GLOBAL EXISTENCE OF SMOOTH SHEARING MOTIONS OF A NONLINEAR VISCOELASTIC FLUID

By

Deborah Brandon

and

William J. Hrusa

IMA Preprint Series # 654

June 1990
Global Existence of Smooth Shearing
Motion of a Nonlinear Viscoelastic Fluid

by

DEBRAH BRANDON

Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

and

WILLIAM J. HRUSA

Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Dedicated to John Nohel on the occasion of his sixty-fifth birthday.

1. Introduction.

The aim of this note is to establish global existence of smooth rectilinear shearing motions of incompressible nonlinear viscoelastic fluids of K-BKZ type. These fluids, which are described by constitutive relations of integral type, were introduced independently by Kaye [7] and Bernstein, Kearsley and Zapas [1]. Global existence theorems have been obtained previously by Kim [8] and by Renardy, Hrusa and Nohel [10; Section IV.5]. Kim discusses a situation in which the fluid occupies all of \( \mathbb{R}^3 \) and the nonlinearity in the constitutive equation has a special form. Renardy, Hrusa and Nohel study spatially periodic three dimensional motions with a general nonlinearity in the constitutive equation. In [8] and [10] the initial data are assumed to be smooth and small and the kernel of the constitutive relation is assumed to be smooth on \([0, \infty)\).

The equation of motion that we shall consider is

\[
(1.1) \quad u_{tt}(x, t) = \int_0^\infty a'(s) g(u_x(x, t) - u_x(x, t - s))_x \, ds + f(x, t), \quad x \in B, t \geq 0,
\]

where subscripts \( x \) and \( t \) indicate partial derivatives and \( a' \) denotes the derivative of \( a \). Here the unknown \( u \) is a component of the displacement, \( f \) is a forcing function, \( a : [0, \infty) \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are smooth constitutive functions, and \( B \subset \mathbb{R} \) is an interval. It follows from symmetry considerations that \( g \) is an odd function, i.e., \( g(-\xi) = -g(\xi) \) for all \( \xi \in \mathbb{R} \). We refer to Coleman and Noll [4] and to Sections 2 and 3 of Coleman and Gurtin [3] for a general discussion of shearing motions of incompressible viscoelastic fluids. The monograph [10] contains relevant information on K-BKZ fluids as well as derivation of (1.1) for rectilinear shearing motions.

If the displacement \( u \) is sufficiently regular then we may rewrite (1.1) in the form

\[
(1.2) \quad \bar{v}_t(x, t) = \int_0^\infty a'(s) g(\bar{\nu}_x(x, s))_x \, ds + f(x, t), \quad x \in B, t \geq 0,
\]
where \( v := u_t \) is the velocity and \( \bar{v}^t \) is the summed history up to time \( t \) of \( v \), i.e.

\[
(1.3) \quad \bar{v}^t(x, s) := \int_{t-s}^t v(x, \lambda) \, d\lambda.
\]

For our purposes, it will be convenient to work with (1.2) in place of (1.1). We assume that the fluid has been at rest prior to time \( t = 0 \) and that an initial velocity is prescribed at \( t = 0^+ \). We treat in detail the case when \( B = [0, 1] \) and nonslip boundary conditions are imposed:

\[
(1.4)_1 \quad v_t(x, t) = \int_0^\infty a'(s)g(\bar{v}^s_x(x, s))_x \, ds + f(x, t), \quad x \in [0, 1], t \geq 0,
\]

\[
(1.4)_2 \quad v(0, t) = v(1, t) = 0, \quad t \geq 0,
\]

\[
(1.4)_3 \quad v(x, \tau) = 0, \quad x \in [0, 1], \tau < 0,
\]

\[
(1.4)_4 \quad v(x, 0) = v_0(x), \quad x \in [0, 1].
\]

We assume that the constitutive functions \( a \) and \( g \) satisfy

\[
(1.5) \quad a, a', a'' \in L^1(0, \infty), \quad a \text{ is strongly positive},
\]

\[
(1.6) \quad g \in C^3(\mathbb{R}), \quad g \text{ is odd, } \exists \gamma > 0 \text{ such that } g'(\xi) \leq -\gamma \quad \forall \xi \in \mathbb{R}.
\]

The derivatives appearing in (1.5) and throughout the remainder of the paper should be interpreted in the sense of distributions. The definition of a strongly positive kernel is given in the next section. For now, we note that (1.5) implies

\[
(1.7) \quad a \in C^1[0, \infty), \quad a(0) > 0, \quad a'(0) < 0;
\]

moreover if \( a \) satisfies

\[
(1.8) \quad a \in C^2[0, \infty), \quad a \geq 0, \quad a' \leq 0, \quad a'' \geq 0, \quad a'(0) < 0
\]

then \( a \) is strongly positive. Regarding the smoothness of \( v_0 \) and \( f \) we require

\[
(1.9) \quad v_0 \in H^2(0, 1), \text{ i.e. } v_0, v_0', v_0'' \in L^2(0, 1),
\]

\[
(1.10) \quad f, f_x, f_t \in C_b([0, \infty); L^2(0, 1)) \cap L^2([0, \infty); L^2(0, 1)), \quad f_{tt} \in L^2([0, \infty); L^2(0, 1)),
\]

where \( C_b([0, \infty); L^2(0, 1)) \) denotes the set of all \( w : [0, 1] \times [0, \infty) \rightarrow \mathbb{R} \) such that the mapping \( t \mapsto w(\cdot, t) \) is bounded and continuous from \([0, \infty)\) to \( L^2(0, 1) \). We also assume that \( v_0 \) and \( f \) are compatible with the boundary conditions in the sense that

\[
(1.11) \quad v_0(0) = v_0(1) = f(0, 0) = f(1, 0) = 0.
\]

Assumptions (1.5) and (1.6) ensure that equation (1.4) is of hyperbolic type and that the memory has a dissipative effect. However, we cannot expect (1.4) to have a globally defined smooth solution unless some restrictions are placed on the “sizes” of \( v_0 \) and \( f \). (Cf. Coleman and Gurtin [3] who show that in shearing motions of a general nonlinear viscoelastic fluid acceleration waves of small amplitude decay, but waves of large amplitude can explode in finite time.)

To “measure” \( v_0 \) and \( f \) we define

\[
(1.12) \quad V_0(v_0) := \int_0^1 \{ v_0(x)^2 + v'_0(x)^2 + v''_0(x)^2 \} \, dx
\]

\[
F(f) := \sup_{t \geq 0} \int_0^1 \{ f^2 + f_x^2 + f_t^2 \}(x, t) \, dx \, dt.
\]
There are some superfluous terms in (1.12) and (1.13) that can be eliminated because of Poincaré's inequality and the Sobolev embedding theorem. However, for our proof of global existence, it is convenient to define $V_0$ and $F$ as above.

We shall not obtain a time-independent bound for $\bar{v}_x$. Consequently, for technical reasons, we need to make an assumption concerning the rate of growth of $g$ at infinity relative to the rate of decay of $a$. Precisely we assume that there are constants $K > 0$ and $k > 1$ such that

$$|g^{(j)}(\xi) - g^{(j)}(0)| \leq K(|\xi| + |\xi|^k) \quad j = 1, 2, 3, \forall \xi \in \mathbb{R}$$

and

$$\int_0^\infty |a'(z)|z^{\frac{k}{2}+1} dz, \int_0^\infty |a''(z)|z^{\frac{k}{2}} dz < \infty.$$  

**Theorem 1.1.** Assume that (1.5), (1.6), (1.14) and (1.15) hold. Then there is a number $\delta > 0$ such that for every $v_0$ and $f$ satisfying (1.9), (1.10), (1.11) and

$$V_0(v_0) + F(f) \leq \delta,$$

the problem (1.4) has a unique solution $v$ with

$$v, v_x, v_t, v_{xx}, v_{xt}, v_{tt} \in C_b([0, \infty); L^2(0,1)) \cap L^2([0, \infty); L^2(0,1)).$$

**Remark 1.1.** It follows from (1.17) and standard embedding theorems that $v \in C^1([0,1] \times [0,\infty))$ and $v, v_x, v_t \to 0$ uniformly on $[0,1]$ as $t \to \infty$.

A similar theorem holds when $B = \mathbb{R}$, i.e. for the problem

$$v_t(x,t) = \int_0^\infty a'(s)g(v(x,s)) v_x(x,t) ds + f(x,t), \quad x \in \mathbb{R}, t \geq 0,$$

$$v(x,\tau) = 0, \quad x \in \mathbb{R}, \tau < 0,$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}.$$  

In place of (1.9), (1.10) and (1.11), we assume that

$$v_0 \in H^2(\mathbb{R})$$

and use

$$F^*(f) := \sup_{t \geq 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{f^2 + f_x^2 + f_t^2\}(x,t) dx dt + \int_0^{\infty} \int_{-\infty}^{\infty} \{f_x^2 + f_t^2 + f_{tt}^2\}(x,t) dx dt$$

$$+ \left( \int_0^{\infty} \left( \int_{-\infty}^{\infty} f(x,t)^2 dx \right)^{\frac{1}{2}} dt \right)^2$$

(1.22)

to measure the data.
Theorem 1.2. Assume that (1.5), (1.6), (1.14) and (1.15) hold. Then there is a number $\delta^* > 0$ such that for every $v_0$ and $f$ satisfying (1.19), (1.20) and

$$V_0^*(v_0) + F^*(f) \leq \delta^*,$$

the problem (1.18) has a unique solution $v$ with

$$v, v_x, v_{xt}, v_{xx}, v_{xt}, v_{tt} \in C_b([0, \infty); L^2(\mathbb{R})), \quad v_x, v_t, v_{xx}, v_{xt}, v_{tt} \in L^2([0, \infty); L^2(\mathbb{R})).$$

Remark 1.2. It follows from (1.24) and standard embedding theorems that $v \in C^1(\mathbb{R} \times [0, \infty))$ and that $v, v_x, v_t \to 0$ uniformly, $v_x, v_t \to 0$ in $L^2(\mathbb{R})$ as $t \to \infty$.

Similar existence theorems for viscoelastic solids have been established by various authors (cf., e.g., [5],[10] and the references cited therein). (For solids, one generally can also obtain precise information on the behavior of the displacement as $t \to \infty$.) Moreover, for nonlinear constitutive equations several authors have shown that smooth solutions can develop singularities in finite time if the data are too large; such results have been obtained both for fluids and solids (cf., e.g. [10],[11]). In the theorems concerning formation of singularities it is assumed that the kernel is smooth on $[0, \infty)$. We refer to the paper of Engler [6] for some interesting results concerning global existence of weak solutions for equations with singular kernels and data of unrestricted size.

The next section contains some preliminary material concerning strong positivity of the kernel. In Section 3 we prove Theorem 1.1 and point out the modifications needed to prove Theorem 1.2.

2. Preliminaries.

This section contains some preliminary material concerning the kernel $a$. Since the notion of strong positivity plays an important role in our analysis we briefly recall a few basic concepts. Let $b \in L^1_{loc}(0, \infty)$ be given. We say that $b$ is a kernel of positive type if

$$\int_0^t y(s) \int_0^s b(s - \tau)y(\tau) \, d\tau \, ds \geq 0, \quad \forall t \geq 0$$

for every $y \in C[0, \infty)$; we say that $b$ is of strongly positive type if there exists $\epsilon > 0$ such that the kernel $t \mapsto b(t) - \epsilon e^{-t}$ is of positive type. As the terminology suggests, strong positivity of $b$ implies positivity of $b$.

The above definitions generally are not easy to check directly. We note that if

$$b \in C^2[0, \infty), \quad b \geq 0, \quad b' \leq 0, \quad b'' \geq 0, \quad b' \neq 0,$$

then $b$ is strongly positive (cf., e.g., Corollary 2.2 of [9]). Strong positivity does not imply any global sign conditions (e.g., $e^{-t}$cost is strongly positive). However, if a strongly positive function is sufficiently regular then statements can be made regarding its pointwise behavior near zero. In particular, (1.5) implies

$$a(0) > 0, \quad a'(0) < 0.$$

For each $T > 0$ and $w \in C([0, T]; L^2(0, 1))$ let us put

$$Q(w, t, a) = \int_0^t \int_0^1 w(x, s) \int_0^s a(s - \tau)w(x, \tau) \, d\tau \, dx \, ds \quad \forall t \in [0, T].$$

Strong positivity of $a$ implies some very useful estimates of coercive type for $Q(w, t, a)$. Our first lemma, which was established by Brandon (cf. Lemma 2.4 of [2]), generalizes an inequality that was used by Dafermos and Nohel [5] to obtain global estimates for a one-dimensional nonlinear
viscoelastic solid. The basic idea is that a time–independent bound for $Q(w, t, a) + Q(w_t, t, a)$ yields a time–independent bound for

$$
\int_0^1 w(x, t)^2 \, dx + \int_0^t \int_0^1 w(x, s)^2 \, dx \, ds.
$$

For technical reasons, we use difference operators

$$
(\Delta_h w)(x, t) := w(x, t + h) - w(x, t)
$$

in place of derivatives.

**Lemma 2.1.** Assume that (1.5) holds. Then there is a constant $L > 0$ such that for every $T > 0$ and every $w \in C([0, T]; L^2(0, 1))$ we have

$$
\int_0^1 w(x, t)^2 \, dx + \int_0^t \int_0^1 w(x, s)^2 \, dx \, ds \\
\leq L \left\{ \int_0^1 w(x, 0)^2 \, dx + Q(w, t, a) + \liminf_{h \to 0} \frac{1}{h^2} Q(\Delta_h w, t, a) \right\} \quad \forall t \in [0, T].
$$

Our next lemma is due to Staffans (cf. Lemma 4.2 of [12]).

**Lemma 2.2.** If $a$ satisfies (1.5) then for every $T > 0$ and every $w \in C([0, T]; L^2(0, 1))$ we have

$$
\int_0^1 \left( \int_0^t a(t - \tau) w(x, \tau) \, d\tau \right)^2 \, dx \leq 2a(0)Q(w, t, a) \quad \forall t \in [0, T].
$$

Lemmas 2.1 and 2.2 will be used to obtain a priori bounds for the (local) solution $w$ of (1.4) directly from energy integrals. We shall need an additional bound that can be obtained by expressing $v_{xx}$ in terms of $v_{tt}$ through an inverse Volterra operator.

If (1.5) holds then for each $h \in L^1_{loc}(0, \infty)$ the Volterra equation

$$
a(0)y(t) + \int_0^t a'(t - \tau)y(\tau) \, d\tau = h(t), \quad t \geq 0
$$

has a unique solution $y \in L^1_{loc}(0, \infty)$; the solution can be expressed in the form

$$
y(t) = \frac{1}{a(0)} \left( h(t) + \int_0^t r(t - \tau)h(\tau) \, d\tau \right),
$$

where $r$ is the resolvent kernel associated with (2.8), i.e. the unique solution of

$$
a(0)r(t) + \int_0^t a'(t - \tau)r(\tau) \, d\tau = -a'(t).
$$

It follows from the Paley–Wiener theorem that $r \notin L^1(0, \infty)$. However, we do have the following result.
Lemma 2.3. Assume that (1.5) holds and let \( r \) be the resolvent kernel associated with (2.8), i.e. the solution of (2.10). Then \( r \) is locally absolutely continuous on \([0, \infty)\) and \( r' \in L^1(0, \infty) \).

See, for example, Lemma 2.3 of [2] for a proof. (In [2] it is also assumed that \( a''' \in L^1(0, \infty) \). However, this assumption is not needed for the proof of Lemma 2.3.)

Remark 2.1: For a viscoelastic solid the analog of equation (2.8) has a resolvent kernel that belongs to \( L^1(0, \infty) \). Nonintegrability of the resolvent kernel is one of key reasons why it is generally more difficult to obtain estimates for a viscoelastic fluid than for a viscoelastic solid.

Remark 2.2: Lemmas 2.1 and 2.2 remain valid if the spatial interval \([0, 1]\) is replaced by \((-\infty, \infty)\) throughout.

3. Proof of Theorem 1.1.

Our proof is based on essentially the same line of argument as the one employed in Section 3 of [2]. Therefore, some of the steps that are virtually identical will not be repeated here.

The local existence of a smooth solution can be established by a routine contraction–mapping argument. The relevant result is recorded below without proof. We refer the reader to Chapter III of [10] and Lemma 2.1 of [2] for proofs of similar results.

Proposition. Assume that (1.5), (1.6), (1.9)–(1.11), (1.14) and (1.15) hold. Then the initial–value problem (1.4) has a unique solution \( v \), defined on a maximal time interval \([0, T_0)\), \( T_0 > 0 \), satisfying

(3.1) \[ v, v_x, v_{tt}, v_{xx}, v_{xt}, v_{tt} \in C([0, T_0); L^2(0, 1)). \]

Moreover, if

(3.2) \[ \sup_{t \in [0, T_0]} \int_0^1 (v^2 + v_x^2 + v_{tt}^2 + v_{xx}^2 + v_{xt}^2 + v_{tt}^2)(x, t) \, dx < \infty \]

then \( T_0 = \infty \).

In order to prove that (1.4) has a solution which is defined globally in time, it suffices to show that if (1.16) holds for \( \delta > 0 \) sufficiently small, then the local solution satisfies (3.2). For this purpose it is convenient to introduce the quantities

(3.3) \[ E(t) := \sup_{s \in [0, t]} \int_0^1 (v^2 + v_x^2 + v_{xx}^2 + v_{xt}^2 + v_{tt}^2)(x, s) \, dx \]

and

(3.4) \[ v(t) := \sup_{x \in [0, 1]} \left( v^2 + v_x^2 + v_{xx}^2 + \left( \int_0^t (v_x(x, s))^2 \, ds \right)^{1/2} \right), \quad t \in [0, T_0). \]

It is also convenient to rewrite (1.4), as

(3.5) \[ v_t(x, t) + g'(0) \int_0^t a(t - s) v_{xx}(x, s) \, ds = f(x, t) + \int_0^t v_{xx}(x, s) \int_{t-s}^{\infty} a'(x)[g'(v^i_x(x, z)) - g'(0)] \, dz \, ds, \quad x \in [0, 1], t \in [0, T_0). \]

Our aim is to establish inequality (3.28) below; to do so we use energy methods. Most of the estimates are derived from energy integrals; some additional ones are obtained from equation
(3.5) by inverting the Volterra operator. In the energy integrals, the left-hand side of (3.5) yields positive definite terms, and the right-hand side leads to expressions which are under control near equilibrium.

In the calculations below we frequently use the inequalities

\[(3.6) \quad \left( \sum_{i=1}^{n} A_i \right)^2 \leq n \sum_{i=1}^{n} A_i^2 \quad A_1, \ldots, A_n \in \mathbb{R}, \]

\[(3.7) \quad |AB| \leq \frac{A^2}{4\lambda} + \lambda B^2 \quad A, B \in \mathbb{R}, \lambda > 0, \]

and

\[(3.8) \quad \|A \ast B\|_{L^p((0,T);L^2(0,1))} \leq \|A\|_{L^1(0,\infty)} \|B\|_{L^p((0,T);L^2(0,1))} \]

for every \( T > 0, A \in L^1(0,\infty) \) and \( B \in L^p((0,T);L^2(0,1)) \), where \( p \in [1,\infty] \) and \( \ast \) denotes convolution in the time variable. We use \( \Gamma \) to denote a (possibly large) positive generic constant, independent of \( v_0, f \) and \( T_0 \).

The first energy integral is obtained by multiplying (3.5) by \( v \) and integrating over \([0,1] \times [0,t], t \in [0,T_0] \). After integration by parts, we obtain the identity

\[(3.9) \quad \frac{1}{2} \int_0^1 v^2(x,t) \, dx - g'(0)Q(v_x,t,a)
 = \frac{1}{2} \int_0^1 v_0^2(x) \, dx + \int_0^t \int_0^1 v(x,s)f(x,s) \, dx \, ds
 + \int_0^t \int_0^1 v(x,s) \int_0^s v_{xx}(x,y) \int_{s-y}^\infty a'(z)[g'(v_x^s(x,z)) - g'(0)] \, dz \, dy \, ds, \quad t \in [0,T_0]. \]

We now differentiate (3.5) with respect to \( t \) to obtain

\[(3.10) \quad v_t(x,t) + g'(0)a(0)v_{xx}(x,t) + g'(0) \int_0^t a'(t-s)v_{xx}(x,s) \, ds
 = f_t(x,t) + \frac{\partial}{\partial t} \left\{ \int_0^t v_{xx}(x,s) \int_{t-s}^\infty a'(z)[g'(v_x^s(x,z)) - g'(0)] \, dz \, ds \right\},
 \quad x \in [0,1], \ t \in [0,T_0]. \]

After multiplying (3.10) by \( v_t \) and then integrating over \([0,1] \times [0,t], t \in [0,T_0] \) we obtain the relation

\[(3.11) \quad \frac{1}{2} \int_0^1 v_t^2(x,t) \, dx - g'(0)Q(v_x,t,a)
 = \frac{1}{2} \int_0^1 v_t^2(x,0) \, dx - g'(0) \int_0^t \int_0^1 v(x,s)a(s)v_0^s(x) \, dx \, ds
 + \int_0^t \int_0^1 v_t(x,s)f_t(x,s) \, dx \, ds + \int_0^t \int_0^1 v_t(x,s) \frac{\partial}{\partial s} \left\{ \int_0^s v_{xx}(x,y) \int_{s-y}^\infty a'(z)[g'(v_x^s(x,z))
 - g'(0)] \, dz \, dy \right\} \, dz \, ds, \quad t \in [0,T_0]. \]

Observe that

\[(3.12) \quad v_t(x,0) = f(x,0) \quad x \in [0,1], \]

\[7\]
by virtue of (3.5).

We next apply the forward difference operator $\Delta_h$ to (3.10). We multiply the resulting equation by $\Delta_h v_t$ and integrate over $[0, 1] \times [0, t]$, $t \in [0, T_0]$. We integrate several terms by parts and then divide both sides by $h^2$ and let $h \downarrow 0$ to get

$$\frac{1}{2} \int_0^1 u_t^2(x, t) \, dx - g'(0) \lim_{h \downarrow 0} \frac{1}{h^2} Q(\Delta_h v_{xt}, t, a)$$

$$= \frac{1}{2} \int_0^1 u_t^2(x, 0) \, dx - g'(0) \int_0^t \int_0^1 u_t(x, s) a'(s) v'_t(x, s) \, dx \, ds$$

$$- g'(0) \int_0^t \int_0^1 v_{xt}(x, s) a'(s) v_{xt}(x, 0) \, dx \, ds + g'(0) \int_0^1 a(t) v_{xt}(x, t) v_{xt}(x, 0) \, dx$$

$$- g'(0) \int_0^1 a(0) v^2_{xt}(x, 0) \, dx + \int_0^t \int_0^1 v_{xt}(x, s) f_{xt}(x, s) \, dx \, ds$$

$$+ \int_0^t \int_0^1 v_{xt}(x, s) \int_0^1 v_{xx}(x, y) \int_y^\infty a'(z) \frac{d}{dz} \{ g''(\bar{v}_x^a(x, z)) \} \, dz \, dy \, dx \, ds$$

(3.13)

$$+ 2 \int_0^t \int_0^1 v_{xt}(x, s) \int_0^\infty a'(z) \frac{d}{dz} \{ g'(\bar{v}_x^a(x, z)) \} \{ v_{xx}(x, s) - v_{xx}(x, s - z) \} \, dz \, dx \, ds$$

$$- \frac{1}{2} \int_0^1 v^2_{xt}(x, t) \int_0^\infty a'(s) [g'(\bar{v}_x^a(x, s)) - g'(0)] \, ds \, dx$$

$$+ \frac{1}{2} \int_0^t \int_0^1 v^2_{xt}(x, s) \frac{d}{dz} \{ \int_0^\infty a'(z) [g'(\bar{v}_x^a(x, z)) - g'(0)] \, dz \} \, dx \, ds$$

$$+ \int_0^1 v_{xt}(x, t) \int_0^t a'(s) [g'(\bar{v}_x^a(x, s)) - g'(0)] v_{xt}(x, t - s) \, ds \, dx$$

$$- \int_0^t \int_0^1 v_{xt}(x, s) \int_0^1 a''(z) [g'(\bar{v}_x^a(x, z)) - g'(0)] v_{xt}(x, s - z) \, dz \, dx \, ds$$

$$- \int_0^t \int_0^1 v_{xt}(x, s) v_{xx}(x, s) \int_0^1 a'(z) g''(\bar{v}_x^a(x, z)) v_{xt}(x, s - z) \, dz \, dx \, ds, \quad t \in [0, T_0].$$

The initial values of $v_{tt}$ and $v_{xt}$ can be expressed in terms of $f$ and $v_0$. In particular (3.10) and (3.12) imply

$$v_{tt}(x, 0) = f_t(x, 0) - g'(0) a(0) v''_0(x) \quad x \in [0, 1]$$

and

$$v_{xt}(x, 0) = f_x(x, 0) \quad x \in [0, 1].$$

We note that $\lim_{h \downarrow 0} \frac{1}{h^2} Q(\Delta_h v_{xt}, t, a)$ exists for $t \in [0, T_0]$, since the limit of each of the other terms in the derivation of (3.13) exists. Furthermore, the limit under consideration is nonnegative.

We add (3.9), (3.11) and (3.13) and use Lemma 2.2 to obtain a lower bound for the left-hand side. After making some routine estimations we arrive at

$$\int_0^1 (v^2 + v_x^2 + v_t^2 + v_{xt}^2 + v_{tt}^2)(x, t) \, dx + \int_0^1 \int_0^1 (v_x^2 + v_{xt}^2)(x, s) \, dx \, ds$$

$$\leq \Gamma(V_0 + F) + \Gamma(\sqrt{V_0 + F}) \sqrt{E(t)} + \Gamma(\nu(t) + \nu^{k+2}(t)) \nu(t), \quad \forall t \in [0, T_0).$$
We will now give an indication of how (3.16) was derived. We show the details of the estimation of several terms that arise in (3.9), (3.11) and (3.13). Many of the terms can be estimated in a simple way, e.g.

\[
\left| \int_0^t \int_0^1 v_t(x, s) f_t(x, s) \, dx \, ds \right|
\]

\[
\leq \left( \int_0^t \int_0^1 v_t^2(x, s) \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_0^1 f_t^2(x, s) \, dx \, ds \right)^{\frac{1}{2}}
\]

\[
\leq \Gamma \sqrt{F} \sqrt{\mathcal{E}(t)}, \quad \forall t \in [0, T_0)
\]

and

\[
|g''(0)| \int_0^t \int_0^1 v_t(x, s) a'(s) v_0''(x) \, dx \, ds|
\]

\[
\leq \Gamma(\int_0^t \int_0^1 v_t^2(x, s) \, dx \, ds)^{\frac{1}{2}} (\int_0^t a'(s)^2 \, ds \int_0^1 v_0''(x)^2 \, dx)^{\frac{1}{2}}
\]

\[
\leq \Gamma \sqrt{\mathcal{V}_0} \sqrt{\mathcal{E}(t)}, \quad \forall t \in [0, T_0).
\]

The following two estimations are much more involved than (3.17) and (3.18). Recall here that \(g''(0) = 0\). From (3.13) we consider:

\[
\left| \int_0^t \int_0^1 v_{xt}(x, s)v_x(x, s) \int_0^s a'(z)g''(\bar{v}_x(x, z))v_{xt}(x, s - z) \, dz \, dx \, ds \right|
\]

\[
\leq \sup_{s \in [0, t]} |v_x(x, s)| \int_0^t \int_0^1 |v_{xt}(x, s)| \int_0^s |a'(z)| \cdot |g''(\bar{v}_x(x, z))|
\]

\[
- g''(0)| \cdot |v_{xt}(x, s - z)| \, dz \, dx \, ds
\]

\[
\leq \nu(t) \int_0^t \int_0^1 |v_{xt}(x, s)| \int_0^s |a'(z)|K[|\bar{v}_x(x, z)| + |\bar{v}_x(x, z)|^k]|v_{xt}(x, s - z)| \, dz \, dx \, ds
\]

\[
\leq \Gamma \nu(t) \int_0^t \int_0^1 |v_{xt}(x, s)| \int_0^s |a'(z)|[\sqrt{2}(\int_{s-z}^s v_x^2(x, \xi) \, d\xi)^{\frac{1}{2}}
\]

\[
+ (\sqrt{2})^k(\int_{s-z}^s v_x^2(x, \xi) \, d\xi)^{\frac{1}{2}}]|v_{xt}(x, s - z)| \, dz \, dx \, ds
\]

\[
\leq \Gamma \nu(t)(\int_0^t \int_0^1 v_{xt}^2(x, s) \, dx \, ds)^{\frac{1}{2}}(\int_0^t \int_0^s |a'(z)|[\sqrt{2} \nu(t)
\]

\[
+ (\sqrt{2})^k \nu^k(t)]|v_{xt}(x, s - z)| \, dz \, dx \, ds)^{\frac{1}{2}}
\]

\[
\leq \Gamma \nu(t) \sqrt{\mathcal{E}(t)}(\int_0^t \int_0^1 v_{xt}^2(x, s) \, dx \, ds)^{\frac{1}{2}}\{\nu(t) \int_0^\infty |a'(z)| \sqrt{2} \, dz
\]

\[
+ \nu^k(t) \int_0^\infty |a'(z)|[\sqrt{2}]^k \, dz\}
\]

\[
\leq \Gamma \nu(t)(\nu(t) + \nu^k(t)) \mathcal{E}(t)
\]

\[
= \Gamma(\nu^2(t) + \nu^{k+1}(t)) \mathcal{E}(t), \quad \forall t \in [0, T_0).
\]

The following expression arises after differentiation with respect to \(s\) is carried out in the last term.
of (3.11):

\[
\left| \int_0^t \int_0^1 v_t(x, s) \int_0^s v_{xx}(x, y) \int_\infty^{\infty} a'(z)g''(\bar{v}_x^k(x, z))v_x(x, s - z) \, dz \, dy \, dx \, ds \right|
\]

\[
\leq \sup_{x \in [0,1]} |v_x(x, s)| \int_0^t \int_0^1 |v_t(x, s)| \int_0^s |v_{xx}(x, y)| \int_\infty^{\infty} |a'(z)| |g''(\bar{v}_x^k(x, z))|
\]

\[
- g''(0) |dz| \, dy \, dx \, ds
\]

\[
\leq \nu(t) \int_0^t \int_0^1 |v_t(x, s)| \int_0^s |v_{xx}(x, y)| \int_\infty^{\infty} |a'(z)| |K||\bar{v}_x^k(x, z)|
\]

\[
+ |\bar{v}_x^k(x, z)|^k |dz| \, dy \, dx \, ds
\]

\[
\leq \Gamma \nu(t) \left( \int_0^t \int_0^1 v_t(x, s) \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_0^1 |v_{xx}(x, y)| \int_\infty^{\infty} |a'(z)| |\sqrt{z} \nu(t)|
\]

\[
+ (\sqrt{z})^k \nu^k(t) |dz| \, dy \, dx \, ds \right)^{\frac{1}{2}}
\]

\[
\leq \Gamma (\nu^2(t) + \nu^{k+1}(t)) \sqrt{\mathcal{E}(t)} \left( \int_0^t \int_0^1 v_{xx}(x, s) \, dx \, ds \right)^{\frac{1}{2}} \int_\infty^{\infty} |a'(z)| |\sqrt{z} + (\sqrt{z})^k| \, dz \, ds
\]

\[
\leq \Gamma (\nu^2(t) + \nu^{k+1}(t)) \mathcal{E}(t), \quad \forall t \in [0, T_0).
\]

In the derivation of (3.20) we have used (3.8) with \( A(s) := \int_\infty^{\infty} |a'(z)| |\sqrt{z} + (\sqrt{z})^k| \, dz \). The estimations (3.17)–(3.20) are typical of the calculations necessary to obtain (3.16). All the terms appearing on the right-hand side of (3.9), (3.11) and (3.13) can be estimated in the same spirit as (3.17), (3.18), (3.19) or (3.20).

A temporal \( L^2 \) bound for \( v_t \) follows from Poincaré's inequality and (3.16). However, since Poincaré's inequality cannot be applied in the proof of Theorem 1.2, we shall derive an additional energy integral. We multiply (3.5) by \( v_{xx} \) and integrate over \([0,1] \times [0, t] , t \in [0, T_0] \) to obtain

\[
\frac{1}{2} \int_0^1 v_x^2(x, t) - g'(0)Q(v_{xx}, t, a)
\]

\[
= \frac{1}{2} \int_0^1 v_0^2(x) \, dx + \int_0^t \int_0^1 v_{xx}(x, s)f(x, s) \, dx \, ds
\]

\[
+ \int_0^t \int_0^1 v_{xx}(x, s) \int_0^s v_{xx}(x, y) \int_\infty^{\infty} a'(z)[g'(\bar{v}_x^k(x, z)) - g'(0)] \, dz \, dy \, dx \, ds,
\]

which implies

\[
Q(v_{xx}, t, a) \leq \Gamma V_0 + \Gamma \sqrt{F} \sqrt{\mathcal{E}(t)} + \Gamma (\nu(t) + \nu^k(t)) \mathcal{E}(t), \quad \forall t \in [0, T_0).
\]

We now square (3.5) and integrate over \([0,1] \times [0, t] , t \in [0, T_0] \). The terms resulting from the right-hand side of (3.5) can be controlled near equilibrium; Lemma 2.1 and (3.22) can be employed to estimate the square of the convolution term from the left-hand side of (3.5). Thus we have

\[
\int_0^t \int_0^1 v_t^2(x, s) \, dx \, ds \leq \Gamma (V_0 + F) + \Gamma \sqrt{F} \sqrt{\mathcal{E}(t)} + \Gamma (\nu(t) + \nu^{2k}(t)) \mathcal{E}(t), \quad \forall t \in [0, T_0).
\]
If we set $G$ equal to the right-hand side of (3.5), we can invert (3.10) to express $v_{xx}$ in terms of $v_t, v_{tt}, G,$ and $G_t$ (see (2.8) et seq.), i.e.,

$$g'(0)a(0)v_{xx}(x, t) = G_t(x, t) - v_{tt}(x, t) - \frac{a'(0)}{a(0)}(G(x, t) - v_t(x, t))$$

$$+ \int_0^t r'(t-s)(G(x, s) - v_t(x, s))\, ds, \quad x \in [0,1], \ t \in [0,T_0).$$

(3.24)

We now square (3.24) and integrate over $[0,1]$. Using (3.16) and Lemma 2.3 we arrive at

$$\int_0^1 v_{xx}^2(x, t)\, dx \leq \Gamma(V_0 + F) + \Gamma(\sqrt{V_0} + \sqrt{F})\sqrt{\mathcal{E}(t)} + \Gamma(\nu(t) + \nu^{2k+2}(t))\mathcal{E}(t), \quad \forall t \in [0,T_0).$$

(3.25)

We next multiply (3.24) by $v_{tt}$ and integrate over $[0,1] \times [0,t], \ t \in [0,T_0)$ to obtain

$$\int_0^t \int_0^1 v_{tt}^2(x, s)\, ds\, dx \leq \Gamma(V_0 + F) + \Gamma(\sqrt{V_0} + \sqrt{F})\sqrt{\mathcal{E}(t)} + \Gamma(\nu(t) + \nu^{k+2}(t))\mathcal{E}(t), \quad \forall t \in [0,T_0).$$

(3.26)

Here, we made crucial use of Lemma 2.3 and inequality (3.7). We again square (3.24), but now we integrate over $[0,1] \times [0,t], \ t \in [0,T_0)$ to establish a temporal $L^2$ estimate for $v_{xx}$. We employ (3.16), (3.23), (3.26) and Lemma 2.3 to get

$$\int_0^t \int_0^1 v_{xx}^2(x, s)\, ds\, dx \leq \Gamma(V_0 + F) + \Gamma(\sqrt{V_0} + \sqrt{F})\sqrt{\mathcal{E}(t)} + \Gamma(\nu(t) + \nu^{2k+2}(t))\mathcal{E}(t), \quad \forall t \in [0,T_0).$$

(3.27)

Therefore, by combining (3.16), (3.23), (3.25), (3.26), (3.27) and using Poincaré's inequality to obtain a temporal $L^2$ bound for $v$, we arrive at

$$\mathcal{E}(t) \leq \Gamma(V_0 + F) + \Gamma(\sqrt{V_0} + \sqrt{F})\sqrt{\mathcal{E}(t)} + \Gamma(\nu(t) + \nu^{2k+2}(t))\mathcal{E}(t), \quad \forall t \in [0,T_0).$$

(3.28)

It follows from (3.28) and (3.7) (with $\lambda$ sufficiently small) that

$$\mathcal{E}(t) \leq \overline{\Gamma}(V_0 + F) + \overline{\Gamma}(\nu(t) + \nu^{2k+2}(t))\mathcal{E}(t), \quad \forall t \in [0,T_0),$$

(3.29)

where $\overline{\Gamma}$ denotes a fixed positive constant which is independent of $v_0, f$ and $T_0$. We choose $\overline{\mathcal{E}}, \delta > 0$ such that

$$\overline{\Gamma}(\sqrt{2\overline{\mathcal{E}}} + (\sqrt{2\overline{\mathcal{E}}})^{k+2}) \leq \frac{1}{4}, \quad \overline{\delta} \leq \frac{1}{4}\overline{\mathcal{E}}.$$

(3.30)

Suppose that (1.16) holds for the above choice of $\delta$. By the Sobolev embedding theorem

$$\nu(t) \leq \sqrt{2\overline{\mathcal{E}}(t)}, \quad \forall t \in [0,T_0).$$

(3.31)

Hence (3.29) implies that for any $t \in [0,T_0)$ with $\mathcal{E}(t) \leq \overline{\mathcal{E}}$, we actually have $\mathcal{E}(t) \leq \frac{1}{2}\overline{\mathcal{E}}$; thus by continuity, if $\mathcal{E}(0) \leq \frac{1}{2}\overline{\mathcal{E}}$ then

$$\mathcal{E}(t) \leq \frac{1}{2}\overline{\mathcal{E}} \quad \forall t \in [0,T_0).$$

(3.32)
Clearly, one can choose a smaller $\delta > 0$ (if necessary) so that (1.16) yields $\mathcal{E}(0) \leq \frac{1}{2} \mathcal{E}$. Therefore, for $\delta > 0$ sufficiently small, (3.32) holds and $T_0 = \infty$. Furthermore, (1.17) follows directly from (3.32).

The proof of Theorem 1.1 is now complete.

The proof of Theorem 1.2 is almost identical. The only significant difference is that we do not obtain a bound for $v$ in $L^2([0, \infty); L^2(\mathbb{R}))$ because Poincaré's inequality fails on unbounded spatial intervals. Since we do obtain a bound for $v \in L^\infty([0, \infty); L^2(\mathbb{R}))$, the assumption $f \in L^1([0, \infty); L^2(\mathbb{R}))$ allows us to control the term $\int_0^t \int_{-\infty}^\infty f v$. The last term on the right-hand side of (3.9) also requires special attention. To treat this integral we integrate by parts with respect to $x$ and then with respect to $z$. 
Acknowledgements

The work of D. Brandon was initiated at the Institute for Mathematics and its Applications and completed with partial support from the Office of Naval Research under grant number N00014–88–K-0417 and from DARPA grant F4920–87–C–0116. W. J. Hrusa was supported by the Air Force Office of Scientific Research under grant number AFOSR 88–0265.
References


<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>580</td>
<td>Werner A. Stahel</td>
<td>Robust Statistics: From an Intellectual Game to a Consumer Product</td>
</tr>
<tr>
<td>581</td>
<td>Avner Friedman and Fernando Reitich</td>
<td>The Stefan Problem with Small Surface Tension</td>
</tr>
<tr>
<td>582</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Separation of Variables Methods for Systems of Differential Equations in Mathematical Physics</td>
</tr>
<tr>
<td>583</td>
<td>Mitchell Luskin and George R. Sell</td>
<td>The Construction of Inertial Manifolds for Reaction-Diffusion Equations by Elliptic Regularization</td>
</tr>
<tr>
<td>584</td>
<td>Konstantin Mischaikow</td>
<td>Dynamic Phase Transitions: A Connection Matrix Approach</td>
</tr>
<tr>
<td>585</td>
<td>Philippe Le Floch and Li Tatsien</td>
<td>A Global Asymptotic Expansion for the Solution to the Generalized Riemann Problem</td>
</tr>
<tr>
<td>586</td>
<td>Matthew Witten, Ph.D.</td>
<td>Computational Biology: An Overview</td>
</tr>
<tr>
<td>587</td>
<td>Matthew Witten, Ph.D.</td>
<td>Peering Inside Living Systems: Physiology in a Supercomputer</td>
</tr>
<tr>
<td>588</td>
<td>Michael Renardy</td>
<td>An existence theorem for model equations resulting from kinetic theories of polymer solutions</td>
</tr>
<tr>
<td>590</td>
<td>Luigi Preziosi</td>
<td>An Invariance Property for the Propagation of Heat and Shear Waves</td>
</tr>
<tr>
<td>591</td>
<td>Gregory M. Constantine and John Bryant</td>
<td>Sequencing of Experiments for Linear and Quadratic Time Effects</td>
</tr>
<tr>
<td>592</td>
<td>Prabir Daripa</td>
<td>On the Computation of the Beltrami Equation in the Complex Plane</td>
</tr>
<tr>
<td>593</td>
<td>Philippe Le Floch</td>
<td>Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form</td>
</tr>
<tr>
<td>595</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Numerical computation and continuation of invariant manifolds connecting fixed points</td>
</tr>
<tr>
<td>596</td>
<td>Scott J. Spector</td>
<td>Linear Deformations as Global Minimizers in Nonlinear Elasticity</td>
</tr>
<tr>
<td>597</td>
<td>Denis Serre</td>
<td>Richness and the classification of quasilinear hyperbolic systems</td>
</tr>
<tr>
<td>598</td>
<td>L. Preziosi and F. Rosso</td>
<td>On the stability of the shearing flow between pipes</td>
</tr>
<tr>
<td>599</td>
<td>Avner Friedman and Wenxiong Liu</td>
<td>A system of partial differential equations arising in electrophotography</td>
</tr>
<tr>
<td>600</td>
<td>Jonathan Bell, Avner Friedman, and Andrew A. Lacey</td>
<td>On solutions to a quasilinear diffusion problem from the study of soft tissue</td>
</tr>
<tr>
<td>601</td>
<td>David G. Schaeffer and Michael Shearer</td>
<td>Loss of hyperbolicity in yield vertex plasticity models under nonproportional loading</td>
</tr>
<tr>
<td>602</td>
<td>Herbert C. Kranzer and Barbara Lee Keyfitz</td>
<td>A strictly hyperbolic system of conservation laws admitting singular shocks</td>
</tr>
<tr>
<td>603</td>
<td>S. Laederich and M. Levi</td>
<td>Qualitative dynamics of planar chains</td>
</tr>
<tr>
<td>604</td>
<td>Milan Miklavčič</td>
<td>A sharp condition for existence of an inertial manifold</td>
</tr>
<tr>
<td>605</td>
<td>Charles Collins, David Kinderlehrer, and Mitchell Luskin</td>
<td>Numerical approximation of the solution of a variational problem with a double well potential</td>
</tr>
<tr>
<td>606</td>
<td>Todd Arbogast</td>
<td>Two-phase incompressible flow in a porous medium with various nonhomogeneous boundary conditions</td>
</tr>
<tr>
<td>607</td>
<td>Peter Poláčik</td>
<td>Complicated dynamics in scalar semilinear parabolic equations in higher space dimension</td>
</tr>
<tr>
<td>608</td>
<td>Bei Hu</td>
<td>Diffusion of penetrant in a polymer: a free boundary problem</td>
</tr>
<tr>
<td>609</td>
<td>Mohamed Sami ElBialy</td>
<td>On the smoothness of the linearization of vector fields near resonant hyperbolic rest points</td>
</tr>
<tr>
<td>610</td>
<td>Max Jodeit, Jr. and Peter J. Olver</td>
<td>On the equation $\nabla f = M \nabla g$</td>
</tr>
<tr>
<td>611</td>
<td>Shui-Nee Chow, Kening Lu, and Yun-Qiu Shen</td>
<td>Normal form and linearization for quasiperiodic systems</td>
</tr>
<tr>
<td>612</td>
<td>Prabir Daripa</td>
<td>Theory of one dimensional adaptive grid generation</td>
</tr>
<tr>
<td>613</td>
<td>Michael C. Mackey and John G. Milton</td>
<td>Feedback, delays and the origin of blood cell dynamics</td>
</tr>
<tr>
<td>614</td>
<td>D.G. Aronson and S. Kamin</td>
<td>Disappearance of phase in the Stefan problem: one space dimension</td>
</tr>
<tr>
<td>615</td>
<td>Martin Krupa</td>
<td>Bifurcations of relative equilibria</td>
</tr>
<tr>
<td>616</td>
<td>D.D. Joseph, P. Singh, and K. Chen</td>
<td>Couette flows, rollers, emulsions, tall Taylor cells, phase separation and inversion, and a chaotic bubble in Taylor-Couette flow of two immiscible liquids</td>
</tr>
<tr>
<td>617</td>
<td>Artemio González-López, Niky Kamran, and Peter J. Olver</td>
<td>Lie algebras of differential operators in two complex variables</td>
</tr>
<tr>
<td>618</td>
<td>L.E. Fraenkel</td>
<td>On a linear, partly hyperbolic model of viscoelastic flow past a plate</td>
</tr>
<tr>
<td>619</td>
<td>Stephen Schecter and Michael Shearer</td>
<td>Undercompressive shocks for nonstrictly hyperbolic</td>
</tr>
</tbody>
</table>
conservation laws

Xinfu Chen, Axially symmetric jets of compressible fluid

J. David Logan, Wave propagation in a qualitative model of combustion under equilibrium conditions

M.L. Zeeman, Hopf bifurcations in competitive three-dimensional Lotka-Volterra Systems

Allan P. Fordy, Isospectral flows: their Hamiltonian structures, Miura maps and master symmetries

Daniel D. Joseph, John Nelson, Michael Renardy, and Yuriko Renardy, Two-Dimensional cusped interfaces

Avner Friedman and Bei Hu, A free boundary problem arising in electrophotography

Hamid Bellout, Avner Friedman and Victor Isakov, Stability for an inverse problem in potential theory

Barbara Lee Keyfitz, Shocks near the sonic line: A comparison between steady and unsteady models for change of type

Barbara Lee Keyfitz and Gerald G. Warnecke, The existence of viscous profiles and admissibility for transonic shocks

P. Szmolyan, Transversal heteroclinic and homoclinic orbits in singular perturbation problems

Philip Boyland, Rotation sets and monotone periodic orbits for annulus homeomorphisms

Kenneth R. Meyer, Apollonius coordinates, the N-body problem and continuation of periodic solutions

Chjan C. Lim, On the Poincare-Whitney circuitspace and other properties of an incidence matrix for binary trees


Stanley Minkowitz and Matthew Witten, Periodicity in cell proliferation using an asynchronous cell population

M. Chipot and G. Dal Maso, Relaxed shape optimization: The case of nonnegative data for the Dirichlet problem

Jeffery M. Franke and Harlan W. Stech, Extensions of an algorithm for the analysis of nongeneric Hopf bifurcations, with applications to delay-difference equations

Xinfu Chen, Generation and propagation of the interface for reaction-diffusion equations

Philip Korman, Dynamics of the Lotka-Volterra systems with diffusion

Harlan W. Stech, Generic Hopf bifurcation in a class of integro-differential equations

Stephane Laederich, Periodic solutions of non linear differential difference equations

Peter J. Olver, Canonical Forms and Integrability of BiHamiltonian Systems

S.A. van Gils, M.P. Krupa and W.F. Langford, Hopf bifurcation with nonsemisimple 1:1 Resonance

R.D. James and D. Kinderlehrer, Frustration in ferromagnetic materials

Carlos Rocha, Properties of the attractor of a scalar parabolic P.D.E.

Debra Lewis, Lagrangian block diagonalization

Richard C. Churchill and David L. Rod, On the determination of Ziglin monodromy groups

Xinfu Chen and Avner Friedman, A nonlocal diffusion equation arising in terminally attached polymer chains

Peter Gritzmann and Victor Klee, Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces

P. Szmolyan, Analysis of a singularly perturbed traveling wave problem

Stanley Reiter and Carl P. Simon, Decentralized dynamic processes for finding equilibrium

Fernando Reitich, Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions

Russell A. Johnson, Cantor spectrum for the quasi-periodic Schrödinger equation

Wenxiong Liu, Singular solutions for a convection diffusion equation with absorption

Deborah Brandon and William J. Hrusa, Global existence of smooth shearing motions of a nonlinear viscoelastic fluid

James F. Reineck, The connection matrix in Morse-Smale flows II

Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay, Simple resonance regions of torus diffeomorphisms

Willard Miller, Jr., Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar

Calvin H. Wilcox, Lecture notes in radar/sonar: Sonar and Radar Echo Structure

Richard E. Blahut, Lecture notes in radar/sonar: Theory of remote surveillance algorithms