prime with at least one of the components of \(p\).

As stated in the introduction, in order to find such orbits, we need to assume a boundary condition for our map. It is better expressed in terms of the generating function \(S\).

BOUNDARY CONDITIONS. We suppose that \(S\) can be written:

\[
S(x, X) = S_0(x, X) + R(x, X), \text{ with }
\]

\[
(\partial)
S_0(x, X) = \langle A(X - x), (X - x) \rangle
\]
where $A$ is a symmetric nondegenerate matrix and $R$ assumes sublinear growth in its derivative:

$$\lim_{\|X-x\| \to \infty} \frac{\|\partial_{\alpha} R(x, X)\|}{\|X - x\|} = 0, \quad \alpha = 1, 2$$

As stated in the introduction, this boundary condition implies that the map is somewhat comparable to a completely integrable one (generated by $S_0$) at infinity. However, this boundary condition is flexible enough to allow $R(x, X) = \nabla V(x)$ for any periodic $V$, and it is enough that this be satisfied for $X - x$ large. This includes, in particular, maps which are in the standard family (see the examples below). In particular, this includes twist maps without any invariant (homotopically non trivial) circles. \textit{Note that $A$ is not assumed to be positive definite.} When it is not, $F$ falls into the class of indefinite maps considered by Herman [He].

\textbf{Continuation setting.} We say that a monotone map $F$ with generating function $S$ continues to a completely integrable $F_0$ generated by $S_0$ if there is a family

$$S_\lambda(x, X) = S_0(x, X) + R_\lambda(x, X)$$

generating monotone maps with $R_0 = 0$ and $S_1 = S$.

We can now state our main result.

\textbf{Theorem 1}. Let $F$ be a monotone map continuing to a completely integrable one and satisfying the boundary condition $(\partial)$. Then $F$ has at least $n + 1$ distinct $(p,q)$-periodic orbit for each prime $(p,q)$ in $\mathbb{Z}^n \times \mathbb{Z}$, and $2^n$ of them if they are nondegenerate.

Remember that a $q$-periodic orbit is nondegenerate if $\det(DF^q(z) - \text{Id}) \neq 0$ for any $z$ in the orbit. The above theorem remains true for compositions of $F_i$'s satisfying the conditions of the theorem, as long as their matrices $A_i$ commute and are of the same sign on their common eigenspaces.

If we do not assume the continuation setting, we have the following:

\textbf{Theorem 1'}. Let $F = F_1 \circ \ldots \circ F_s$ where each $F_i$ is a monotone map of $\mathbb{A}^n$ generated by a global generating function $S_i$. Suppose that each $F_i$ satisfies the boundary condition $\partial$ and that the corresponding set $\{A_i\}_{i=1}^s$ of symmetric, nondegenerate matrices is such that

$$[A_i, A_j] = 0 \quad \forall i, j \in \{1, \ldots, s\},$$

and such that all the $A_i$'s are positive definite (resp. negative definite) in the same subspaces of dimension $k_0$ (resp. $n - k_0$). Then:

1. $F$ has at least $n + 1$ geometrically distinct $(p,q)$-periodic orbits for each prime $(p,q)$ in $\mathbb{Z}^n \times \mathbb{Z}$. 

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Suppose in the following that all \((p,q)\)-periodic orbits are nondegenerate. Then:

(2) If either \(q\) is odd, \(s\) is even, or \(k_0\) is even or equal to \(n\) or \(0\), then \(F\) has at least \(2^n\) \((p,q)\)-periodic orbits. Otherwise \(F\) has at least \(2^{n-1}\) of them.

Note that theorem 1' is stronger than theorem 1 only in the sense that it does not assume the continuation setting. Otherwise, we see that some information is lost when we are seeking nondegenerate orbits in this more general setting. The proof of 1' is noticeably more complicated than the one of 1, which has also the advantage to yield to the concept of ghost torus. It would be interesting to find out if all monotone maps continue to a completely integrable one.

**RELATED RESULTS.** The above theorems are in direct line with the Poincaré Birkhoff theorem for twist maps, see [G-H] for the historical background and recent generalizations. The Birkhoff-Lewis theorem finds infinitely many periodic orbits in the neighborhood of an elliptic fixed point of a symplectic map. Transcribed into polar coordinates, it is a theorem of existence of periodic orbits for the perturbation of a completely integrable map. Moser [Mo 2] gave a complete and elegant proof of this theorem. The breakthrough for global results in higher dimensions came with the theorem of Conley and Zehnder [C-Z 1]. They consider the setting of time-one maps of time periodic Hamiltonian systems on \(T^n \times \mathbb{R}^n\) (as well as on \(T^{2n}\)) and assume that they are completely integrable outside a bounded set. They find \(n + 1\) (or \(2^n\) nondegenerate) homotopically trivial periodic orbits of period 1. Josellis [J] refined this result by proving a theorem very much like ours, in the Hamiltonian setting.

Bernstein and Katok [B-K] used the discrete variational approach to find \(n+1\) periodic orbits of any given type for the perturbation of a "convex" monotone map. They came close to proving a result of regularity on the orbits they find which would have enabled them to find points with irrational rotation vector. Chen studied the indefinite case and found \(n + 1\) periodic points in an otherwise similar setting to that of [B-K].

**EXAMPLES.**

(1) Completely integrable and standard monotone maps:

\[
F_0(x, y) = (x + y, y)
\]

has the generating function \(S_0(x, x') = \frac{1}{2}(x' - x)\) and its energy flow (see next section) is given by the infinite system of O.D.E.'s:

\[
\dot{x}_k = -2x_k + x_{k-1} + x_{k+1},
\]

i.e. a discretised heat equation.

One can add a "potential" to \(S_0\) : let

\[
S(x, x') = \frac{1}{2}(x, x') - g(x).
\]
Then $S$ generates $F(x, y) = (x + y + h(x), y + h(x))$ where $h(x) = g'(x)$. The associated flow is given by:

$$\dot{x}_k = -2x_k + x_{k-1} + x_{k+1} + h(x_k).$$

Angenent remarked that this was the discretisation of $x_s = x_{tt} + h(t, x)$ and proved important results for twist maps by developing this analogy. When $n = 1$ and

$$g(x) = -\frac{k}{4\pi^2} \cos(2\pi x),$$

$F$ is called the standard map (or standard family).

(2) Hamiltonian systems:

Let $H(x, y, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be periodic in $x$ and $t$. If we assume

$$\text{Det} \frac{\partial^2 H}{\partial y^2} \neq 0,$$

and that, for $|y| > a$, $H(x, y) = \frac{1}{2} < Ay, y > + < C, y >$

for $A$ symmetric, nondegenerate, and $C \in \mathbb{R}^n$ (see Conley-Zehnder, Thm 3 (C-Z 2)) then the time-1 map $F$ can always be decomposed into monotone $F_i$'s. Moreover, they trivially satisfy the boundary conditions of our theorem. Also note that such Hamiltonian systems fall into the domain of those studied by Josellis [J]

2. The variational setting.

The function $S$ generates a monotone $F$ in the following (classical mechanic) sense:

$$y = -\partial_1 S(x, X)$$

$$Y = \partial_2 S(x, X)$$

Let $z_k = F^k(z_0) = (x_k, y_k)$. The orbit $\{z_k\}$ is completely determined by the sequence $x_k$ of $(\mathbb{R})^\mathbb{Z}$. Indeed, from (1.8), we deduce:

$$y_k = -\partial_1 S(x_k, x_{k+1}) = \partial_2 S(x_{k-1}, x_k)$$

This can be written:

$$\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k) = 0$$

Equation 1.10 can be formally interpreted as:

$$\nabla W(x) = 0, \text{ for } \sum_{-\infty}^{+\infty} S(x_k, x_{k+1}) \text{ and } x \in (\mathbb{R}^n)^\mathbb{Z}.$$
What we have proven here is that there is a 1-1 correspondence between points of $A^n$ and the critical points of $\nabla W$ in $\mathbb{R}^Z$. In particular, invariant tori for the map correspond to completely critical tori for the vector field $\nabla W$.

One can think of the above construction as a discrete version of the classical one: the map $(x, X) \rightarrow (x, X)$ is the analog to the Legendre transformation ($X - x$ is the discretised velocity) and equation (1.11) is a formulation of the "least action principle".

Of course, $W$ is not well defined, since the sum is not convergent. However, "$\nabla W$" is well defined and generates a flow on $(\mathbb{R}^n)^Z$ that we call the energy flow.

More precisely when $S$ is a $C^p$ function, the infinite system of O.D.E's:

$$-(\nabla W(x))_k = \dot{x}_k = -[\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k)]$$

defines a $C^{p-1}$ local flow $\zeta^t$ on $(\mathbb{R}^n)^Z$ (with the usual product topology) whose critical points are in one to one correspondence with the points of $A$ and their orbit under the map. This flow can also be made global by assuming relevant boundary conditions (e.g. that the map be completely integrable outside a bounded strip, or at infinity).

Because $F$ is a lift, $S$ is periodic: $S(x + m, X + m) = S(x, X)$, for any $m \in \mathbb{Z}^n$ and hence the flow $\zeta^t$ can be defined on $(\mathbb{R}^n)^Z/\mathbb{Z}$

3. Sketch of the proof and Ghost tori.

To look for $(p,q)$-periodic points, one restricts the flow $\zeta^t$ to the set of periodic sequences:

$$X_{p,q}(\mathbb{R}^n) = X_{p,q} := \{ x \in (\mathbb{R}^n)^Z \mid x_{k+q} = x_k + p \}$$

$$\cong \{ (x_0, \ldots, x_{q-1}) \in (\mathbb{R}^n)^q \}$$

Because of the periodicity of $S$, it is easy to see that $X_{p,q}$ is invariant under $\zeta^t$. Moreover $\zeta^t$ is in fact the gradient flow of the well defined function:

$$W_{p,q}(x) = \sum_{k=0}^{q-1} S(x_k, x_{k+1})$$

with respect to the coordinates $(x_0, \ldots, x_{q-1})$ of $X_{p,q}$ (with the convention that $x_q = x_0 + p$).

Hence critical points of $\zeta^t$ in $X_{p,q}$ correspond 1-1 to $(p,q)$-points.

In our thesis, we prove that, given the boundary condition $\partial$, the following set is an isolating block for $\zeta^t$, whenever $C$ is large enough:

$$B(C) = \{ (x) \in X_{p,q} \mid \sup_{k \in \{1, \ldots, n \}} \sum_{i=1}^{q} (x_i^k - x_{i-1}^k)^2 \leq C^2 \}$$
It is not hard to see that $B(C)$ is homeomorphic to the product of $\mathbb{T}^n$ with the product of closed disks. To show that it is an isolating block, one has to show that the flow enters some of these disks and exit others (in the positive definite case it always enters, i.e. we have an attracting block). This situation is quite similar to the one encountered in the original proof of Conley and Zehnder [C-Z 1], where they use their generalised Morse theory to find $n + 1$ critical points or $2^n$ if nondegenerate for a gradient flow with a similar isolating block.

However, a problem arises when one wants to count the distinct periodic orbits instead of points. One has to take a quotient of $X_{p,q}$ by the shift $\sigma$ on sequences $\{\sigma x\}_i = x_{i+1}$ which acts as $\mathbb{Z}_q$ on $X_{p,q}$). In our thesis, we proved a theorem of equivariance of the Conley index to study the topology of the quotient $B(C)$ of $B(C)$. Information is lost when counting nondegenerate points with the Conley-Zehnder Morse inequalities whenever the quotient map is not a trivial fibration. In these cases, we have to take a double cover, which explains the $2^{n-1}$ in theorem 1'.

In the continuation setting, we take another route: instead of relying on the topology of the index pair $(B(C), B^-(C))$ (where $B^-(C)$ is the exit set), we use the idea of Floer ([Fl], see also [F-Z]) in his thesis to prove the following a topological continuation theorem for the invariant set contained inside $B(C)$.

**Definition 4.** We denote by $G^\lambda_{pq}$ and call ghost torus the maximum invariant set in $B(C)$ for the energy flow induced by the generating function $S_\lambda$.

**Theorem 2 (Continuation of the ghost tori).**

1. $G^0_{pq}$ is the critical torus corresponding to the $F_0$-invariant torus in $\mathbb{A}^n$ with rotation vector $p/q$.
2. $G^0_{pq}$ is normally hyperbolic for the flow induced by $S_0$.
3. There is a retraction $r : X_{p,q} \to G^0_{pq}$.
4. Since $C$ can be chosen so that $B(C)$ is an isolating block for the energy flows induced by each of the $S_\lambda$’s, the ghost tori $G^\lambda_{pq}$ are related by continuation in the sense of Conley [Co].
5. In conclusion, Floer’s result implies:

$$\left(r \mid G^\lambda_{pq}\right)^* : H^*(G^0_{pq}) \to H^*(G^\lambda_{pq})$$

is an injective map from the cohomology of $\mathbb{T}^n$ into that of $G^\lambda_{pq}$.
6. All of the above remain true when taking the quotient by the action of $\sigma$.

In other words, the topology of $G^\lambda_{pq}$ can only get more complicated than the one of $\mathbb{T}^n$ as the parameter varies. For this reason we will in general refer to a compact invariant set with this property as a ghost torus.
The important advantage of this approach is that all the above feature are conserved when we quotient by the action of \( \sigma \). Hence we can look at the ghost tori \( G^\lambda_{pq} \) in the quotient space: their topology remain always as complicated as that of \( T^n \). As a corollary, using Conley-Zehnder Morse theory [C-Z 1.2] on the invariant set \( G^\lambda_{pq} \), we get at least \( cl(T^n) = n + 1 \) critical points and \( sb(T^n) = 2^n \) if they are nondegenerate. These critical points correspond to distinct orbits of \( F \). In the next sections, we will see how the ghost tori may be of interest in themselves.

4. Irrational rotation vectors?

One way to attack the problem of finding orbits of all rotation vectors would be to consider the flow \( \dot{x} = -\nabla W(x) = \partial_1 S(x_i, x_{i+1}) + \partial_2 S(x_{i-1}, x_i) \),

\[
Y = \bigcup_{\omega \in \mathbb{R}^n} Y_\omega \text{ where } \]

\[
Y_\omega = \{(x) \in (\mathbb{R}^n)^2 \mid |x_j - j\omega| < \infty\}
\]

On \( e \) can check that \( Y_\omega \) is invariant under the flow given by \( \nabla W_\lambda \), and diffeomorphic to \( l^\infty \).

The first way to exploit this situation would be to prove that rational \( G^\lambda_{pq} \)'s continue to some invariant set \( G^\lambda_\omega \) in \( Y_\omega \), in the sense of Conley. Critical points in \( G^\lambda_\omega \) would automatically have rotation vector \( \omega \), as elements of \( Y_\omega \). There the main stumbling block is: even though \( Y \subset (\mathbb{R}^n) \) can be endowed with the product topology, for which \( \nabla L_\lambda \) is continuous, the “slices” \( Y_\omega \) are not closed in that topology and hence do not constitute a local product parametrization for the flow \( \nabla L_\lambda \) (with parameter \( \omega \in \mathbb{R}^n \): for a definition of local product parametrization, i.e. the setting in which Conley’s continuation is defined, see [C],[Sa]).

Another way to exploit this setting is the following. When we consider more restrictive boundary conditions (\( R = 0 \) outside a bounded strip), we can find in \( Y_\omega \) a set \( P \) which has all the features of an isolating block except for compactness. Furthermore, \( P \) is the intersection of a compact set in \( \mathbb{R}^2 \) with \( Y_\omega \), it is a retraction of \( Y_\omega \), and finally it contains all the critical points in \( Y_\omega \) (if there are any) and the critical torus corresponding to the KAM torus of rotation vector \( \omega \) (when it exists). This gives us hope that some version of a continuation theorem may occur in \( Y_\omega \):

**Conjecture.** There exists a ghost torus in \( Y_\omega \), for each \( \omega \) in \( \mathbb{R}^n \)

By ghost torus here we mean a compact invariant set whose cohomology contains the one of \( T^n \). Note that, supposing one could settle this conjecture positively, the problem of finding critical points would not be solved: the flow is not automatically gradient like on \( Y_\omega \), making it hard to use the Conley-Zehnder Morse inequalities to find critical points. However, we think it would be a first step in finding critical points.
We should note here the results of Mather [Ma 2] on existence and regularity of minimal invariant measures for positive definite Lagrangian systems. Katok [K 3] proves that, for perturbations of completely integrable convex monotone maps there are infinitely many rotation vectors for which KAM tori do not exist but minimal orbits exist. Finally, Chen, McKay and Meiss [C-McK-M] found a whole class of (convex, monotone) 4D symplectic maps which have, for each irrational vector in a domain, cantori on which the motion is semi-conjugated to a rigid translation by that vector. These maps are perturbation of a discontinuous (sawtooth) map and are very far from completely integrable.

In the next section, we concentrate on twist maps \( n = 1 \) and prove the conjecture in that case.

5. Ghost circles and twist maps.

In this section, we concentrate on Twist maps and in particular answer positively to the conjecture of the last section for the case \( n = 1 \). When \( n = 1 \), the energy flow \( \zeta^t \) has the remarkable property of being monotone with respect to the partial order on sequences. This fact was noticed by Angenent [An], whose work influenced the one exposed in this section a lot.

Let us recall a few facts about monotone flows (see, e.g. [Hi], [Mto]).

Let \( X \) be a Banach space. A partial order on \( X \) is given by a convex, closed cone \( V_+ \) such that, if we denote by \( -V_+ = V_- \), we have: \( V_- \cap V_+ = \{0\} \). For \( x, y \) in \( X \) we define:

\[
\begin{align*}
  x \leq y & \iff y \in x + V_+ \overset{\text{def}}{=} V_+(x) \\
  x < y & \iff x \leq y \text{ and } x \neq y \\
  [x, y] & = \{ z \in X \mid x < z < y \} \\
  (x, y) & = \{ z \in X \mid x < z < y \text{ and } x \neq z \neq y \}
\end{align*}
\]

**Definition 5.** A map \( A : X \to A \) is called **monotone** if:

\[
x < y \Rightarrow Ax < Ay
\]

A flow \( \zeta^t \) in \( X \) is monotone if for all positive \( t \), the map \( \zeta^t \) is monotone.

One can also define the notion of strong monotonicity by replacing the cone \( V_+ \) by its interior in the above definition (the corresponding order is then denoted by \( \ll \)). \( \mathbb{R}^\mathbb{Z} \) is endowed with the natural partial order on sequences defined by the following: Let \( x \) and \( x' \) be elements of \( \mathbb{R}^\mathbb{Z} \), with \( k \)th terms denoted by \( x_k \) and \( x'_k \). Then:

\[
(1.15) \quad V_+ := \{ y \in \mathbb{R}^\mathbb{Z} \mid y_k \geq 0, \forall k \in \mathbb{Z} \}.
\]
and similarly for $<$.  

The order in $X_{p,q}$ is induced by the one on $\mathbb{R}^Z$. The positive cone is just the positive quadrant of $\mathbb{R}^Z \cong X_{p,q}$, we also use the notation:

\begin{equation}
V_{++} := \{ y \in \mathbb{R}^Z \mid y_k > 0, \forall k \in \mathbb{Z} \},
\end{equation}

\[ x < x' \iff x' \in x + V_{++} \]

and we will say that a map $A : \mathbb{R}^Z \to \mathbb{R}^Z$ is strictly monotone when

\begin{equation}
x < x' \Rightarrow Ax < Ax'.
\end{equation}

It is clear that in $X_{p,q}$ the notion of strong monotonicity and strict monotonicity are equivalent.

**Lemma 1.** The energy flow is strongly monotone in $\mathbb{R}^Z$ (and hence in $X_{p,q}$).

In [Go 3] (Lemma 1.22), we present a proof of this which was communicated to us by S. Angenent. It does not rely on the classical theorem of Kamke, but rather proves that the solution operator of the linearised equation is strictly positive.

Remember that $\mathbb{Z}$ acts on $\mathbb{R}^Z$ in two ways:

\[
T, \sigma : \mathbb{R}^Z \to \mathbb{R}^Z
\]

\[
\{ \sigma x \}_k = x_{k+1}
\]

\[
\{ Tx \}_k = x_k + 1
\]

We will sometimes refer to the action of $T$ as the $\mathbb{Z}$ action, to $\sigma$ as the shift. There are natural projections of $\mathbb{R}^Z$ into $A \cong \mathbb{R}^2$. We denote by:

\[
\pi_{01}(x) = (x_0, x_1)
\]

The results in this section concern the following objects:

**Definitions 6 (σ-Aubry-Mather sets, Ghost circles).**

(1) A **σ-Aubry-Mather set** (or σAM set) is a closed, completely ordered, $\mathbb{Z}$-invariant, $\sigma$-invariant subset of $\mathbb{R}^Z$.

(2) An **Aubry-Mather set** is a completely critical σAM set. We will say that an Aubry-Mather set is saturated if it is not strictly contained inside another one.

(3) A **ghost circle** is a $\zeta^t$-invariant, connected σAM set.
COMMENTS. σAM sets are in a sense all the potential Aubry-Mather sets for all twist maps. The definition that we give here of an Aubry-Mather set is equivalent to the one of a \( F \)-invariant, \( T \)-invariant closed subset of \( A \) on which \( F \) preserves the order in the \( x \) coordinate and on which the first projection \( \pi_x \) is injective. (see \([K1]\)).

The ghost circles as defined here are more than the \( n = 1 \) case of the ghost tori defined in the previous section. They are contained in the latter, sometimes strictly: consider the case when there exists a non ordered orbit of some rotation number. \( F \)-invariant circle are seen in \( \mathbb{R}^2 \) as completely critical ghost circles.

In \([Go3]\), we prove the following theorems of existence:

**Theorem 3 (Existence of ghost circles).** Let \( f \) be any twist map of \( S^1 \times \mathbb{R} \). Then for all \( \omega \) in \( \mathbb{R} \) there is a ghost circle \( G_\omega \) with intrinsic rotation number \( R(G_\omega) = \omega \).

Furthermore, any Aubry-Mather set can be embedded into a ghost circle.

These ghost circles project diffeomorphically into the annulus to circles which are graphs over the \( x \)-axis and such that their images by \( f \) are also graphs. Intersections of such a circle and its image only occur at the critical points.

The projection mentioned is given by \( x \rightarrow (x_0, x_1) \rightarrow (x_0, y_0) \) the last arrow being given by the diffeomorphism \((3)\) in the definition of monotone maps. See theorem 5 for the significance of \( R \).

**Theorem 4 (\( C^1 \) rational ghost circles).** Suppose that the function \( W_{p,q} \) attached to a twist map is Morse (proven to be a generic property in \([Go3]\)). Then:

1. There exists a ghost circle in \( W_{O_{p,q}} \) containing all absolute minima of \( W_{p,q} \) and made of unstable manifolds of mountain pass points between them. These unstable manifolds join \( C^1 \) at the minima.
2. Alternatively, one can construct a ghost circle containing the extended orbits of an absolute minimum and its minimax. This ghost circle is also made out of unstable manifolds of mountain passes.

In both cases, these ghost circles project diffeomorphically in the annulus to \( C^1 \) circles which are graphs over the \( x \)-axis and such that their images by \( f \) are also graphs. Intersections of such a circle and its image only occur at the periodic points.

The proof of theorem 3 derives immediately from a recent generalization of a theorem of Matano \([Mto]\) by Dancer and Hess \([D-H]\) on monotone flows in Banach spaces. This theorem states the existence of a monotone connecting orbit between two critical points \( x < y \) whenever the order interval \((x, y)\) does not contain any
other critical points. The idea to apply their theorem is to fill the gaps in a saturated Aubry-Mather set with monotone connections. The $\sigma$ and $T$ invariance is given by the equivariance of the flow under these maps.

For theorem 4, we work in $X_{p,q}$ where we saturate the minimum periodic orbit with local minima that are ordered with it. We prove that between any two critical points in this order saturated set, there must be a mountain pass point of index 1 (note that this differs from Mather [Ma1] mountain pass lemma in that the extreme points of our order interval are only assumed to be local minima). We use properties of the unstable manifold of such points studied by Angenent [An] to prove that they join $C^1$ at the minima.

As far as the $\sigma AM$ sets are concerned, they provide the appropriate setting for the study of Hausdorff limits, as well as the rotation number. Remember that the rotation number of a sequence $x$ in $\mathbb{R}^Z$ is defined by:

\begin{equation}
\rho(x) = \lim_{k \to \pm \infty} \frac{x_k}{k}
\end{equation}

**Theorem 5.**

1. If $E$ is a connected $\sigma AM$ set, then it is homeomorphic to $\mathbb{R}$ and $E/T = E/\mathbb{Z}$ is homeomorphic to a circle. $\sigma$ induces a circle homeomorphism on $E/T$.

2. Any $\sigma AM$ set can be embedded in a connected $\sigma AM$ set and therefore is homeomorphic to a closed invariant set of a circle homeomorphism.

3. The rotation number $R(E)$ of $\sigma |_E$ is a continuous function of $E$ in the set of $\sigma AM$ sets.

4. Let $x$ be any element of $E$. Then $\rho(x)$ exists and $R(E) = \rho(x)$ for all $x$ in $E$. Moreover, if $R(E) = \omega$, then

$$
E \subset Y_\omega \overset{\text{def}}{=} \{ x \in \mathbb{R}^Z \mid \sup_{k \in \mathbb{Z}} |x_k - k\omega| < \infty \}
$$

The last point of theorem 5 together with theorem 2 proves the conjecture of the last section in the case $n = 1$: the maximum compact invariant set in $Y_\omega$ actually contains a circle. It would be interesting to find out if the ghost tori that we find in higher dimensions contain actual tori.

**6. Applications of ghost circles.**

As a first application, we have a formula for the flux of the map $F$ through the projection $G'$ onto the annulus of a ghost circle $G$. This flux is defined as the area above $G'$ which is also below $F(G')$. The point being that this is well defined since both these sets are graphs over the $x$ coordinate. Consider the set of connections
between critical points in $G$ and let $K$ be the quotient of this set by the actions of $\sigma$ and $T$. Let $x(t)$ denote the class of a connection $x(t)$ under this quotient. Letting $\lim_{t \to \pm \infty} x(t) = x^\pm$ one gets:

$$\text{Flux}(G^*) = \frac{1}{2} \sum_{x(t) \in E} \left| \sum_{k \in \mathbb{Z}} S(x^+_k, x^+_{k+1}) - S(x^-_k, x^-_{k+1}) \right|$$

Because of its geometric interpretation, the above sum converges. One can get rid of the factor $1/2$ and the absolute value by choosing only the elements of $E$ that give us positive terms. One can see the flux as the global variation of the energy $W$ on $G$ (even though $W$ is not necessarily well defined there). When there is a direct connection $x(t)$ between a min and a minimax, the sum corresponding to $x(t)$ is the $\Delta W$ of Mather [Ma 1].

Let us remark that one can compute this flux by just knowing the critical points and that the setting is global, i.e. $F$ can be as far as we want from completely integrable. We hope that this can become a valuable tool in the theory of transport. Could the higher dimensional ghost tori be used for this purpose as well?

Our second application offers a new setting for the study of the $\Delta W$. We now look at $\Delta W$ as a function on the set of rational ghost circles:

**Definition 7.** Let $G_{pq}$ be a ghost circle in $X_{p,q}$ for a given twist map $f$. Then:

$$\Delta W(G_{pq}) = \max_{x, x' \in G_{pq}} |W_{p,q}(x) - W_{p,q}(x')|$$

This is related to the definition of Mather by:

$$\Delta W_{p,q} \leq \Delta W(G_{pq}).$$

Consider a converging sequence of ghost circles $\{G_{p_kq_k}\}_{k \in \mathbb{Z}}$ with

$$G_{p_kq_k} \to G_\omega \text{ and } p_k/q_k \to \omega,$$

Then we have the following:

**Theorem 6.**

1. If $\lim_{k \to \infty} \Delta W(G_{p_kq_k}) = 0$ then $G_\omega$ is a completely critical ghost circle (and hence projects to an $F$-invariant circle in $A$).

2. Conversely, if $G_\omega$ is completely critical then any sequence of rational ghost circles $G_{p_kq_k}$ with $G_{p_kq_k} \to G_\omega$ will satisfy: $\lim_{k \to \infty} \Delta W(G_{p_kq_k}) = 0$
This inspired by the criterion of nonexistence of invariant circles of Mather [Ma ?], see also [K2].

If $G_\omega$ is transitive, then it is unique with this rotation number. Then any sequence $G_{p_kq_k}$ with $p_k/q_k \to \omega$ has $\Delta W(G_{p_kq_k}) \to 0$. Another way to say this is the following criterion:

If $G_{p_kq_k}$ is a converging sequence of ghost circles and if $\Delta W(G_{p_kq_k}) \neq 0$, then there is no transitive invariant circles of rotation number $\omega = \lim_{k \to \infty} p_k/q_k$.

We could refine this criterion by erasing "transitive" in the above criterion if we could prove the unicity of $G_\omega$ when it is a completely critical set, not necessarily transitive. Also, we think that $\Delta W$ can actually be defined on $\sigma AM$ sets and that it is continuous there. The discontinuity at rational values of the $\Delta W$ of Mather would arise from the non unicity of the rational ghost circles.

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