LOAD BALANCING IN INFINITE NETWORKS

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A set of locations and a set of consumers are given, and to each consumer there corresponds a subset of the locations. Each consumer has a demand, which is a load to be distributed among the locations corresponding to the consumer. The load at a location is the sum of the loads placed on the location by all consumers. Consumers assign load in such a way that no single consumer can reassign its load to make the total load more balanced. The model is motivated by the performance study of dynamic allocation algorithms in large systems.

The set of possible balanced load vectors is examined for infinite networks with deterministic or random demands. The balanced load vector is shown to be unique for rectangular lattice networks, and a method for computing the load distribution is explored for tree networks. An FKG type inequality is proved. The concept of load percolation is introduced and is shown to be associated with infinite sets of locations with identical load.

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1 PROBLEM FORMULATION

A consumer-demand network is a triple \((U, V, C)\) where \(U\) and \(V\) are finite or countably infinite sets, and \(C = (C_{u,v} : u \in U, v \in V)\) where \(0 \leq C_{u,v} \leq +\infty\) for all \(u, v\). Defining \(N(u)\) by \(N(u) = \{v \in V : C_{u,v} > 0\}\), we assume that \(N(u)\) is finite for each \(u \in U\), and similarly that \(\{u \in U : C_{u,v} > 0\}\) is finite for each \(v \in V\). Elements of \(U\) are called consumers, elements of \(V\) are called locations, and \(C_{u,v}\) is an upper bound on the load of consumer \(u\) that can be assigned to location \(v\). A demand vector \(m\) for \((U, V, C)\) has the form \(m = (m_u : u \in U)\) where \(0 \leq m_u \leq \sum_v C_{u,v}\) for \(u \in U\), and a baseload vector \(b\) for \((U, V, C)\) has the form \(b = (b_v : v \in V)\) where \(b_v\) is real valued for each \(v\).

An admissible assignment vector is a vector \(f, f = (f_{u,v} : u \in U, v \in V)\), such that \(0 \leq f_{u,v} \leq C_{u,v}\). The interpretation is that \(f_{u,v}\) denotes the load at location \(v\) due to consumer \(u\). An assignment vector \(f\) meets the demand if

\[
\sum_{v \in V} f_{u,v} = m_u \quad \text{for } u \in U, \tag{1.1}
\]

and the total load at location \(v\), \(x(v)\), is given by

\[
x(v) = b_v + \sum_{u \in U} f_{u,v} \quad \text{for } v \in V. \tag{1.2}
\]

A vector \(x = (x(v) : v \in V)\) so arising from an admissible assignment vector \(f\) meeting the demand is called a load vector. A load vector \(x\) is said to be balanced, if for some corresponding \(f\), the following conditions hold: For all \(u \in U\) and all \(v, v' \in V\), \(f_{u,v}(C_{u,v'} - f_{u,v'}) = 0\) whenever \(x(v) > x(v')\). If these conditions are satisfied we say \(x\) is a balanced load vector and that \(f\) balances. If \(C_{u,v} \geq m_u\) for all \(u, v\) with \(v \in N(u)\), then all that is relevant about \(C\) is summarized in the set of sets \(N = (N(u) : u \in U)\) and we call \((U, V, N)\) a consumer-demand network with no capacity constraints. The balancing condition simplifies to: For all \(u \in U\) and all \(v, v' \in N(u)\), \(f_{u,v} = 0\) whenever \(x(v) > x(v')\).

An example network with no capacity constraints is pictured in Figure 1. There are five consumers and four locations. The upper picture in the figure shows the admissible assignment \(f\) such that the demand of each consumer is split evenly among its two possible locations. Assignment \(f\)
does not balance because, for example, the first consumer assigns positive load to \( v_1 \), even though \( x(v_1) > x(v_2) \). The lower picture in the figure shows an assignment vector \( \hat{f} \) which balances the load. Note that there are other assignment vectors that balance. For example, there is an assignment that balances such that the first consumer places all of its load on the first location. Since each consumer in the example happens to be permitted two locations, the network and demand vector can be conveniently represented as the graph in Figure 2, where the demand vector is represented as edge weights. In this paper particular attention is given to consumer-demand networks which arise in the same way from infinite rectangular lattices and tree graphs.

Figure 1: Two assignments for a network with given demand.

The main questions addressed in this paper can be stated in broad terms as follows. How can the set of balanced load vectors be characterized? It is not difficult to show that balanced load vectors exist, but are they unique? What is the distribution of the load at a given location for a balanced load vector when the demand vector is random? Finally, the notion of balancing implies
that each consumer does its best to balance the load. If each consumer is permitted to place load on only a small set of locations, the effort to balance is thus local in nature. What “global” or long-range effects can be observed in balanced load vectors?

The model in this paper was studied for finite networks in [4]. A motivation of this work is to determine the effectiveness of local balancing mechanisms. The emergence of large, even infinite, sets of locations with identical load indicates that global balancing is induced by local adjustment. The present paper deals with a static problem. Related dynamic problems studied for finite networks are the problem of computing a balanced load vector [4], and the problem of dynamically allocating traffic in a loss network in balanced fashion [2].

This paper is organized as follows. The first half of the paper (Sections 2-5) concerns networks with deterministic demand vectors and the second half (Sections 6-8) concerns networks with random demand vectors. Section 2 is intended as an introduction to load balancing in infinite networks—tree networks and rectangular lattice networks with unit demands are examined. Finite networks are considered in Section 3. General considerations for infinite networks, including the introduction of boundary conditions and the definition of load percolation, are given in Section 4. The goal of Section 5 is to establish a necessary condition for a consumer-demand network to have multiple balanced load vectors for a single demand vector. General considerations for infinite networks with random demand are given in Section 6, and Sections 7 and 8 are devoted to tree networks and rectangular lattice networks, respectively, with a random demand vector. A summary of the highlights of this paper and open problems are given in Section 9.
2 TREE AND RECTANGULAR LATTICE NETWORKS WITH UNIT DEMANDS

Tree and rectangular lattice networks with unit demand are examined in this section in order to provide some insight into the behavior of infinite consumer-demand networks. Let \( d \geq 1 \). An infinite regular tree graph with degree \( d+1 \) is defined as follows. The set of nodes \( V \) is given by \( V = \bigcup_{k \geq 0} R_k \) where \( R_0 = \{ \Delta \} \), and \( R_k = \{(a_1, \ldots, a_k) : 1 \leq a_1 \leq d+1, 1 \leq a_i \leq d \text{ for } 2 \leq i \leq k \} \). The set of edges \( U \) is the set of unordered pairs of the form \([(a_1, \ldots, a_{k-1}), (a_1, \ldots, a_k)]\). A consumer-demand network with no capacity constraints \((U, V, N)\) is formed from the graph \((U, V)\) by identifying each edge \( u \) with \( N(u) \): that is if \( u = [v, v'] \) then \( N(u) = \{ v, v' \} \). Edges in the graph are consumers in the network and nodes in the graph are locations in the network. This section concerns unit demands, that is \( m_u = 1 \) for all \( u \). Tree networks with independent, identically distributed demands are considered in Section 7.

**Proposition 2.1** The set of possible values for each coordinate of a balanced load vector is the interval \([1, d]\). In particular, the balanced load vector is unique if \( d = 1 \) and is not unique if \( d \geq 2 \).

**Proof.** Let \( F \) be a finite, connected subset of nodes. Then \( \sum_{v \in F} x(v) \) is bounded above (below) by the number of edges with at least one (with both) endpoints in \( F \). This yields the following bounds on the average load carried by nodes in \( F \):

\[
\frac{k-1}{k} \leq \frac{\sum_{v \in F} x(v)}{k} \leq \frac{kd + 1}{k},
\]

where \( k \) is the cardinality of \( F \). As \( k \) tends to infinity, the lower and upper bounds in (2.1) tend to 1 and \( d \), respectively.

Next we argue that \( x(v) \in [1, d] \) for all \( v \) and any balanced load vector. For the sake of argument by contradiction, suppose that there exists \( \epsilon > 0 \) such that the set \( \{ v : x(v) \geq d + \epsilon \} \) is nonempty. By the observation in the previous paragraph, this set cannot have arbitrarily large connected subsets. Therefore, it must have a finite component (where a component is a maximal connected subset), which we denote by \( F \). The definition of balanced load implies that any edge that connects \( F \) and \( F^c \) assigns its entire unit load to the node in \( F^c \). Thus, the average load in \( F \) is \( \frac{k-1}{k} \), where
$k$ is the cardinality of $F$, which contradicts the assumption that $x(v) > d + \epsilon$ for $v \in F$. Since $\epsilon$ is arbitrary, $x(v) \leq d$ for all $v$. A similar argument shows that $x(v) \geq 1$ for all $v$, so that the coordinates of any balanced load vector lie within the interval $[1, d]$.

On the other hand, for any $\alpha \in [1, d]$ it is not difficult to show that $x(v) = \alpha$ for all $v$ is a balanced load vector. A corresponding assignment vector is found as follows. Seek an assignment such that $f_{u,v} = z_k$ whenever $u = [v, v']$ with $v \in R_k$ and $v' \in R_{k+1}$. Total load $\alpha$ is assigned to each node if $z_0 = \frac{\alpha}{4+1}$ and $1 - z_{k-1} + dz_k = \alpha$ for $k \geq 1$. By induction on $k$, $z_k \in [0, 1]$ for all $k$, so that a valid assignment vector $f$ is constructed. □

We remark that there is another assignment vector giving rise to $x(v) = 1$ for all $v$. Consider first setting $f_{u,v} = 0$ and $f_{u,v'} = 1$ whenever $u = [v, v']$ and $v'$ is further from $\Delta$ than $v$. Then $x(\Delta) = 0$ and $x(v) = 1$ for all $v \neq \Delta$. By swapping the load from one node to the other for each edge along an infinite path starting at $\Delta$, a new assignment vector is obtained with $x(v) = 1$ for all $v$. An assignment vector satisfying $x(v) = d$ for all $d$ can be constructed by a similar method.

A rectangular lattice network is defined as follows. Let $d \geq 1$. The $d$-dimensional rectangular lattice network with no capacity constraints is the network $(U, V, N)$ where $V = \mathbb{Z}^d$, $U = \{v, v' : |v - v'| = 1\}$, where $|v - v'|$ denotes the Euclidean distance between $v$ and $v'$, and $N(u) = \{v, v'\}$ for $u = [v, v']$. Suppose the baseload vector is zero and that the demand vector is given by $m_u = 1$ for all $u$. A balancing assignment vector is given by $f(u, v) = 1/2$ whenever $v \in N(u)$, and the corresponding balanced load vector is given by $x(v) = d$ for all $d$.

If $V_n$ is an $n$-cube (i.e. a subset of $n^d$ locations which is a translation of $\{1, \ldots, n\}^d$) then the average load per location in $V_n$ satisfies the bounds

$$d - \frac{d}{n} \leq \frac{\sum_{v \in V_n} x(v)}{|V_n|} \leq d + \frac{d}{n} \quad (2.2)$$

The lower (resp. upper) bound corresponds to setting $f_{u,v} = 0$ (resp. $f_{u,v} = 1$) whenever $u = [v, v'] \in U$, $v \in V_n$, and $v' \in V_n^c$. As $n$ tends to infinity, the average load in an $n$-cube tends to $d$. This suggests that the balanced load vector is unique. Proposition 5.1 implies that there is a unique balanced load vector for a rectangular lattice network and any demand vector. Roughly speaking, for rectangular lattice networks, the effect of “boundary conditions” asymptotically vanishes as
larger and larger sets of locations are considered. A notion of boundary condition is defined in Section 4.

On the other hand, given an $n$-cube $V_n$ and unit demand vector, it is not difficult to show there exists a load vector such that $x(v) = d - \frac{d}{n}$ for all $v \in V_n$, and another such that $x(v) = d + \frac{d}{n}$ for all $v \in V_n$. The fact these two load vectors are distinct for any finite $n$ indicates that the boundary condition has an effect as long as $n$ is finite. In the terminology introduced in Section 4, load percolation is said to occur for the rectangular lattice network with unit demand vector. As required by Corollary 4.1, the component of the set $\{v : x(v) = x(0)\}$ is infinite (in fact it is all of $V$). Rectangular lattice networks with random demand are discussed in Section 8.

3 FINITE NETWORKS

Suppose that $U$ and $V$ are finite, and that $C$, $m$ and $b$ are given. Let $\phi$ be a convex function on the real line, and define $J(f)$ for an assignment vector $f$ by

$$J(f) = \sum_{v \in V} \phi(x(v))$$

where $x$ is the load vector, given by Eq. (1.2). Define a convex optimization problem $P$ by

**Problem** $(P)$: $\min \{ J(f) : f$ is an admissible assignment meeting demand $m \}$

**Proposition 3.1** (Assume $U$ and $V$ are finite.) If $f$ is a solution to problem $P$ for a strictly convex function $\phi$ then $f$ balances. Conversely, if an assignment vector $f$ balances, it solves problem $P$. There exists a unique balanced load vector $x$ and any $f$ corresponding to $x$ balances. In the componentwise ordering of vectors, the balanced load vector $x$ is increasing in $m$ and $b$.

**Proof.** Suppose that $\phi$ is strictly convex and that $f$ is an admissible assignment vector meeting the demand $m$. If $f$ does not balance then there are $u \in U$ and $v, v' \in V$ such that $f_{u,v} > 0$, $C_{u,v'} > f_{u,v}$ and $x(v) > x(v')$. Hence decreasing $f_{u,v}$ by $\epsilon$ and increasing $f_{u,v'}$ by $\epsilon$ for a sufficiently small value of $\epsilon$ leads to an admissible assignment with strictly smaller cost, implying that $f$ does
not solve $P$. Thus, if $f$ does not balance, it does not solve problem $P$. Equivalently, if $f$ solves problem $P$, then $f$ balances.

Conversely, suppose that $f$ balances. It is shown next that $f$ solves $P$ if $\phi$ is (not necessarily strictly) convex. Since $\phi$ can be approximated arbitrarily closely uniformly on bounded sets by continuously differentiable convex functions and since the coordinates of $x$ are bounded as $f$ ranges over assignments meeting the demand $m$, it is without loss of generality that we assume $\phi$ to be continuously differentiable. For each $u \in U$ define $\hat{x}(u)$ and $\pi_u$ by $\hat{x}(u) = \max \{ x(v) : v \in V, f_{u,v} > 0 \}$ and $\pi_u = \phi'(\hat{x}(u))$. (The constants $(\pi_u : u \in U)$ are Lagrange multipliers.)

Let $f + h$ be any admissible assignment meeting the demand $m$. To prove that $J(f + h) \geq J(f)$, since $J(f + \epsilon h)$ is convex in $\epsilon$, it is enough to show that the limit of $(J(f + \epsilon h) - J(f))/\epsilon$ as $\epsilon$ decreases to zero is nonnegative. The limit is equal to $\sum_{u,v} h_{u,v} \phi'(x(v))$, which, since $\sum_v h_{u,v} = 0$ for $u \in U$, is equal to

$$\sum_{u,v} h_{u,v} \phi'(x(v)) - \pi_u.$$  \hspace{1cm} (3.1)

Consider first a pair $u,v$ such that $h_{u,v} > 0$. Then $f_{u,v} < C_{u,v}$ since $f + h$ is admissible, so that $\hat{x}(u) \leq x(v)$ since $x$ is balanced, which in turn implies that $\pi_u \leq \phi'(x(v)))$. On the other hand, if $h_{u,v} < 0$, then $f_{u,v} > 0$ since $f + h$ is admissible, so that $\hat{x}(u) \geq x(v)$ by the definition of $\hat{x}(u)$, which in turn implies that $\pi_u \geq \phi'(x(v))$. Therefore all terms of the sum in (3.1) are nonnegative. Thus, if $f$ balances it solves $P$.

The function $J$ in problem $P$ depends on $f$ only through the load vector $x$. In addition, the set of load vectors, obtained as the image of all admissible assignments $f$ meeting the demand, is a compact, convex set. Thus if $\phi$ is strictly convex and continuous, problem $P$ is equivalent to maximizing a strictly convex, continuous function over a compact, convex set. Thus, a balanced load vector exists and it is unique. Any assignment $f$ corresponding to the balanced load vector solves $P$, and hence balances.

It remains to prove the claimed monotonicity. Suppose that $x$ and $\hat{x}$ are the balanced load vectors for demand vectors and baseloads $(m, b)$ and $(\check{m}, \check{b})$, respectively, such that $b \leq \check{b}$ and $m \leq \check{m}$. Let $f$ and $\hat{f}$ denote corresponding balancing assignments.
It must be shown that \( x \leq \tilde{x} \). For the sake of argument by contradiction, suppose \( \epsilon = \max_v x(v) - \tilde{x}(v) > 0 \). Let \( F = \{ v : x(v) - \tilde{x}(v) = \epsilon \} \). Lemma 3.1 below implies that for \( u \in U \),

\[
\sum_{v \in F} f_{u,v} \leq \sum_{v \in \tilde{F}} \tilde{f}_{u,v}. \tag{3.2}
\]

Summing each side of (3.2) over \( u \in U \) and using the ordering of baseloads yields \( \sum_{v \in F} x(v) \leq \sum_{v \in \tilde{F}} \tilde{x}(v) \), which contradicts our assumption that \( \epsilon > 0 \).

\[
\square
\]

**Lemma 3.1** Fix \( u \in U \). Suppose \( (x, f) \) satisfies the following conditions: \( f \) is an admissible assignment for \((U, V, C)\) and \( f_{u,v}(C_{u,v} - f_{u,v}) = 0 \) whenever \( x(v) > x(v') \). (The vector \( x \) need not be the load vector for \( f \).) Suppose \( (\tilde{x}, \tilde{f}) \) satisfies the same conditions and that \( m_u = \sum_v f_{u,v} \leq \tilde{m}_u = \sum_v \tilde{f}_{u,v} \). Given \( \epsilon \geq 0 \), suppose \( F \) is a set of locations such that

\[
x(v) - \tilde{x}(v) \geq \epsilon \quad \text{if} \quad v \in F \cap N(u)
\]

\[
x(v) - \tilde{x}(v) < \epsilon \quad \text{if} \quad v \in F^c \cap N(u)
\]

Then \( \sum_{v \in F} f_{u,v} \leq \sum_{v \in \tilde{F}} \tilde{f}_{u,v} \).

**Proof.** Let \( r = \max\{x(v) : f_{u,v} > 0\} \). The definitions of \( F \) and \( r \) imply the following two relations:

\[
\{ v \in F^c : x(v) < r \} \cap N(u) \supset \{ v \in F^c : \tilde{x}(v) \leq r - \epsilon \} \cap N(u) \tag{3.3}
\]

\[
\{ v \in F : x(v) \leq r \} \cap N(u) \subset \{ v \in F : \tilde{x}(v) \leq r - \epsilon \} \cap N(u) \tag{3.4}
\]

To complete the proof, the following three inequalities are justified below.

\[
\sum_{v \in F} f_{u,v} \leq (m_u - \sum_{v \in F^c : x(v) < r} C_{u,v}) \land (\sum_{v \in F : x(v) \leq r} C_{u,v}) \tag{3.5}
\]

\[
\leq (\tilde{m}_u - \sum_{v \in F^c : \tilde{x}(v) \leq r - \epsilon} C_{u,v}) \land (\sum_{v \in F : \tilde{x}(v) \leq r - \epsilon} C_{u,v}) \tag{3.6}
\]

\[
\leq \sum_{v \in \tilde{F}} \tilde{f}_{u,v}. \tag{3.7}
\]

Inequality (3.5) follows from the definition of \( r \) and the balancing conditions for \((x, f)\). Inequality (3.6) follows from the assumption \( m_u \leq \tilde{m}_u \), (3.3) and (3.4). Finally, inequality (3.7) follows from the balancing condition for \((\tilde{x}, \tilde{f})\).

\[
\square
\]
Given a threshold \( L \) and a load vector \( x \), the load in excess of \( L \) is defined by \( \sum_v (x(v) - L)_+ \), the maximum load is \( \max_v x(v) \), and the minimum load is \( \min_v x(v) \).

**Corollary 3.1** The balanced load vector minimizes the load in excess of \( L \) for any \( L \). It also minimizes the maximum load and maximizes the minimum load.

**Proof.** The first assertion of the corollary follows from Proposition 3.1 with the choice \( \phi(z) = (z - L)_+ \). Since the resulting cost is zero if and only if the maximum load is less than or equal to \( L \), it follows that the balanced load vector minimizes the maximum load. Similarly, since the balanced load vector minimizes \( J \) for \( \phi(z) = (L - z)_+ \) and any choice of \( L \), it must also maximize the minimum load.  

\( \Box \)

4 BOUNDARY CONDITIONS, EXISTENCE OF BALANCED LOAD VECTORS, AND LOAD PERCOLATION

A simple method to produce balanced load vectors for consumer-demand networks with infinitely many locations is to construct balanced load vectors on finite subsets of locations and then take convergent subsequences. Different limits can sometimes result if different boundary conditions are placed on the finite subsets. Hence, a notion of boundary conditions is presented before the existence of balanced load vectors is considered. The section concludes with a notion of long-range influence, called load percolation, and a demonstration that load percolation can occur only if there are infinite sets of locations with equal load for some balanced load vector.

Let \((U, V, C)\) be a consumer-demand network with demand vector \( m \) and baseload vector \( b \). Let \( f \) be an admissible assignment vector satisfying the demand \( m \) and let \( x \) denote the corresponding load vector. Given a subset \( V_o \) of \( V \) we say that \( f \) balances in \( V_o \) (with unspecified boundary condition) if the following condition holds: for all \( u \in U \) and all \( v, v' \in V \), \( f_{u,v}(C_{u,v'} - f_{u,v'}) = 0 \) whenever \( x(v) > x(v') \) and \( v, v' \in V_o \). We say that \( f \) balances in \( V_o \) with \( \beta \)-boundary condition for some \( \beta \in [-\infty, +\infty] \), if: For all \( u \in U \) and all \( v, v' \in V \), \( f_{u,v}(C_{u,v'} - f_{u,v'}) = 0 \) whenever one of the following three conditions holds:
\[ x(v) > x(v') \text{ and } v, v' \in V_o \]
\[ x(v) > \beta, v \in V_o \text{ and } v' \in V_o^c \]
\[ \beta > x(v'), v \in V_o^c \text{ and } v' \in V_o. \]

If \( f \) balances the load in \( V_o \) with \( \beta \)-boundary condition, then \( x \) is said to be balanced in \( V_o \) with \( \beta \)-boundary condition.

**Proposition 4.1** Assume \( V_o \) is finite. There exists a load vector \( x^\beta \) that balances in \( V_o \) with \( \beta \) – boundary condition. The restriction of \( x^\beta \) to \( V_o \), \( (x^\beta(v) : v \in V_o) \), is unique (for given \( \beta \)) and is nondecreasing in \( m, b \) and \( \beta \). If \( x \) is balanced in \( V_o \) with unspecified boundary condition then \( x^{-\infty}(v) \leq x(v) \leq x^{+\infty}(v) \) and all \( v \in V_o \). For all \( v \in V_o \), and \( \beta \in [-\infty, +\infty] \)

\[
x^\beta(v) = \begin{cases} 
  x^{+\infty}(v) & \text{if } x^{+\infty}(v) \leq \beta \\
  \beta & \text{if } x^{-\infty}(v) \leq \beta \leq x^{+\infty}(v) \\
  x^{-\infty}(v) & \text{if } x^{-\infty}(v) \geq \beta 
\end{cases}
\] (4.1)

(In other words, for each \( v \in V_o \), \( x^\beta(v) \) is the point in the interval \([x^{-\infty}(v), x^{+\infty}(v)]\) nearest to \( \beta \).)

**Proof.** Define the cost function \( J_{\beta,V_o} \) by \( J_{\beta,V_o}(f) = \sum_{v \in V_o} (x(v) - \beta)^2 \). This function is a strictly convex and continuous function of \((x(v) : v \in V_o)\), and the set of possible values of \((x(v) : v \in V_o)\) as \( f \) varies is a compact, convex set. There thus exists a minimum, and the resulting load vector restricted to \( V_o \) is unique. Reasoning as in the proof of Proposition 3.1, it can be shown that \( f \) balances the load in \( V_o \) with \( \beta \)-boundary condition if and only if \( f \) minimizes \( J_{\beta,V_o} \) over the set of admissible assignments meeting demand \( m_u \). The existence of \( x^\beta \) and the uniqueness of \((x^\beta(v) : v \in V_o)\) are established.

Let \( x \) be an arbitrary load vector that balances in \( V_o \). To establish that \( x^{-\infty}(v) \leq x(v) \) for \( v \in V_o \), we view the restrictions of \( x \) and \( x^{-\infty} \) to \( V_o \) as balanced load vectors for a new network with set of locations \( V_o \). The demand vector \( \tilde{m} \) for \( x \) restricted to \( V_o \) is given by \( \tilde{m}_u = m_u - \sum_{v \in V_o^c} f_{u,v} \), and the demand vector \( \tilde{m}^{-\infty} \) for \( x^{-\infty} \) restricted to \( V_o \) is defined similarly. The baseloads for the two load vectors are the same: the restriction of \( b \) to \( V_o \). The \( \beta \)-boundary conditions with \( \beta = -\infty \)
imply that \( \tilde{m}_u \geq \tilde{m}_u^{-\infty} \) for all \( u \), so by Proposition 3.1 it follows that \( x^{-\infty}(v) \leq x(v) \) for all \( v \in V_0 \). The same argument shows that \( x(v) \leq x^{+\infty}(v) \) for all \( v \in V_0 \).

A proof that \( x^\beta \) is nondecreasing in \( m \) and \( b \) follows the proof of monotonicity given for Proposition 3.1. Specifically, suppose that \( x \) and \( \tilde{x} \) are balanced in \( V_0 \) with \( \beta \) boundary conditions for \( (m, b) \) and \( (\tilde{m}, \tilde{b}) \) respectively, where \( b \leq \tilde{b} \) and \( m \leq \tilde{m} \). Modify \( x \) and \( \tilde{x} \) by setting \( x(v) = \tilde{x}(v) = \beta \) for all \( v \in V_0^c \). (Of course the modified vectors may no longer satisfy (1.2) unless \( v \in V_0 \), but that isn’t important.) It must be shown that \( x \leq \tilde{x} \). The proof of this fact is word for word identical to the last two paragraphs of the proof of Proposition 3.1. The proof that \( x^\beta \) is nondecreasing in \( \beta \) follows from (4.1), which is established below.

It remains to establish (4.1). Fix a finite value of \( \beta \). Let \( V_1 = \{ v \in V_0 : x^\beta(v) > \beta \} \). The balancing conditions satisfied by \( x^\beta \) imply that \( x^\beta \) balances in \( V_1 \) with \(-\infty\)-boundary condition. By the part of the Proposition already proved, it follows that \( x^\beta(v) \leq x(v) \) for all \( v \in V_1 \) and any \( x \) that balances in \( V_1 \) with unspecified boundary conditions. Taking \( x = x^{-\infty} \) thus establishes that \( x^\beta(v) \leq x^{-\infty}(v) \) for all \( v \in V_1 \). On the other hand, \( x^\beta(v) \geq x^{-\infty}(v) \) for all \( v \in V_1 \) (in fact for all \( v \in V_0 \)) so that \( x^\beta(v) = x^{-\infty}(v) \) for all \( v \in V_1 \). The same argument shows that \( x^\beta(v) = x^{+\infty}(v) \) for all \( v \in V_0 \) such that \( x^\beta(v) < \beta \). Thus, (4.1) is proven for all \( v \in V_0 \) such that \( x^\beta(v) \neq \beta \). The relation (4.1) is also true for \( v \in V_0 \) such that \( x^\beta(v) = \beta \), by the extremality of \( x^{-\infty} \) and \( x^{+\infty} \) already proven. Thus, (4.1) is established for all \( v \in V_0 \).

\[ \square \]

**Proposition 4.2** Given a consumer-demand network \((U, V, C)\) with demand vector \( m \) and baseload vector \( b \), there exist balanced load vectors \( x^{-\infty} \) and \( x^{+\infty} \) such that for any balanced load vector \( x \), \( x^{-\infty} \leq x \leq x^{+\infty} \) coordinatewise. For any finite \( \beta \) the relation (4.1) defines a balanced load vector \( x^\beta \). The load vector \( x^\beta \) is nondecreasing in \( m \), \( b \) and \( \beta \).

**Proof.** Take a sequence of finite subsets \((V_n : n \geq 0)\) such that \( V_n \nneq V \), which by definition means that \( V_n \subset V_{n+1} \) for each \( n \) and \( \cup_{n\geq1} V_n = V \). By Proposition 4.1, for each \( n \) there exist load vectors \( x^{-\infty}_n \), \( x^\beta_n \), and \( x^{+\infty}_n \) which are balanced in \( V_n \) with \(-\infty\)-boundary condition, \( \beta \)-boundary condition and \(+\infty\)-boundary condition, respectively. Since \( 0 \leq f_{u,v} \leq m_{u,v} \) for each \( u, v \) and any admissible assignment vector \( f \) meeting demand \( m \), there is a subsequence of \( n \to \infty \) such that, along the
subsequence, each coordinate of the assignment vectors $f_n^{-\infty}, f_n^\beta,$ and $f_n^{+\infty}$ converges. Let $f^{-\infty}, \ f^\beta,$ and $f^{+\infty}$ denote the respective limits, and let $x^{-\infty}, \ x^\beta,$ and $x^{+\infty}$ denote the corresponding load vectors. The balancing conditions satisfied by $f_n^{-\infty}, f_n^\beta,$ and $f_n^{+\infty}$ for each $n$ imply that the limits $x^{-\infty}, \ x^\beta,$ and $x^{+\infty}$ are balanced load vectors with corresponding assignment vectors $f^{-\infty}, \ f^\beta,$ and $f^{+\infty}.$ The load vectors obtained for $V_n$ satisfy the relation (4.1) for $v \in V_n,$ so that (4.1) is satisfied by the limits $x^{-\infty}, \ x^\beta,$ and $x^{+\infty}$ for all $v \in V.$ Similarly, the load vectors obtained for $V_n,$ when restricted to $V_n,$ are nondecreasing in $m$ and $b,$ so that the limits $x^{-\infty}, \ x^\beta,$ and $x^{+\infty}$ are also monotone in $m$ and $b.$ Equation (4.1) implies that $x^\beta$ is also nondecreasing as a function of $\beta.$

Finally, let $x$ be an arbitrary balanced load vector. The extremality properties of $x_n^{-\infty}$ and $x_n^{+\infty}$ imply that $x_n^{-\infty} \leq x \leq x_n^{+\infty}$ on $V_n$ for each $n.$ In the limit $n \to \infty$ this yields $x^{-\infty} \leq x \leq x^{+\infty}$ as desired.

The extremality properties of $x_n^{-\infty}$ and $x_n^{+\infty}$ imply that $x_{n+1}^{-\infty} \geq x_n^{-\infty}$ on $V_n$ and $x_{n+1}^{+\infty} \leq x_n^{+\infty}$ on $V_n.$ Thus, for $v \in V$ and for $n$ large enough that $v \in V_n,$ the sequence $x_n^{-\infty}(v)$ is nondecreasing in $n$ and the sequence $x_n^{+\infty}(v)$ is nonincreasing in $n.$ These facts, together with (4.1) for set $V_n,$ imply that for any $\beta \in [-\infty, +\infty]$ and $v \in V,$ $\lim_{n \to \infty} x_n^\beta(v) = x^\beta(v),$ and the convergence is monotone for $n$ large enough that $v \in V_n.$ Of course the corresponding assignment vectors need not converge without passing to a subsequence.

The concept of load percolation for the given network $(U, V, C),$ demand vector $m$ and baseload vector $b$ is defined as follows. Load percolation is said to occur at a fixed location $v_o$ if $x_n^{-\infty}(v_o) < x_n^{+\infty}(v_o)$ for all sufficiently large $n$ (equivalently, for all $n$ such that $v_o \in V_n$). A sufficient but not necessary condition for load percolation at $v_o$ is $x^{-\infty}(v_o) < x^{+\infty}(v_o).$ The property of load percolation does not depend on which sequence of finite subsets $(V_n : n \geq 0)$ is used to define the sequences $(x_n^{-\infty})$ and $(x_n^{+\infty}).$ In fact, a condition equivalent to load percolation at $v_o$ is that $x^{-\infty}_A(v_o) < x^{+\infty}_A(v_o)$ whenever $A$ is a finite set of locations containing $v_o$ and $x^{-\infty}_A$ and $x^{+\infty}_A$ are load vectors that balance in $A$ with $-\infty$ and $+\infty$ boundary conditions respectively.

Some additional notation is introduced in order to describe an implication of load percolation. Locations $v$ and $v'$ are called neighbors if $v, v' \in N(u)$ for some consumer $u.$ A path from $v$ to $v'$ is a sequence $s$ of locations $s = (v_1, \ldots, v_{|s|})$ so that $v = v_1, v' = v_{|s|},$ and $v_i$ and $v_{i+1}$ are neighbors for
$1 \leq i \leq |s| - 1$, and $|s|$ denotes the length of the path. The distance between two locations $v$ and $v'$ is the minimum length of a path from $v$ to $v'$, unless no path exists in which case the distance is infinite. The boundary of a set $V_o$ of locations, written $\partial V_o$, is defined by

$$\partial V_o = \{ v' \in V - V_o : v' \text{ is a neighbor of } v \text{ for some } v \in V_o \}$$

A set $V_o$ is connected if for any $v, v' \in V_o$, there is a path from $v$ to $v'$. Finally, a component of a set $V_o$ is a maximal connected subset of $V_o$.

**Proposition 4.3** Suppose load percolation occurs at location $v_o$. Let $\beta \in [x^{-\infty}(v_o), x^{+\infty}(v_o)]$. Then the component of $\{ v : x^\beta(v) = \beta \}$ containing $v_o$ is infinite.

**Proof.** We prove the contrapositive. Let $V^*$ denote the component in question, and suppose that $V^*$ is finite. The goal is to establish that load percolation does not occur at $v_o$. Consider the partition of $\partial V^*$ into two sets $\partial_+ V^*$ and $\partial_- V^*$ given by

$$\partial_+ V^* = \{ v \in \partial V^* : x^\beta(v) > \beta \}$$

(4.2)

$$\partial_- V^* = \{ v \in \partial V^* : x^\beta(v) < \beta \}.$$  

(4.3)

Note that by the relation (4.1), $x^{+\infty}(v) = x^\beta(v)$ for $v \in \partial_- V^*$ and $x^{-\infty}(v) = x^\beta(v)$ for $v \in \partial_+ V^*$. Since $\cup_{n \geq 1} V_n = V$ and $x_n^{+\infty}(v) \rightarrow x^{+\infty}(v)$ as $n \rightarrow \infty$ for all $v \in V$, if $n$ is sufficiently large then $V^* \cup \partial V^* \subset V_n$ and $x_n^{+\infty}(v) < \beta$ for $v \in \partial_- V^*$.

We argue next that for such $n$, $x_n^{+\infty}(v) = x^\beta(v)$ for $v \in V^*$. On the one hand, $x_n^{+\infty}(v) \geq x^\beta(v)$ for $v \in V^*$, in fact for $v \in V_n$, since $x_n^{+\infty}$ dominates on $V_n$ any load vector that balances on $V_n$. On the other hand, it will be shown at the end of the proof that for any $u$,

$$\sum_{v \in V^*} f_{u,v} \leq \sum_{v \in V^*} f^\beta_{u,v},$$

(4.4)

where $f$ and $f^\beta$ denote assignment vectors corresponding to $x_n^{+\infty}$ and $x^\beta$, respectively. Summing each side of (4.4) over $u$ shows that $\sum_{v \in V^*} x_n^{+\infty}(v) \leq \sum_{v \in V^*} x^\beta(v)$ for $v \in V^*$. This proves that $x_n^{+\infty}(v) = x^\beta(v)$ for $v \in V^*$ and sufficiently large $n$.

Similarly, $x_n^{-\infty}(v) = x^\beta(v)$ for $v \in V^*$ and sufficiently large $n$. Therefore, $x_n^{+\infty}(v) = x_n^{-\infty}(v)$ for sufficiently large $n$ and all $v \in V^*$, in particular for $v = v_o$. Thus, if $V^*$ is finite, load percolation does not occur at $v_o$. 

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It remains to establish (4.4) for given \( u \). Without loss of generality, suppose \( N(u) \cap V^* \neq \emptyset \). Since \( f \) and \( f^\beta \) each satisfy the same demand,

\[
\sum_{v \in V^*} (f_{u,v}^\beta - f_{u,v}) = T_+ + T_- - T_+^\beta - T_-^\beta
\]

(4.5)

where

\[
T_+ = \sum_{v \in \partial_+ V^*} f_{u,v} \quad T_- = \sum_{v \in \partial_- V^*} f_{u,v} \\
T_+^\beta = \sum_{v \in \partial_+ V^*} f_{u,v}^\beta \quad T_-^\beta = \sum_{v \in \partial_- V^*} f_{u,v}^\beta
\]

Note that if \( v \in V^* \cup \partial_+ V^* \) and \( v' \in \partial_- V^* \), then \( x_n^+ (v) \geq x^\beta (v) \geq \beta > x_n^+ (v') \) and \( x^\beta (v) > x^\beta (v') \). Therefore, \( T_- = T_-^\beta = m_u \land \sum_{v \in \partial_- V^*} C_{u,v} \). Note also that \( x^\beta (v) > x^\beta (v') \) if \( v \in \partial_+ V^* \) and \( v' \in V^* \cup \partial_- V^* \). Therefore

\[
T_+ \geq T_+^\beta = (m_u - \sum_{v \in V^* \cup \partial_- V^*} C_{u,v})_+
\]

In summary, \( T_- = T_-^\beta \) and \( T_+ \geq T_+^\beta \) which in view of (4.5) implies (4.4). \( \square \)

The following corollary is immediate from Proposition 4.3 and the fact that \( x^\beta \equiv x \) for all \( \beta \) if there is a unique balanced load vector \( x \).

**Corollary 4.1** If there is a unique balanced load vector \( x \) and if load percolation occurs at a location \( v_o \), then the component of \( \{v : x(v) = x(v_o)\} \) containing \( v_o \) is infinite.

5 A NECESSARY CONDITION FOR NONUNIQUENESS

Let \( (U, V, C) \) be a consumer-demand network such that for some constants \( A \) and \( B \), \( |N(u)| \leq A \) for \( u \in U \) and \( |\{u \in U : C_{u,v} > 0\}| \leq B \) for \( v \in V \). Suppose a demand vector \( m \) is given with \( m_u \geq 0 \) for all \( u \) as usual. Throughout this section the baseload is taken to be zero. Fix a location \( v_o \) and let \( S_n \) denote the set of locations at distance less than or equal to \( n \) from \( v_o \). Let \( x \) and \( \tilde{x} \) denote balanced load vectors for \( (U, V, C) \) and \( m \).

**Proposition 5.1** If \( x(v_o) > \tilde{x}(v_o) \), then there is a constant \( a > 1 \) such that \( |S_n| \geq a^n \) for all \( n \).
Lemma 5.1 Proposition 5.1 is true under the additional assumption $m_u \leq L$ for all $u \in U$ for some $L < \infty$.

Proof. Let $\epsilon = x(v) - \bar{x}(v)$, which by assumption is positive, and let $G$ denote the component of the set $\{v : x(v) - \bar{x}(v) \geq \epsilon\}$ containing $v$. For $n \geq 1$ set

$$s_n = \sum_{v \in S_n \cap G} x(v) - \bar{x}(v).$$

By the relation between assignment vectors and load vectors,

$$s_n = \sum_u \left( \sum_{v \in S_n \cap G} f_{u,v} - \hat{f}_{u,v} \right) \quad (5.1)$$

Consider the sum in parentheses on the righthand side of (5.1). If $N(u) \cap S_n \cap G = \emptyset$, the sum is zero, so consider $u$ such that $N(u) \cap S_n \cap G \neq \emptyset$. If $N(u) \cap \partial S_n \cap G = \emptyset$ then $N(u) \subset (S_n \cap G) \cup G^c$, so the sum is less than or equal to zero by Lemma 3.1 with $F = S_n \cap G$. For any $u$ the sum is less than or equal to $m_u$, which is at most $L$. Thus,

$$s_n \leq L|\{u : N(u) \cap \partial S_n \cap G \neq \emptyset\}| \leq BL|\partial S_n \cap G|$$

On the other hand, $s_n \geq \epsilon|S_n \cap G|$. Thus, $|\partial S_n \cap G| \geq \frac{\epsilon}{BL}|S_n \cap G|$ for all $n \geq 1$. Since $S_{n+1}$ is the union of the disjoint sets $S_n$ and $\partial S_n$, $|S_{n+1} \cap G| \geq (1 + \frac{\epsilon}{BL})|S_n \cap G|$. By induction on $n$, $|S_n| \geq |S_n \cap G| \geq (1 + \frac{\epsilon}{BL})^n$ for all $n$. \qed

Given $D > 0$, define $\overline{m}_u = m_u \land (AD)$ for $u \in U$.

Lemma 5.2 There exists a balanced load vector $\overline{x}$ for demand $\overline{m}$ such that $\overline{x}(v) \land D = x(v) \land D$ for all $v \in V$.

Proof. Assume without loss of generality that $V$ is connected so that $S_n \not\supset V$. First a (not necessarily balancing) assignment vector $\hat{f}$ meeting demand $\overline{m}$ is specified. Consider $u \in U$. If $m_u \leq AD$ let $\hat{f}_{u,v} = f_{u,v}$ for $v \in N(u)$. Otherwise, let $\hat{f}_{u,v} = f_{u,v} \land \gamma_u$ for $v \in N(u)$, where $\gamma_u$ is the unique number such that $\sum_{v \in N(u)} \hat{f}_{u,v} = AD$. Observe that $\gamma_u \geq D$. The vector $\hat{f}$ is an
admissible assignment meeting demand \( \overline{m} \). Let \( \hat{x} \) denote the load vector for \( \hat{f} \). Although \( \hat{x} \) is not necessarily balanced, it satisfies \( \hat{x}_v \wedge D = x_v \wedge D \) for all \( v \in V \). Let \( F = \{ v : x(v) \geq D \} \). Note that if \( v, v' \in N(u) \) for some \( u \) and if \( v \in F \) and \( v' \in F^c \), then \( f_{u,v}(C_{u,v} - f_{u,v}) = \hat{f}_{u,v}(C_{u,v} - \hat{f}_{u,v}) = 0 \) since \( x \) is balanced, \( f_{u,v} \geq \hat{f}_{u,v} \) and \( f_{u,v'} = \hat{f}_{u,v'} \).

For \( n \geq 1 \) consider the finite network with set of locations \( S_n \cap F \), set of consumers \( U \), capacity vector equal to the restriction of \( C \) to \( U \times (S_n \cap F) \) and demand \( m_u = \sum_{v \in (S_n \cap F)} \hat{f}(u,v) \) for \( u \in U \). Note that \( \hat{x} \) restricted to \( S_n \cap F \) is a load vector for the finite network. Let \( x^{(n)} \) be a balanced load vector for the finite network. Since \( \hat{x} \) satisfies \( \hat{x}(v) \geq D \) for all \( v \in S_n \cap F \), the same is true for \( x^{(n)} \) by the minimum maximizing property (cf Corollary 3.1) of balanced load vectors for finite networks.

Let \( f^{(n)} \) denote the assignment vector for \( x^{(n)} \). Since each coordinate of \( f^{(n)} \) is bounded uniformly in \( n \), there is a subsequence of \( n \to \infty \) such that along the subsequence the vectors \( f^{(n)} \), and hence also the vectors \( x^{(n)} \), converge coordinatewise. Denote the limits by \( \overline{f} \) and \( \overline{x} \). Note that \( \overline{x}(v) \geq D \) for \( v \in F \) since \( x^{(n)}(v) \geq D \) for all \( v \in F \cap S_n \). Extend \( \overline{f} \) to an assignment vector for \( (U,V,C) \) by setting \( \overline{f}_{u,v} = \hat{f}_{u,v}(= f_{u,v}) \) if \( v \in F^c \), and extend \( \overline{x} \) to be the load vector for \( \overline{f} \). Then \( \overline{x} \) is a balanced load vector and \( \overline{x}_u \wedge D = x_u \wedge D \) for all \( u \in U \). The lemma is proved.

**Proof of Proposition 5.1.** Choose \( D = x(v_o) \). Then \( x(v_o), \hat{x}(v_o) \leq D \), so by Lemma 5.2, applied to both \( x \) and \( \hat{x} \), there exist two balanced load vectors for \( (U,V,C) \) with demand \( \overline{m} \), one with load at \( v_o \) equal to \( x(v_o) \) and one with load at \( v_o \) equal to \( \hat{x}(v_o) \). Hence by Lemma 5.1 with \( L = AD \), there exists \( a > 1 \) such that \( |S_n| \geq a^n \) for all \( n \).

### 6 RANDOMLY LOADED NETWORKS

Let \( (U,V,C) \) denote a consumer-demand network. Suppose that \( b = (b_v : v \in V) \) is a random baseload vector and that \( m = (m_u : u \in U) \) is a random demand vector such that \( 0 \leq m_u \leq \sum_{v \in V} C_{u,v} \) for all \( u \) with probability one. Then for each \( \beta \in [-\infty, +\infty] \) there is a balanced load vector \( x^\beta \), which is random. Fix a location \( v_o \), and let \( F^\beta \) denote the probability distribution
function of \(x^\beta(v_0)\). Write \(F \prec G\) for two distribution functions \(F\) and \(G\) if \(F(c) \geq G(c)\) for all real \(c\).

**Proposition 6.1** A distribution function \(F\) is the distribution of load at \(v_0\) for some balanced load vector if and only if \(F^{-\infty} \prec F \prec F^{+\infty}\) (assuming the probability space is large enough to support a uniform random variable independent of \((m, b)\) for the “if” statement). In particular, \(F^\beta(c) = F^{+\infty}(c)I_{\{c < \beta\}} + F^{-\infty}(c)I_{\{c \geq \beta\}}\).

**Proof.** The fact that \(F\) is the distribution of \(x(v_0)\) for some balanced load vector only if \(F^{-\infty} \prec F \prec F^{+\infty}\) follows from the minimality of \(x^{-\infty}\) and the maximality of \(x^{+\infty}\). Conversely, let \(F\) be any distribution function satisfying \(F^{-\infty} \prec F \prec F^{+\infty}\). Then by the well-known connection between stochastic ordering and stochastic coupling, if the underlying probability space supports a uniform random variable independent of \((m, b)\), there exists a random variable \(B\) such that \(B \in [x^{-\infty}(v_0), x^{+\infty}(v_0)]\) with probability one and \(B\) has distribution function \(F\). Letting \(x^B\) denote \(x^\beta\) evaluated at \(B = \beta\), we have that \(x^B\) is a balanced load vector and \(x^B(v_0) = B\) with probability one so that \(x^B(v_0)\) has distribution function \(F\). The final statement in the proposition follows from the relations (4.1). \(\square\)

The monotonicity property stated in Proposition 4.2 immediately implies an FKG type inequality for networks with independent demand and baseload variables.

**Proposition 6.2** Suppose \(b_v, v \in V; m_u, u \in U\) are mutually independent random variables. Fix \(\beta \in [-\infty, +\infty]\) (or let \(\beta\) be random and independent of \(m\) and \(b\)). Let \(g\) and \(h\) be nondecreasing, Borel measurable functions mapping \(R^V\) to \(R\) such that \(E[g(x^\beta)^2]\) and \(E[h(x^\beta)^2]\) are finite. Then \(E[g(x^\beta)h(x^\beta)] \geq E[g(x^\beta)]E[h(x^\beta)]\).

Taking \(g(x) = x(v)\) and \(h(x) = x(v')\) for two locations \(v\) and \(v'\), Proposition 6.2 yields that \(\text{Cov}(x^\beta(v), x^\beta(v')) \geq 0\). Of course if there is a unique balanced load vector \(x\) with probability one, then the FKG inequality holds for \(x^\beta\) replaced by \(x\).
Proof. Since by Proposition 4.2 the balanced load vector is a nondecreasing function of $b, m$ and $\beta$, the random variables $g(x^\beta)$ and $h(x^\beta)$ are also nondecreasing functions of $b, m$ and $\beta$. The proposition hence follows from the FKG inequality for nondecreasing functions of independent random variables (see [1], [3, Theorem 2.4]).

\[\square\]

7 TREE NETWORKS WITH RANDOM DEMAND

Tree networks as defined in Section 2 are considered in this section with independent random demands on the edges. First it is shown that the $\tau$-surplus of nodes for finite subsets with boundary conditions can be computed recursively, yielding a method to numerically compute the distribution of load at a given location for finite or infinite tree networks. The basic technique was introduced in [4] for analysis of a random tree network with Poisson degree and unit demands. Two examples are investigated more closely—Bernoulli loads and exponentially distributed loads.

The $\tau$-surplus of a Resource Location. Let $(U, V, C)$ be a finite consumer-demand network with demand vector $m$ and baseload $b$. The $\tau$-surplus of a location is defined as follows. Given a real number $\tau$ and a location $v_o \in V$, define the $\tau$-surplus of $v_o$ to be the unique value $y$ so that if the baseload $b_{v_o}$ is changed to $b_{v_o} - y$ then the load at $v_o$ for a balanced assignment is $\tau$. The load at $v_0$ under a balanced assignment for the original baseload vector $b$ is greater than (resp. less than, equal to) $\tau$ if and only if the $\tau$-surplus of $v_o$ is greater than (resp. less than, equal to) zero. Hence, the $\tau$-surplus of $v_o$ for all $\tau$ determines the load at $v_o$ for a balanced assignment.

Next suppose $V$ is infinite. Fix a node $v_o$ and take a sequence of finite subsets $(V_n : n \geq 0)$ with $V_n \not\subseteq V$. Consider the $\tau$-surplus at $v_0$ for the balancing problem in $V_n$ as $n$ tends to infinity. Specifically, let $Y_{\tau,n}^+ (\text{resp. } Y_{\tau,n}^-)$ denote the $\tau$-surplus of node $v_0$ for the balancing problem in $V_n$ with $+\infty$-boundary condition ($-\infty$-boundary condition), respectively. The variables $Y_{\tau,n}^+$ (resp. $Y_{\tau,n}^-$) are nonincreasing (resp. nondecreasing) in $n$ and are bounded. Let $Y_{\tau}^+$ (resp. $Y_{\tau}^-$) denote the corresponding limits. We thus can write $Y_{\tau,n}^+ \downarrow Y_{\tau}^+$ and $Y_{\tau,n}^- \not\nearrow Y_{\tau}^-$, where the notation $a_n \downarrow a$ (resp. $a_n \not\nearrow a$) denotes that the sequence $(a_n)$ is nonincreasing (resp. nondecreasing) with limit $a$. Observe that $Y_{\tau,n}^+ \geq Y_{\tau,n}^-$ for all $n$, so that $Y_{\tau}^+ \geq Y_{\tau}^-$.
Proposition 7.1 For any finite \( \tau \),

\[
P[x^{+\infty}(v_0) < \tau] = P[Y^{+\infty}_\tau < 0] = \lim_{n \to -\infty} P[Y^{+\infty}_{\tau,n} < 0] \tag{7.1}
\]

\[
P[x^{-\infty}(v_0) \leq \tau] = P[Y^{-\infty}_\tau \leq 0] = \lim_{n \to -\infty} P[Y^{-\infty}_{\tau,n} \leq 0] \tag{7.2}
\]

Proof. As in the proof of Proposition 4.2, let \( x^{+\infty}_n \) denote a load vector that is balanced in \( V_n \) with \(+\infty\)-boundary condition. As discussed after the proof of Proposition 4.2, \( x^{+\infty}_n(v_0) \setminus x^{+\infty}(v_0) \). We thus have

\[
\{x^{+\infty}_n(v_0) < \tau\} \not\supset \{x^{+\infty}(v_0) < \tau\} \text{ and } \{x^{+\infty}_n(v_0) < \tau\} = \{Y^{+\infty}_{\tau,n} < 0\} \not\supset \{Y^{+\infty}_\tau < 0\}.
\]

Consequently, \( x^{+\infty}(v_0) < \tau \) if and only if \( Y^{+\infty}_\tau < 0 \), and \( \{Y^{+\infty}_{\tau,n} < 0\} \not\supset \{Y^{+\infty}_\tau < 0\} \), which implies (7.1), and (7.2) is established similarly. \( \square \)

Corollary 7.1 The distribution of \( x(v_0) \) is not unique (i.e. not the same for all equilibrium load vectors \( x \)) if and only if there exists an interval of positive length such that \( P[Y^{+\infty}_\tau < 0] < P[Y^{-\infty}_\tau \leq 0] \) for all \( \tau \) in the interval. In particular, the distribution of \( x(v_0) \) is not unique if there exists an interval of positive length such that \( P[Y^{+\infty}_\tau < 0] < P[Y^{-\infty}_\tau < 0] \) for all \( \tau \) in the interval.

Proof. The distribution of \( x(v_0) \) is unique if and only if \( x^{+\infty}(v_0) \) and \( x^{-\infty}(v_0) \) have the same distribution function. If they do have the same distribution function then whenever \( \tau \) is not a point of discontinuity of the function, \( P[Y^{+\infty}_\tau < 0] = P[x^{+\infty}(v_0) < \tau] = P[x^{-\infty}(v_0) \leq \tau] = P[Y^{-\infty}_\tau \leq 0] \). Since a distribution function has at most countably many points of discontinuity, there is no interval of positive length such that \( P[Y^{+\infty}_\tau < 0] < P[Y^{-\infty}_\tau \leq 0] \) for all \( \tau \) in the interval. Conversely, if \( x^{+\infty}(v_0) \) and \( x^{-\infty}(v_0) \) have different distribution functions, then there is an interval of positive length such that the left-continuous modification of the distribution function of \( x^{+\infty}(v_0) \) lies strictly below the distribution function of \( x^{-\infty}(v_0) \) throughout the interval. Consequently, \( P[Y^{+\infty}_\tau < 0] < P[Y^{-\infty}_\tau \leq 0] \) for all \( \tau \) in that interval. The first statement in the corollary is proved. The second statement in the corollary is an immediate consequence of the first and the fact that \( P[Y^{-\infty}_\tau < 0] \leq P[Y^{-\infty}_\tau \leq 0] \) for all \( \tau \). \( \square \)

The following lemma is useful for calculating the \( \tau \)-surplus of nodes in a tree network.
Lemma 7.1 Let \((U, V, C)\) be a finite network with no capacity constraints. Suppose there is a partition \(\{\{u_1, \ldots, u_d\}, U_1, U_2, \ldots, U_d\}\) of \(U\) and a partition \(\{\{v_1\}, V_1, V_2, \ldots, V_d\}\) of \(V\) such that \(N(u_i) = \{v_{x_i}, v_{y_i}\}\) where \(v_i \in V_i\) for all \(i\), and \(N(v_i) \subset V_i\) for all \(u \in U_i\) and all \(i\). Let \(\tau\) be a real number. Let \(y\) denote the \(\tau\)-surplus of \(v_x\) relative to \((U, V)\), and for \(1 \leq i \leq d\) let \(y_i\) denote the \(\tau\)-surplus of the node \(v_i\) relative to the subnetwork \((U_i, V_i)\). For brevity, let \(m_i = m_{u_i}\). Then

\[
y = b_{v_o} + \sum_{i=1}^{d} [y_i + m_i]_{0}^{m_i} - \tau
\]

(7.3)

where we use the notation \([x]_{a}^{b}\) for the number in \([a, b]\) closest to \(x\).

Proof. Construct an assignment vector \(f\) for \((U, V, N)\) meeting the demand, and balanced with baseload \(\bar{b}\) defined by \(\bar{b}_{v_o} = b_{v_o} - y\) and \(\bar{b}_v = b_v\) for \(v \neq v_o\), as follows. Set \(f_{u_i, v_i} = [-y_i]_{0}^{m_i}\) and \(f_{u_i, v_o} = m_i - [-y_i]_{0}^{m_i} = [y + m_i]_{0}^{m_i}\) for \(1 \leq i \leq d\). Choose \(f_{u, v}\) for \(u \in U_i, v \in V_i\) for each \(i\) so that the demand is met for each \(u \in U_i\) and so that the load is balanced in \(V_i\) for each \(i\) (with unspecified boundary conditions). We claim that \(f\) is a balanced assignment for \((U, V, N)\). By construction, \(f\) is an admissible assignment vector that meets the demand \(m\). Since the load is balanced within \(V_i\) for each \(i\) it remains to check the balance conditions for \(u_1, \ldots, u_d\). The load at node \(v_o\) for assignment \(f\) and baseload \(\bar{b}\) is \(b_{v_o} - y + \sum_{i=1}^{d} f_{u_i, v_o}\), which by (7.3) is equal to \(\tau\). For each \(i\) one of the following three applies: (1) If \(y_i > 0\) then \(f_{u_i, v_i} = 0\) and \(x(v_i) > \tau\); (2) If \(y_i < -m_i\) then \(f_{u_i, v_i} = m_i\) and \(x(v_i) < \tau\); (3) If \(-m_i \leq y_i \leq 0\) then \(x(v_i) = \tau\). In any case, the balance condition holds for \(u_i\). Thus, \(f\) balances with baseload \(\bar{b}\) and demand \(m\) as claimed, and the load at \(v_o\) is \(\tau\). The proof of the Lemma is complete. \(\square\)

For the remainder of this section, suppose \((U, V, C)\) is a tree network with degree \(d + 1\) where \(d \geq 1\). Suppose the baseload is zero and that demand on each edge has distribution function \(F_m\) and the demands on different edges are mutually independent.

A generalized distribution function \(F\) corresponds to a probability measure on the interval \([-\infty, +\infty]\). Such a function is assumed to be nondecreasing and right continuous, and the limits \(\lim_{x \to -\infty} F(x)\) and \(\lim_{x \to +\infty} 1 - F(x)\) must be nonnegative—the values of the limits are the probability assigned to \(-\infty\) and \(+\infty\), respectively. Write \(F \prec G\) if \(F(x) \geq G(x)\) for all \(x\). We write \(I_{\{x \leq -\infty\}}\) (resp. \(I_{\{x \leq +\infty\}}\)) to denote the minimal (resp. maximal) generalized distribution function.
Given \( \tau \geq 0 \) and an integer \( d \geq 1 \) let \( \Gamma_{d,\tau} \) denote a mapping from the space of generalized distribution functions to the space of ordinary distribution functions defined as follows. Given a generalized distribution function \( F \), let \( \Gamma_{d,\tau} F \) denote the distribution function of \( Y \), where

\[
Y = \sum_{i=1}^{d} [Y_i + m_i]_0^{m_i} - \tau \tag{7.4}
\]

where \( m_1, \ldots, m_d, Y_1, \ldots, Y_d \) are mutually independent, and for each \( i \), \( m_i \) has distribution function \( F_m \) and \( Y_i \) has distribution function \( F \). Since \( Y \) in (7.4) is a nondecreasing, continuous function of \( Y_i \) for each \( i \), it follows that the mapping \( \Gamma_{d,\tau} \) is monotone, in the sense that if \( F < G \) then \( \Gamma_{d,\tau} F < \Gamma_{d,\tau} G \). Also \( \Gamma_{d,\tau} \) is a continuous mapping if the space of generalized distribution functions is endowed with the usual weak topology. Thus, there is a unique minimal solution and a unique maximal solution to the fixed point equation \( F = \Gamma_{d,\tau} F \).

Fix a node \( v_o \in V \) and let \( V_n \) be the set of nodes at distance less than or equal to \( n \) from \( v_o \). By Lemma 7.1 and induction on \( n \), it follows that \( Y_{\tau,\tau}^{\leq n} \) has distribution function \( \Gamma_{d+1,\tau}^{\leq n} \Gamma_{d,\tau}^{n*} \{1_{\tau} \leq +\infty \} \), where \( \Gamma_{d,\tau}^{n*} \) denotes the \( n \)-fold composition of \( \Gamma_{d,\tau} \) with itself. The distribution \( 1_{\tau} \leq +\infty \) appears here since the \( +\infty \)-boundary condition on \( V_n \) corresponds to \( \tau \)-surplus \( +\infty \) for all nodes in \( V_n^{c} \). Take \( n \) to infinity to obtain the following proposition.

**Proposition 7.2** For given \( \tau \), \( Y_{\tau}^{+\infty} \) has distribution \( \Gamma_{d+1,\tau} F^{+\infty} \), where \( F^{+\infty} \) is the maximal solution to \( \Gamma_{d,\tau} F = F \). Similarly \( Y_{\tau}^{-\infty} \) has distribution \( \Gamma_{d+1,\tau} F^{-\infty} \) where \( F^{-\infty} \) is the minimal solution to \( \Gamma_{d,\tau} F = F \).

**Trees with Exponentially Distributed Loads.** Suppose that the distribution of the demand \( m_u \) for each edge \( u \) in the tree has the exponential distribution with mean one: \( F_m(c) = 1 - e^{-c} \) for \( c \geq 0 \). The following relation is helpful in determining the set of solutions \( F \) to the fixed point equation \( F = \Gamma_{d,\tau} F \). If \( m \) is exponentially distributed with mean one and \( Y \) is independent of \( m \) and has an arbitrary distribution, then for any \( c > 0 \)

\[
P[Y + m]_0^m \geq c \] \[= P[Y + m]_0^m \geq c | Y \geq 0]P[| Y \geq 0] + P[Y + m]_0^m \geq c | 0 > Y \geq -m]P[0 > Y \geq -m]
\]

\[= P[m \geq c | Y \geq 0]P[| Y \geq 0]
\]

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\[ + P[Y + m \geq c | 0 > Y \geq -m] P[0 > Y \geq -m] \]
\[ = e^{-c} P[Y \geq 0 \text{ or } 0 > Y \geq -m] \]
\[ = e^{-c} P[Y + m \geq 0] \]

Thus, the distribution of \([Y + m]_0^n\) depends on the distribution of \(Y\) only through the parameter \(q = P[Y + m \geq 0]\). If \(F_n\) is a sequence of distribution functions such that \(F_{n+1} = \Gamma_{d,\tau} F_n\) and if \(q^n\) is the parameter \(q\) associated with \(F_n\), then \(q^{n+1} = \gamma_{d,\tau}(q^n)\), where \(\gamma_{d,\tau}(q) = P[m + Z_1 + \ldots + Z_d - \tau \geq 0]\)

where \(Z_1, \ldots, Z_d\) are random variables with \(P[Z_i \geq c] = q e^{-c}\) for \(c > 0\) and \(P[Z_i = 0] = 1 - q\), \(m\) is an exponentially distributed random variable with mean one, and \(m, Z_1, \ldots, Z_d\) are mutually independent. If \(U\) denotes the number of \(i\) such that \(Z_i > 0\), then \(U\) has the binomial distribution with parameters \(d\) and \(q\). The probability that the sum of \(u + 1\) exponentially distributed random variables is greater than or equal to \(\tau\) is equal to \(P[\text{Poi}(\tau) \leq u]\), where \(\text{Poi}(\tau)\) represents a Poisson random variable with mean \(\tau\). Therefore,

\[ \gamma_{d,\tau}(q) = P[\text{Poi}(\tau) \leq Bi(d, q)] \]

where \(Bi(d, q)\) represents a Poisson random variable with parameters \(d\) and \(q\) which is independent of \(\text{Poi}(\tau)\).

Let \(q^{+\infty}\) (resp. \(q^{-\infty}\)) denote the maximal (resp. minimal) fixed point of \(\gamma_{d,\tau}\). Then \(q^{+\infty}\) and \(q^{-\infty}\) are the parameters associated to \(F^{+\infty}\) and \(F^{-\infty}\), the maximal and minimal solutions to \(F = \Gamma_{d,\tau} F\). Therefore \(Y_{\tau}^{+\infty}\) has the same distribution as \(Z_1 + Z_2 + \ldots + Z_{d+1} - \tau\) where \(Z_1, \ldots, Z_{d+1}\) are independent random variables with \(P[Z_i \geq c] = q^{+\infty} e^{-c}\) for \(c > 0\) and \(P[Z_i = 0] = 1 - q^{+\infty}\), so that

\[ P[Y_{\tau}^{+\infty} < 0] = 1 - P[Z_1 + Z_2 + \ldots + Z_{d+1} \geq \tau] = 1 - \tilde{\gamma}_{d+1,\tau}(q^{+\infty}). \]

where \(\tilde{\gamma}_{d+1,\tau}(q) = P[\text{Poi}(\tau) < Bi(d + 1, q)]\)

Combining this and a similar equation for \(Y_{\tau}^{-\infty}\) with Proposition 7.1 yields the following proposition.
Proposition 7.3 For a tree network with independent, exponentially distributed demands of mean one on the edges,

\[
P[x^{+\infty}(v_o) < \tau] = 1 - \tilde{\gamma}_{d+1,\tau}(q^{+\infty}) \\
P[x^{-\infty}(v_o) \leq \tau] = 1 - \tilde{\gamma}_{d+1,\tau}(q^{-\infty}).
\]

(7.5)  
(7.6)

Thus, the distributions of \(x^{+\infty}(v_o)\) and \(x^{-\infty}(v_o)\) are different if and only if the minimal and maximal fixed points of \(\gamma_{d,\tau}\) are distinct for \(\tau\) in some interval of nonzero length.

Fixed points of the function \(\gamma_{d,\tau}\) are easily computed numerically for small \(d\) and the qualitative behavior of the fixed points is easy to determine for large \(d\). The distribution functions are pictured in Figure 3 for \(1 \leq d \leq 8\). For each integer \(d\) in the range \(1 \leq d \leq 5\), the upper and lower distributions are identical, and for \(d \geq 6\) the upper and lower distributions are different in some interval. The function \(\gamma_{\tau,d}\) is pictured in Figures 4 and 5 for the two “borderline” cases, \(d = 5\) and \(d = 6\), respectively, for selected values of \(\tau\). The derivative of the distribution function for \(d = 5\) is less than 35 throughout the interval \([0, 1]\). The function \(\gamma_{d,\tau}(q)\) is decreasing in \(\tau\) for each value of \(d\) and \(q\) fixed. There are either one, two or three fixed points of \(\gamma_{d,\tau}\) for \(d, \tau\) fixed.

Trees with Bernoulli Loads. Suppose the demand \(m_u\) is 1 with probability \(p\) and 0 with probability \(1 - p\) for all \(u\), and that demands for distinct consumers are independent.

Proposition 7.4 If \(pd \leq 1\) then with probability one there is a unique balanced load vector and load percolation does not occur at any node. If \(pd > 1\) then for any fixed node \(v_o\), the load at \(v_o\) is not unique with positive probability.

Proof. If \(pd \leq 1\), then the components of the graph induced by edges \(u\) with \(m_u = 1\) are all finite, with probability one, as is well known from the theory of branching processes. Hence the balanced load vector is unique and load percolation does not occur, with probability one. The first statement of the proposition is proved.

It is shown next that \(P[x^{-\infty}(v_o) > 1] = 0\) for any location \(v_o\) and any value of \(p\) with \(0 \leq p \leq 1\). If in (7.4), \(\tau = 1\), \(Y_i \leq -1\), and \(m_i \in \{0, 1\}\) for \(1 \leq i \leq d\), then \(Y = -1\). Thus the minimal solution to the equation \(F = \Gamma_{d,\tau} F\) for \(\tau = 1\) is \(F^{-\infty}(x) = I_{\{x \geq -\tau\}}\). Moreover, \(F^{-\infty} = \Gamma_{d+1,\tau} F^{-\infty}\).
Figure 3: Distribution functions of load for $1 \leq d \leq 8$. For $d = 6, 7, 8$ the distribution function for both the minimal and maximal loads are shown, with dashed lines indicating where they diverge from each other.

Therefore by (7.2) and Proposition 7.2, $P[x^{-\infty}(v_o) \leq 1] = F^{-\infty}(0) = 1$, as was to be proved. (Another proof of this result is the following. It was proved for $p = 1$ in Section 2, and it can thus be proved for all $p$ by exploiting the fact that $x^{-\infty}$ is stochastically increasing in $p$.)

To complete the proof of the proposition we show that $P[x^{+\infty}(v_o) > 1] > 0$ if $pd > 1$. Suppose that $\tau = \frac{2t+1}{2t}$ for some integer $t$, and $d \geq 2$. Define a new mapping $\bar{\Gamma}_{d,\tau}$ as follows. Given a distribution function $F$, let $\bar{\Gamma}_{d,\tau}F$ denote the distribution function of $\bar{Y}$, where

$$\bar{Y} = \begin{cases} 0 & \text{if } [Y_i + m_i]_o^{m_i} \geq \frac{t + 1}{2t} \text{ for exactly two values of } i \\
 -\frac{k+1}{2t} & \text{if } [Y_i + m_i]_o^{m_i} = 1 - \frac{k}{2t} \text{ for some } i \text{ and some } k \in \{0, \ldots, t - 2\}, \\
 & \text{and } [Y_j + m_j]_o^{m_i} \leq \frac{t}{2t} \text{ for } j \neq i \\
 -\tau & \text{else.} \end{cases}$$

and where $m_1, \ldots, m_d, Y_1, \ldots, Y_d$ are the same as in the definition of $\Gamma_{d,\tau}F$. Note that $\bar{Y} \leq Y$ for
any value of $m_1, \ldots, m_d, Y_1, \ldots, Y_d$ and that $\tilde{Y}$, like $Y$, is a nondecreasing function of $Y_1, \ldots, Y_d$. Hence, if $F < G$ then $\tilde{F}_{d, r} F < \Gamma_{d, r} G$. It follows that $\tilde{F}_{d, r}$ has a maximal fixed point, $\tilde{F}^{+\infty}_{d, r}$, and that $\tilde{F}^{+\infty}_{d, r}$ is stochastically smaller than the maximal fixed point of $\Gamma_{d, r}$. It will be shown that $\tilde{F}^{+\infty}_{d, r}(0) > 0$ if $pd > 1$, so that indeed $P[x^{+\infty}(v_o) > 1] > 0$ if $pd > 1$.

The probability mass corresponding to the distribution function $\tilde{F}^{+\infty}_{d, r}$ is concentrated on the finite set \{0, $-\frac{1}{2t}, \ldots, -\frac{t-1}{2t}\} \cup \{-r\}$. Setting $\pi_j$ equal to the probability mass assigned by $\tilde{F}^{+\infty}_{d, r}$ to $-\frac{j}{2t}$, the fixed point equation of $\tilde{F}_{d, r}$ becomes

$$\pi_0 = \binom{d}{2} (ps)^2 (1 - ps)^{d-2}$$

$$\pi_{j+1} = \pi_j [dp(1 - ps)^{d-1}] \quad 0 \leq j \leq t - 2$$

$$s = \pi_0 + \ldots + \pi_{t-1}.$$  \hspace{1cm} (7.7) \hspace{1cm} (7.8) \hspace{1cm} (7.9)

Solve for $\pi_1, \ldots, \pi_{t-1}$ in terms of $\pi_0$. Assume that $dp(1 - ps)^{d-1} \neq 1$ and sum a partial geometric series to obtain

$$s = \frac{1 - [dp(1 - ps)^{d-1}]^t}{1 - [dp(1 - ps)^{d-1}]} \pi_0.$$  \hspace{1cm} (7.10)

Using (7.7) to substitute in for $\pi_0$ and cancelling a factor of $s$ we obtain the following equation for $s$:

$$sp^2 \binom{d}{2} (1 - ps)^{d-2} \{1 - [dp(1 - ps)^{d-1}]^t\} = 1 - dp(1 - ps)^{d-1}. $$

Let $s_o$ denote the unique value in the interval $(0, 1)$ satisfying $dp(1 - ps_o)^{d-1} = 1$. Such $s_o$ exists assuming that $dp > 1$. Then $s = s_o$ is a solution to equation (7.10). However, in deriving (7.10) we assumed that $dp(1 - ps)^{d-1} \neq 1$, so the value $s_o$ does not necessarily represent a solution to the original equations (7.7)-(7.9). Let $D_{\text{left}}$ (resp. $D_{\text{right}}$) denote the derivative of the quantity on the lefthand side (resp. righthand side) of (7.10), evaluated at $s_o$. The lefthand side of (7.10) is 0 at $s = 0$ and the righthand side is negative at $s = 0$. Thus if $D_{\text{left}} > D_{\text{right}}$ then there must exist a solution $s$ to (7.10) in the interval $(0, s_o)$. By straightforward calculation, find that $D_{\text{left}} > D_{\text{right}}$ if and only if

$$t > \frac{2(1 - s_0 p)}{(d - 1)s_0 p} $$

Thus, for $t$ satisfying (7.11), the maximal fixed point $F^{+\infty}$ of $\Gamma_{d, r}$, where $\tau = \frac{2t+1}{2t}$, assigns positive probability to $[0, \infty)$. Hence, $P[x^{+\infty}(v_o) \geq \tau] > 0$ if $dp > 1$ and $\tau = \frac{2t+1}{2t}$ where $t$ satisfies (7.11).
8 RECTANGULAR LATTICE NETWORKS WITH RANDOM DEMAND

Consider a rectangular lattice network with parameter \( d \geq 1 \), as defined in Section 2. Suppose the baseload vector is zero. Let \( 0 \leq p \leq 1 \) and suppose the demands \( m_u \) are independent Bernoulli random variables with parameter \( p \). By Proposition 5.1, the balanced load vector \( x \) is unique for each realization of \( m \). As indicated in Section 2, load percolation occurs in the special case \( p = 1 \). Whether load percolation occurs at a given node with positive probability for some \( p \) strictly less than one is an open problem. Of course \( p \) has to be large enough \((p > 0.5 \text{ if } d = 2)\) so that there is positive probability of percolation in the classical sense [3]. Simulation results concerning the problem are presented here. One approach to simulation, motivated by Corollary 4.1, is to simulate a finite rectangular grid network with \( \beta \)-boundary condition, and vary \( \beta \) in order to maximize the size of components of the network with equal load. A similar approach, which we decided to use, is to simulate the load on a torus, rather than on a finite two dimensional grid. Intuitively speaking, using a torus is similar to using a finite grid network and imposing boundary conditions which favor the formation of large sets of nodes with equal load.

Simulation data for a \( 60 \times 60 \) torus is shown in Figures 6-8 for \( p = 0.6, 0.75 \) and \( 0.90 \) respectively. The figures were produced as follows. A demand vector was generated using a pseudo-random number generator, and the balanced load vector was computed. The multiplicity of each load value was determined and the three values of highest multiplicity were identified. Finally, the locations with loads of the three highest multiplicity are indicated in the three figures. For \( p = 0.6 \) note that only a small fraction of the locations have an identical load. This was observed in other simulations for \( p \) less than 0.6, if load values of 0 and 0.5 are neglected. These values are not associated with load percolation but are quite frequent for small values of \( p \). At the other extreme, for \( p \geq 0.90 \), we find that the balanced load vector is constant over a vast majority of the locations. Figure 8 is representative of this case. As \( p \) was increased from 0.6 to 0.9, we observed the emergence and growth of clusters of locations having the same load values. Figure 7 is representative of this intermediate range of values.

The simulation data leads to the conjecture that percolation of load occurs for the infinite two-dimensional lattice with Bernoulli demands if \( p \) is sufficiently close to 1.0. Moreover, the data
suggests that for $d = 2$ there is a corresponding cutoff value somewhere in the interval 0.75-0.90, but it seems unwise to draw any definite conclusions. Let us endeavor to speculate even further. Load percolation is likely to occur for smaller values of $p$ in higher dimensions. The total load within a cube of side $n$ in $d$ dimensions has standard deviation on the order of $n^{d/2}$, whereas the number of edges crossing the boundary of such a cube is on the order of $n^{d-1}$. In two dimensions these are equal, so that, intuitively speaking, the balancing capability of edges might just be able to smooth out load fluctuations, whereas in three or more dimensions there seems to be more than enough edges available to smooth the load over long ranges. Perhaps load percolation does not occur for any $p < 1$ unless $d \geq 3$.

9 SUMMARY AND OPEN PROBLEMS

The highlights of this paper are summarized as follows. The concept of load balancing on an infinite network is introduced. Minimal and maximal balanced load vectors are shown to exist, and the idea of load balancing in finite subsets with boundary conditions is used to exhibit a one-parameter family of balanced load vectors whenever the balanced load vector is not unique. It is shown that the balanced load vector is unique for a wide class of networks including rectangular lattice networks. The concept of $r$-surplus is used to characterize the possible distributions of the load at a location in a tree network with independent, identically distributed demands. The case of Bernoulli demand and exponentially distributed demands are investigated in some detail. Finally a notion of long range interaction, load percolation, is introduced. Load percolation is shown to imply infinite connected sets of locations with identical load.

A number of questions are raised by the paper. Is there a way to compute the distribution of the load at node 0 in a $d$-dimensional lattice network for some $d \geq 2$ and independent demands on edges with some interesting distribution. Does load percolation occur with positive probability for a $d$-dimensional rectangular lattice and Bernoulli loads for some $p < 1$? Whether or not load percolation occurs, what are the typical size or shape of the connected components consisting of nodes with equal load for a balanced load vector?
Some questions related to a $d + 1$ dimensional tree network with no baseload and independent, identically distributed demands on edges are the following. Is it true that for any $d_o \geq 1$ there is a distribution of demands such that the balanced load vector is unique for $d \leq d_o$? Stephen Turner of U. Cambridge conjectures that for any given demand distribution, there is nonuniqueness for sufficiently large $d$. How can one characterize the possible sets of fixed points of $\gamma_{d,r}$ as the demand distribution varies. Is there a simple test for when load percolation occurs in tree networks in cases the balanced load vector is unique, in particular for exponentially distributed demands with $2 \leq d \leq 5$?

References


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Figure 4: Graph of $\gamma_{\tau,d}$ for $d = 5$ and $\tau=3.0, 3.05, 3.10, 3.15, 3.20$ and $3.25$.

Figure 5: Graph of $\gamma_{\tau,d}$ for $d = 6$ and $\tau=3.4, 3.5, 3.6$, and $3.7$. 
Figure 6: Plot of balanced load vector for $60 \times 60$ torus network with Bernoulli ($p=0.60$) demands.

The highest multiplicity values are 1.0000000, 1.3333333, and 1.2500000, with multiplicities 740, 303, and 140, respectively.
Figure 7: Plot of balanced load vector for $60 \times 60$ torus network with Bernoulli ($p=0.75$) demands. The highest multiplicity values are 1.5722121, 1.5865169, and 1.5000000, with multiplicities 547, 445, and 318, respectively.

Figure 8: Plot of balanced load vector for $60 \times 60$ torus network with Bernoulli ($p=0.90$) demands. The highest multiplicity values are 1.8059028, 1.7986348, and 1.8053097, with multiplicities 2880, 293, and 113, respectively.
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