Designing Periodic Structures with Specified Low-Frequency Scattered Far-Field Data

David C. Dobson
Institute for Mathematics and its Applications
University of Minnesota
514 Vincent Hall
Minneapolis, MN 55455-0436

Abstract

We consider the problem of designing a periodic interface between two different materials in such a way that monochromatic waves scattered from the structure have a specified far-field pattern. The waves are assumed to be low-frequency with respect to the period of the structure.

We pose the optimal design problem as a “relaxed” minimization problem, which allows continuously varying impedance profiles near the interface. A modified gradient descent minimization algorithm, designed to be more efficient than standard gradient descent for this problem, is described. Examples from numerical experiments are given.

1 Introduction

Consider a periodic interface between two materials with different acoustic or electromagnetic impedance properties. We assume that neither material absorbs energy. A time-harmonic plane wave incident on the interface “scatters”; part of the energy of the wave is reflected and part of the energy is transmitted through the interface. Both the reflected and transmitted components of the scattered wave can be expressed away from the interface as infinite sums of plane waves. However, only a finite number of the plane waves propagate outward. The remaining modes are exponentially damped, or evanescent. Thus in the far-field, only the propagating modes are detectable. The magnitudes and phases of the outward propagating waves depend upon the shape or structure of the interface.

In this paper, we study the problem of designing the periodic interface in such a way that the outward propagating modes have a specified pattern. One application area is in micro-optics, where it is often desired to construct diffraction gratings which scatter monochromatic electromagnetic waves (say, from a laser) in a specified way. Such structures are increasingly used in advanced optical devices. These devices have very small features, on the same order as the wavelength of the incident radiation (around 0.5 μm). Thus although the optical frequencies are very large, the waves are low-frequency with respect to the length scale of the structure.

In a previous article [?], we studied the problem of designing a periodic interface in such a way that the far-field reflected energy is as small as possible for a chosen range of incidence angles—the problem of designing an antireflective structure. The techniques in [?] follow those of Achdou and Pironneau [?], [?] where the optimal design of a photocell is considered. In the present paper, we use the same mathematical framework to study the design of the more general periodic diffraction structures.
The outline of the paper is as follows. In the next section we describe the direct scattering problem—the Helmholtz equation in a periodic structure in $\mathbb{R}^2$—formulated as an equivalent variational problem over a bounded region with "transparent" boundary conditions. In Section 3, we pose a "relaxed" design problem which allows continuously varying material parameters near the interface. The design problem is formulated as a least-squares minimization problem. In Section 4, we describe a minimization algorithm, based on gradient descent, for solving the design problem. The idea of the algorithm is similar to "infeasible point" methods from constrained optimization, where the constraints are only required to be satisfied in the limit as the method converges. The algorithm is designed to be more efficient than standard gradient descent for this problem. In Section 5, we briefly describe our numerical implementation, then present the results of a numerical experiment in which a periodic "beam splitter" structure is designed.

Although this work is motivated mainly by the applications in optics, the mathematics is the same for linear acoustics (the underlying model is the Helmholtz equation), and we suspect that similar design problems may exist in acoustics. Also, some of the methods described here may be applicable to inverse problems in acoustics in which periodic diffractive structures are to be determined from far-field scattered data.

2 Formulation of the direct scattering problem

In this section, we formulate the direct scattering problem as a variational problem over a bounded region. A somewhat more detailed derivation can be found in [?]; see also [?], [?]. For an introduction to electromagnetic scattering through diffraction gratings, see [?].

The interface between the two materials is described as a simple curve $S$, assumed to be $2\pi$-periodic in the $x_1$ direction and imbedded in the strip $\Omega_0 = \{(x_1,x_2) \in \mathbb{R}^2 : -b < x_2 < b\}$, where $b$ is some positive constant. The boundaries $\Gamma_1 = \{x_2 = b\}$, $\Gamma_2 = \{x_2 = -b\}$ separate $\Omega_0$ from the half-planes $\Omega_1$ and $\Omega_2$ above and below $\Omega_0$, respectively. The curve $S$ divides $\Omega_0$ into two connected components: $\Omega_0^+\;\text{above}\;S$, and $\Omega_0^-\;\text{below}\;S$. See Figure 1.

Let $k_1$ and $k_2$ be real, positive numbers corresponding to the indices of refraction for the two materials. The
index of refraction in $\mathbb{R}^2$ thus depends on the curve $S$ and is given by

$$ k_S(x) = \begin{cases} 
  k_1 & \text{in } \Omega_0^+ \cup \Omega_1, \\
  k_2 & \text{in } \Omega_0^- \cup \Omega_2.
\end{cases} $$

We define $a_S = k_S^2$. The propagation of waves through the two materials is governed by the Helmholtz equation

$$ (\Delta + a_S)u = 0 \quad \text{in } \mathbb{R}^2. \quad (1) $$

We let an incoming plane wave $u_* = e^{i\alpha x_1 - i\beta_1 x_2}$ illuminate $S$ from $\Omega_1$. Here, $\alpha = k_1 \sin \theta$, $\beta_1 = k_1 \cos \theta$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is the angle of incidence with respect to the $x_2$-axis.

We look for “quasiperiodic” solutions $u$, that is, solutions $u$ such that $ue^{-i\alpha x_1}$ is 2\pi-periodic. Define $u_\alpha = ue^{-i\alpha x_1}$. If $u$ satisfies (1) then $u_\alpha$ satisfies

$$ (\Delta_\alpha + a_S)u_\alpha = 0 \quad \text{in } \mathbb{R}^2, $$

where $\Delta_\alpha = \Delta + 2i\alpha \partial_\alpha - \alpha^2$. From now on, we assume that $\alpha$ is fixed (the incidence angle has been chosen), and we drop the subindex $\alpha$ from $u_\alpha$. Since the field $u$ and the coefficient $a_S$ are both $2\pi$-periodic in $x_1$, we identify $\Omega_0$ with the quotient space $\Omega_0/(2\pi Z \times \{0\})$, and the boundaries $\Gamma_j$ with the circles $\Gamma_j/2\pi Z$. Thus all functions defined on $\Omega_0$ and $\Gamma_j$ are implicitly $2\pi$-periodic in the $x_1$ variable.

Define the coefficients

$$ \beta_j^n = \begin{cases} 
  \sqrt{k_j^2 - (n + \alpha)^2} & \text{if } k_j^2 \geq (n + \alpha)^2, \\
  i\sqrt{(n + \alpha)^2 - k_j^2} & \text{if } k_j^2 < (n + \alpha)^2.
\end{cases} $$

Assuming that $k_j^2 \neq (n + \alpha)^2$ for all $n \in \mathbb{Z}$, $j = 1, 2$, it follows from knowledge of the fundamental solution, (see eg. [?], or [?]) that inside $\Omega_1$ and $\Omega_2$, $u$ can be expressed as a sum of plane waves:

$$ u|_{\Omega_j} = \sum_{n \in \mathbb{Z}} a_j^n e^{\pm i\beta_j^n x_2 + inx_1}, \quad j = 1, 2, \quad (2) $$

where the $a_j^n$ are complex scalars. The finite number of plane waves in the sum (2) which correspond to real indices $\beta_j^n$ are called propagating modes. The remaining modes grow or decay exponentially. We insist that $u$ is a sum of bounded outgoing plane waves, plus the incident incoming wave $u_*$ in $\Omega_1$. Expanding $u$ in a Fourier series:

$$ u(x_1, x_2) = \sum_{n \in \mathbb{Z}} u_n(x_2)e^{inx_1}, \quad (3) $$

where $u_n(x_2) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1, x_2)e^{-inx_1}dx_1$, this imposes the condition on the Fourier components $u_n$ that

$$ u_n(x_2) = \begin{cases} 
  u_n(b)e^{i\beta_1^n(x_2-b)}, & n \neq 0, \text{ in } \Omega_1, \\
  u_0(b)e^{i\beta_1^n(x_2-b)} + e^{-i\beta_1 x_2} - e^{i\beta_1(x_2-2b)}, & n = 0, \text{ in } \Omega_1, \\
  u_n(-b)e^{-i\beta_2^n(x_2+b)}, & \text{in } \Omega_2.
\end{cases} \quad (4) $$

From (4) we can compute the derivative of $u_n(x_2)$ with respect to the unit outward normal $\nu$ on $\Omega_0$, then use (3) to obtain expressions for the normal derivatives of $u$. This calculation yields

$$ \frac{\partial u}{\partial \nu}_{|_{\Gamma_1}} = T_1(u|_{\Gamma_1}) - 2i\beta_1 e^{-i\beta_1 b}, \quad (5) $$

$$ \frac{\partial u}{\partial \nu}_{|_{\Gamma_2}} = T_2(u|_{\Gamma_2}), \quad (6) $$

where for functions $f \in H^{\frac{1}{2}}(\Gamma_j)$, the operators $T_j$ are defined by

$$ (T_j f)(x_1) = \sum_{n \in \mathbb{Z}} i\beta_j^n f_n e^{inx_1}, $$

for $j = 1, 2$. The operators $T_j$ are

$$ T_1(f)(x_1) = \int_{\Gamma_1} f(y)\nu(y)dy, \quad T_2(f)(x_1) = \int_{\Gamma_2} f(y)\nu(y)dy, $$

for $j = 1, 2$.
where \( f_n = \frac{1}{2\pi} \int_0^{2\pi} f(x_1) e^{-inx_1} \) and equality is in the sense of distributions. The operators \( T_j : H^{\frac{\gamma}{2}}(\Gamma_j) \to H^{-\frac{\gamma}{2}}(\Gamma_j) \) are continuous, and from (5), (6) we see that each \( T_j \) is a Dirichlet-Neumann map. We use the maps \( T_j \) to define transparent boundary conditions on the artificial boundaries \( \Gamma_j \).

The scattering problem can be formulated as follows. Find \( u \in H^1(\Omega_0) \) such that

\[
(\Delta_\alpha + a_s) u = 0 \quad \text{in } \Omega_0, \tag{7}
\]

\[
(T_1 - \frac{\partial}{\partial \nu}) u = 2i \beta_1 e^{-i\beta_1 b} \quad \text{on } \Gamma_1, \tag{8}
\]

\[
(T_2 - \frac{\partial}{\partial \nu}) u = 0 \quad \text{on } \Gamma_2. \tag{9}
\]

Problem (7)-(9) can be restated in the following weak form. Find \( u \in H^1(\Omega_0) \) such that

\[
B_{a_s}(u, v) = f(v), \quad \text{for all } v \in H^1(\Omega_0), \tag{10}
\]

where

\[
B_{a_s}(u, v) = \int_{\Omega_0} \nabla u \cdot \nabla v - \int_{\Omega_0} a_s u \bar{v} - \int_{\Gamma_1} (T_1 u) \bar{v} - \int_{\Gamma_2} (T_2 u) \bar{v},
\]

\[
f(v) = -2i \beta_1 e^{-i\beta_1 b} \int_{\Gamma_1} \bar{v}.
\]

Here \( f_{\Gamma_j} \) represents the dual pairing of \( H^{-\frac{\gamma}{2}}(\Gamma_j) \) with \( H^{\frac{\gamma}{2}}(\Gamma_j) \) and \( \nabla_\alpha = \nabla + i\alpha \). A proof that the variational problem (10) admits a unique solution for low-frequency waves (small \( \|a_s\|_{L^\infty} \)) is given in [7]. See also the related result in [8].

### 3 The optimal design problem

With the incidence angle and the refractive indices of the materials fixed, there are a fixed and finite number of outward propagating reflected and transmitted plane waves scattered from the periodic structure. The outward propagating modes correspond to indices \( n \) for which the propagation constants \( \beta_j^n \) are real-valued. Let us define the sets of indices of propagating modes

\[
\Lambda_j = \{ n \in Z : \text{Im}(\beta_j^n) = 0 \}, \quad j = 1, 2.
\]

In general, larger values of \( k_j \) (higher frequencies relative to the period of the structure) mean more outward propagating modes.

From (2) and (4), the coefficients of each propagating reflected mode are

\[
r_n = u_n(b) e^{-i\beta_1 b} \quad \text{for } n \in \Lambda_1, \quad n \neq 0, \tag{11}
\]

\[
r_0 = u_0(b) e^{-i\beta_1 b} - e^{-2i\beta_1 b} \quad \text{for } n = 0. \tag{12}
\]

The energy of each reflected mode is \( \beta_1^n |r_n|^2 / \beta_1 \). Similarly, the coefficients of each propagating transmitted mode are

\[
t_m = u_n(-b) e^{-i\beta_1 b} \quad \text{for } m \in \Lambda_2. \tag{13}
\]

The energy of each transmitted mode is \( \beta_2^m |t_m|^2 / \beta_1 \). It will be convenient to regard the reflection and transmission coefficients as vectors

\[
r = (r_n)_{n \in \Lambda_1}, \quad t = (t_m)_{m \in \Lambda_2}.
\]

The vectors \( r \) and \( t \) are functions of the interface profile \( S \). For now, we will denote this dependence with the notation \( r(S), t(S) \). The optimal design problem is to find a curve \( S \) such that \( r(S) \) and \( t(S) \) are as close as possible to some specified constants \( r_0 = (r_0^n)_{n \in \Lambda_1}, t_0 = (t_0^m)_{m \in \Lambda_2} \). We know from the conservation of energy...
(see [?]), that the total energy of the reflected and transmitted waves must equal the energy of the incident wave \( u_\star \). Since \( u_\star \) has unit energy, this condition can be written
\[
\frac{1}{\beta_1} \left( \sum_{n \in \Lambda_1} \beta_1^n |r_n|^2 + \sum_{m \in \Lambda_2} \beta_2^m |t_m|^2 \right) = 1.
\] (14)

We assume that the specified reflection and transmission coefficients \( r_n^0 \) and \( t_m^0 \) also satisfy (14).

Asking that \( r_n(S) \) and \( t_m(S) \) are close to \( r_n^0 \) and \( t_m^0 \) in a least-squares sense leads us to consider minimizing the functional
\[
J(S) = \|r(S) - r_0\|^2 + \|t(S) - t_0\|^2
\]
over some class of admissible curves \( S \), where \( \|r(S) - r_0\|^2 = \sum_{n \in \Lambda_1} |r_n(S) - r_n^0|^2 \) and \( \|t(S) - t_0\|^2 = \sum_{m \in \Lambda_2} |t_m(S) - t_m^0|^2 \). Since we do not wish to impose artificial constraints on the problem, we will allow as admissible curves all Jordan curves inside the subset \( \Omega'_0 = \{ x \in \Omega_0 : -b' < x_2 < b' \} \), where \( 0 < b' < b \). The subset \( \Omega' \) is chosen so the curves stay bounded away from the artificial boundaries \( \Gamma_j \).

Define the set
\[
\tilde{\mathcal{A}} = \{ a_S | S \text{ is a Jordan curve, } S \subset \Omega' \},
\]
where \( a_S \) is defined as in Section 1. The set \( \tilde{\mathcal{A}} \) is neither convex nor closed in any natural topology; minimizing over this set would be difficult. We wish to find a “nice” set to minimize over. One way (in the terminology of [?]) is to “relax” the problem by taking the closure of \( \tilde{\mathcal{A}} \) with respect to the reflection and transmission coefficients \( r_n \) and \( t_m \). Let
\[
\mathcal{A} = \{ a = k_2^2 \gamma + k_1^2 (1 - \gamma) : \gamma \in L^\infty(\Omega_0), \ 0 \leq \gamma \leq 1,
\gamma = 0 \text{ above } \Omega'_0; \ \gamma = 1 \text{ below } \Omega'_0 \}.
\]
The set \( \mathcal{A} \) is the weak\(^*\) \( L^\infty(\Omega_0) \) closure of \( \tilde{\mathcal{A}} \).

The weak formulation of the scattering problem (10) is still valid with \( a_S \) replaced by any \( a \in \mathcal{A} \), that is, the variational problem
\[
B_a(u, v) = f(v), \quad \text{for all } v \in H^1(\Omega_0).
\] (15)
has a unique solution \( u \in H^1(\Omega_0) \). The reflection vector \( r \) depends now on \( a \in \mathcal{A} \). We denote this dependence by \( r_n[a] \), so \( r_n[a] \) is defined by (11) where \( u \) solves (15). Define \( t_m[a] \) similarly.

In [?] it is shown that for low frequencies, the set \( \mathcal{A} \) is the closure of \( \tilde{\mathcal{A}} \) with respect to the zero-order reflection coefficient \( r_0 \). Essentially the same argument works in the present more general case. Without giving a detailed argument, we assert that for each \( a \in \mathcal{A} \), there exists a sequence \( a_j \in \tilde{\mathcal{A}} \) such that
\[
\begin{align*}
 r_n[a_j] &\to r_n[a], \quad \text{for all } n \in \Lambda_1, \\
t_m[a_j] &\to t_m[a], \quad \text{for all } m \in \Lambda_2.
\end{align*}
\]
The main idea is to show that \( \| u \|_{H^1(\Omega_0)} \) is bounded independently of \( a \in \mathcal{A} \), then use weak convergence to show that the functions \( r_n, t_m : \mathcal{A} \to C \) are continuous in the weak\(^*\) \( L^\infty(\Omega_0) \) topology. The argument in [?] is based on the results in [?], [?].

Given that \( \mathcal{A} \) is the closure of \( \tilde{\mathcal{A}} \) with respect to \( r \) and \( t \), we pose the relaxed minimization problem
\[
\min_{a \in \mathcal{A}} J[a] = \|r[a] - r_0\|^2 + \|t[a] - t_0\|^2.
\] (16)
Since \( \mathcal{A} \) is weak\(^*\) \( L^\infty \)-compact and \( r, t \) are weak\(^*\) \( L^\infty \)-continuous, the minimization problem (16) must have at least one solution \( a \in \mathcal{A} \).
4 A modified gradient descent method

In this section, we describe an algorithm for finding local minima of problem (20). We begin with a formal derivation of the gradient of the functional $J[a]$.

We denote the linearization of $J[a]$ in the direction $\delta a$ by $ DJ[a](\delta a)$. We have

$$DJ[a](\delta a) = 2Re \left\{ Dr[a](\delta a) \cdot (r[a] - r_0) + Dt[a](\delta a) \cdot (t[a] - t_0) \right\}$$

where $Dr[a](\delta a)$ denotes the linearization of the reflection vector $r[a]$ and $Dt[a](\delta a)$ is the linearization of the transmission vector $t[a]$. From (11)-(13), the components of $Dr$ and $Dt$ are

$$Dr_n[a](\delta a) = \frac{e^{-i\beta_1 b}}{2\pi} \int_{\Gamma_1} \delta u e^{-i m z_1},$$

$$Dt_m[a](\delta a) = \frac{e^{-i\beta_2 b}}{2\pi} \int_{\Gamma_2} \delta u e^{-i m z_1},$$

where $\delta u$ solves the linearized problem

$$(\triangle + a)\delta u = -\delta a \ u \quad \text{in } \Omega_0,$$

$$(T_j - \frac{\partial}{\partial n})\delta u = 0 \quad \text{on } \Gamma_j, \quad j = 1, 2.$$ 

The $L^2$ adjoints of the derivative maps $Dr[a](\cdot)$ and $Dt[a](\cdot)$ are the linear operators $Dr^*[a](\cdot)$ and $Dt^*[a](\cdot)$ defined so that

$$Dr[a](\delta a) \cdot \bar{\psi} = \int_{\Omega_0} \delta a \cdot \overline{Dr[a](\psi)},$$

$$Dt[a](\delta a) \cdot \bar{\phi} = \int_{\Omega_0} \delta a \cdot \overline{Dt[a](\phi)}$$

for all $\psi = (\psi_n)_{n \in \Lambda_1}$ and $\phi = (\phi_m)_{m \in \Lambda_2}$. Let $w \in H^1(\Omega_0)$ solve

$$(\triangle + a)w = 0 \quad \text{in } \Omega_0,$$

$$(T_1 - \frac{\partial}{\partial n})w = -\frac{e^{i\beta_1 b}}{2\pi} \sum_{n \in \Lambda_1} \psi_n e^{i m z_1} \quad \text{on } \Gamma_1,$$

$$(T_2 - \frac{\partial}{\partial n})w = -\frac{e^{i\beta_2 b}}{2\pi} \sum_{m \in \Lambda_2} \phi_m e^{i m z_1} \quad \text{on } \Gamma_2,$$

where $T_j^*f = -\sum i\beta_j e^nf e^{inx}$. Problem (20)-(22) has the weak form $B^*_a(w, v) = f^*(v)$ for all $v \in H^1(\Omega_0)$, where

$$B^*_a(w, v) = \int_{\Omega} \nabla_a w \cdot \nabla_a \bar{v} - \int_{\Omega} a \ w \bar{v} - \int_{\Gamma_1} (T_1^*w)\bar{v} - \int_{\Gamma_2} (T_2^*w)\bar{v},$$

$$f^*(v) = \frac{e^{i\beta_1 b}}{2\pi} \sum_{n \in \Lambda_1} \psi_n \int_{\Gamma_1} e^{i m z_1} \bar{v} + \frac{e^{i\beta_2 b}}{2\pi} \sum_{m \in \Lambda_2} \phi_m \int_{\Gamma_2} e^{i m z_1} \bar{v}.$$ 

Integrating by parts, one can show that

$$Dr[a](\delta a) \cdot \bar{\psi} + Dt[a](\delta a) \cdot \bar{\phi} = \int_{\Omega_0} \delta a \ \overline{w u}.$$ 

From (18), (19) we then make the identification

$$Dr^*[a](\psi) + Dt^*[a](\phi) = \overline{w u}.$$
Since the $L^2$ “gradient” of the functional $J[a]$ is the function $G[a]$ for which $DJ[a](\delta a) = 2Re \int_{\Omega_0} \delta a \cdot \overline{G[a]}$, we define

$$G[a] = Re(\overline{wu}),$$

where $w$ solves (20)–(22) with $\psi = (r[a] - r_0)$ and $\phi = (t[a] - t_0)$.

The standard gradient descent algorithm is to simply step iteratively in the direction of the negative gradient $-G[a]$ until (hopefully) a local minimizer is reached. This procedure is somewhat inefficient for this problem because at each step the partial differential equations (7)–(9) and (20)–(22) must be solved to find the value of the functional $J[a]$ and the gradient $G[a]$, respectively. Let us modify the statement of the minimization problem (16). Since the coefficients $r_m$ and $t_m$ given by (11)–(13) are well-defined for any $u \in H^1(\Omega_0)$, the functional $j[u] = \|r - r_0\|_2^2 + \|t - t_0\|_2^2$ is defined over all $H^1(\Omega_0)$, and we see that $J[a] = j[u]$, where $u$ solves (15). Problem (16) can then be restated

$$\min_{a \in \mathcal{A}} j[u],$$

subject to $B_a(u, v) = f(v)$, for all $v \in H^1(\Omega_0)$.

If we are willing to consider functions $u \in H^1(\Omega_0)$ which almost satisfy the constraint (24), we can save a lot of the work it takes to evaluate the functional $J[a]$ and the gradient $G[a]$, by solving (7)–(9) and (20)–(22) approximately to some given tolerance, say with an iterative method. As the minimization algorithm converges, we can ask for more and more accurate solutions to (7)–(9) and (20)–(22) in hopes that the coefficient $a$ will converge to a local minimizer of (16) while at the same time $u$ converges toward the constraint set. This is the idea of our algorithm.

Before we can state the algorithm, we need to take care of one more detail. Since each step must remain in the admissible set $\mathcal{A}$, we need a way to project inadmissible steps back into $\mathcal{A}$. For this, we define $P : L^\infty(\Omega_0) \to \mathcal{A}$ as follows. Given $a \in L^\infty(\Omega_0)$, let $\gamma = \min \left\{ \max \left\{ \frac{(a-a_1)}{(b_2-a_1)}, 0 \right\}, 1 \right\}$ and define $P[a] = k_1^2(1 - \gamma) + k_2^2(\gamma)$. The operator $P$ selects the pointwise nearest element in $\mathcal{A}$.

Our minimization algorithm can now be stated as follows:

1. Choose an initial coefficient $a_0$, an initial step length $t > 0$, and a tolerance $\eta > 0$. Solve for $u_0$ and $w_0$ accurately.

2. Do $k = 0, \ldots$,convergence:

3. Calculate the norm of the gradient $g_k = \|G[w_k, u_k]\|$.

4. Set $\tilde{a}_k = P[a_k - tG[w_k, u_k]]$.

5. Find $\tilde{u}_k \in H^1(\Omega_0)$ such that $|B_{\tilde{a}_k}(\tilde{u}_k, v) - f(v)| \leq \eta t g_k$ for all $v$.

6. If $j[\tilde{u}_k] < j[u_k]$ then

   $$a_{k+1} = \tilde{a}_k,$$

   $$u_{k+1} = \tilde{u}_k,$$

   find $w_{k+1} \in H^1(\Omega_0)$ such that $|B^*_{a_{k+1}}(w_{k+1}, v) - f^*(v)| \leq \eta t g_k$ for all $v$.

   Else

   $$t := \frac{t}{2}; \quad \eta := \frac{\eta}{2},$$

   increase the accuracy of $w_k$ to $\eta t g_k$,

   increase the accuracy of $u_k$ to $\eta t g_k$,

   go to Step 4.