ON THE UNIQUENESS OF SOLUTIONS OF SOME SECOND ORDER DIFFERENTIAL EQUATIONS

By

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§1 Introduction. In [1] we proved a uniqueness result for the solutions of

\begin{align*}
(I)_a & \quad -u'' = \lambda u - \psi(w(x)|u|)u \quad , \quad a < x < \infty \ , \\
& \quad u(a) \cos \theta - u'(a) \sin \theta = 0 \ , \quad u \in L^2[a, \infty) ,
\end{align*}

where $\lambda > 0$ and $\theta \in [0, \frac{\pi}{2}]$. For notational convenience, let $S^+_{a,n}$ denote the set of $u \in C^2[a, \infty) \cap H^1[a, \infty)$ such that $u$ satisfies $(I)_a$, $u > 0$ in a deleted neighborhood of $x = a$, and $u$ has exactly $n - 1$ simple zeros in $(a, \infty)$, where $a \geq 0$ and $n \in \mathbb{N}$. Similarly $S^-_{a,n}$ denotes the set $u < 0$ in a deleted neighborhood of $x = a$. It has been shown in [1] that problem $(I)_a$ has at most one solution in each nodal class $S^\pm_{a,n}$ provided that the functions $\psi$ and $w$ are continuously differentiable and satisfy the following conditions:

(\psi 1) $\psi(0) = 0$ and for $t > 0$, $\psi'(t) > 0$ and $\psi(t) \geq p \cdot t^q$ for some $p, q > 0$.

(w 1) $w > 0$, $w'(0) \geq 0$ and $\frac{w'}{w}$ is nondecreasing on $[0, \infty)$.

The method we used in [1] was shooting with the aid of comparison. Our aim in this note is to describe a new proof which makes use of comparison arguments only. This proof is more concise and requires less smoothness on the functions $\psi$ and $w$; we can treat problem $(I)_a$ when $\psi$ and $w$ are continuous, but not $C^1$ as in the preceding proof. Moreover, the new proof reveals more information about the locations of the nodes of the solutions (see Lemma 1 and Remark 4). For the existence of solutions of $(I)_a$, we refer to [2–6] and the references therein.

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There are some generalizations about utilizing such comparison arguments for showing uniqueness of solutions of certain boundary value problems. They will be discussed in the final section.

§2 A new proof for uniqueness. Throughout this section, the functions \( \psi \) and \( w \) are assumed to satisfy the following conditions.

\((\psi 2)\) \( \psi(0) = 0 \), \( \psi \) is locally Lipschitz continuous and strictly increasing on \([0, \infty)\).

\((w 2)\) \( w \) is a positive continuous function and for any \( \delta > 0 \), the function \( g_\delta(x) = \frac{w(x+\delta)}{w(x)} \)

is a nondecreasing function of \( x \) for \( x \in [0, \infty) \).

Remark 1. If \( w \) is positive and \( \frac{w'}{w} \) is nondecreasing on \([0, \infty)\), it is easy to check that \((w2)\) is satisfied by invoking the identity \( \frac{w(x+\delta)}{w(x)} = \exp \left( \int_0^{x+\delta} \frac{w'(t)}{w(t)} \, dt \right) \).

Theorem 1. Let \( \theta = 0 \) and \( \lambda \) be a fixed positive number. If \((\psi 2)\) and \((w 2)\) are satisfied, then for every \( a \geq 0 \) and \( n \in \mathbb{N} \), \( S_{a,n}^\pm \) contains at most one element.

Theorem 2. Let \( \theta \in (0, \frac{\pi}{2}] \) and \( \lambda > 0 \) be fixed. In addition to \((\psi 2)\) and \((w 2)\), if \( w \) is a nondecreasing function on \([0, \infty)\), then for every \( a \geq 0 \) and \( n \in \mathbb{N} \), \( S_{a,n}^\pm \) contains at most one element.

Remark 2. (a) It is known that \( u \equiv 0 \) is the only solution of \((I)_a\) if \( \lambda \leq 0 \).

(b) In case \( 0 < \theta \leq \frac{\pi}{2} \), the need for the extra assumption has been indicated in [1, Remark 1.2.(a)]. The key point is that Lemma 1.30(iii) of [2] rests on the same sort of assumption and will be used in the proof.

(c) In fact, Theorems 1 and 2 are valid for an equation having a more general nonlinear
term than that of Eq (1); that is, instead of \( \psi(w(x)|u|)u \), we can consider

\[
F(x, u)u = \begin{cases} 
\psi_1(w(x)|u|)u \quad \text{if} \quad u \geq 0 \\
\psi_2(w(x)|u|)u \quad \text{if} \quad u < 0 
\end{cases}
\]

with both \( \psi_1 \) and \( \psi_2 \) satisfying (\( \psi_2 \)). For simplicity we will carry out the proof for the case \( \psi_1 = \psi_2 \) only since there is no essential difference in the proof when \( \psi_1 \neq \psi_2 \).

(d) In [1] the nonlinear term of Eq (1) was written in the form \( \psi(w(x)|u|^\sigma)u \), where \( \sigma \) is a positive number. However, one can rewrite \( \psi(w(x)|u|^\sigma) \) in a simpler form \( \tilde{\psi}(\tilde{w}(x)|u|) \) with \( \tilde{w}(x) = (w(x))^{1/\sigma} \) and \( \tilde{\psi} = \psi \circ h \), where \( h(t) = t^\sigma \), since \( \psi \) and \( w \) satisfy (\( \psi_2 \)) and (\( w_2 \)) if and only if \( \tilde{\psi} \) and \( \tilde{w} \) do.

(e) With slight modifications in the proofs of Theorems 1 and 2, one can show the same sort of uniqueness results for the solutions of Eq. (1) on the bounded intervals under the boundary conditions \( u(a) \cos \theta - u'(a) \sin \theta = 0 \) and \( u(b) = 0 \).

(f) The new proof covers the case \( n \geq 2 \) only. We recall that the uniqueness of positive and negative solutions of \((I)_a \) as well as its counterpart on the bounded interval case has already been established in [2, Theorem 1.5].

Since the proofs of Theorem 1 and Theorem 2 are similar, only the former will be carried out. The proof of Theorem 1 rests on the following two preliminary lemmas.

**Lemma 1.** Let \( n \) be a fixed positive integer. Suppose for \( 1 \leq l \leq n \) that \( S_{a,l}^\pm \) contains at most one element for every \( a \geq 0 \). For any \( \beta > b \geq 0 \), if \( u \in S_{b,n}^+ \) and \( v \in S_{\beta,n}^+ \) then

\[
|u'(b)| \geq \frac{w(\beta)}{w(b)} |v'(\beta)| .
\]
Moreover, if $z_k$ is the $k$-th zero of $u$ in $(b, \infty)$ and $\tilde{z}_k$ is the $k$-th zero of $\tilde{v}$ in $(\beta, \infty)$ then

$$z_k - z_{k-1} \geq \tilde{z}_k - \tilde{z}_{k-1} \quad (4)$$

for $1 \leq k \leq n - 1$, where $z_0 = b$ and $\tilde{z}_0 = \beta$. The same result holds if $u \in S_{b,n}^-$ and $v \in S_{\beta,n}^-$. 

For what follows in this section, we let $V_+(a,b,\cdot)$ denote the unique positive solution of

$$(I)_{a,b} \quad -u'' = \lambda u - \psi(w(x)|u|)u \quad , \quad a < x < b \quad , \quad u(a) = u(b) = 0$$

if the positive solution exists.

**Lemma 2.** Let $\beta > b > 0$. If $V_+(a,b,\cdot)$ and $V_+(a,\beta,\cdot)$ exist then

$$|V'_+(a,b,b)| < \frac{w(\beta)}{w(b)} \frac{|V'_+(a,\beta,\beta)|}{|V'_+(a,b,b)|} \quad . \quad (5)$$

**Proof of Theorem 1.** We proceed the proof by induction. Since $n = 1$ is true by Remark 2 (f), it suffices to show that

if for every $a > 0$ and $1 \leq l \leq n$, $S_{a,l}^\pm$ contains at most one element

$$\text{then so does } S_{a,n+1}^\pm \quad . \quad (6)$$

We prove (6) indirectly. Suppose that $u, \tilde{u} \in S_{a,n+1}^+$. Let $b$ and $\beta$ be the first zero of $u$ and $\tilde{u}$ in $(a, \infty)$ respectively. Let $u_1(x) = u(x)$ for $x \in [b, \infty)$ and $\tilde{u}_1(x) = \tilde{u}(x)$ for $x \in [\beta, \infty)$. It is clear that $u_1 \in S_{b,n}^-$ and $\tilde{u}_1 \in S_{\beta,n}^-$. By the induction hypothesis we know that $b \neq \beta$. Without loss of generality, we may assume that $\beta > b$. It follows from Lemma 1 that

$$|u'_1(b)| \geq \frac{w(\beta)}{w(b)} |\tilde{u}'_1(\beta)| \quad . \quad (7)$$
By the uniqueness of positive solution of \((I)_{a,b}\) we know that \(u(x) = V_+(a, b, x)\) if \(x \in [a, b]\).

Similarly \(\tilde{u}(x) = V_+(a, \beta, x)\) for \(x \in [a, \beta]\). Invoking Lemma 2 yields

\[
\frac{w(\beta)}{w(b)} |V'_+(a, \beta, \beta)| > |V'_+(a, b, b)|. \tag{8}
\]

Combining (7) with (8) gives

\[
|u'(b)| = |u'_1(b)| \geq \frac{w(\beta)}{w(b)} |\tilde{u}'_1(\beta)| = \frac{w(\beta)}{w(b)} |\tilde{u}'(\beta)| = \frac{w(\beta)}{w(b)} |V'_+(a, \beta, \beta)| > |V'_+(a, b, b)| = |u'(b)|,
\]

which is absurd. Thus \(S^+_{a,n+1}\) contains at most one element.

The proof of \(S^-_{a,n+1}\) is similar.

We are now going to prove Lemma 1 and Lemma 2. We first prove the following lemma which contains a partial result of Lemma 1.

**Lemma 3.** For any \(\beta > b \geq 0\), if \(u \in S^+_{b,1}\) and \(v \in S^+_{\beta,1}\) then

\[
|u'(b)| \geq \frac{w(\beta)}{w(b)} |v'(\beta)|. \tag{9}
\]

The same result holds if \(u \in S^-_{b,1}\) and \(v \in S^-_{\beta,1}\).

**Proof of Lemma 3.** We prove the case \(u \in S^+_{b,1}\) and \(v \in S^+_{\beta,1}\) only. Let \(\delta = \beta - b\) and \(\eta(x) = v(x + \delta)\) for \(x \in [b, \infty)\). Then by direct computation, we get

\[
-\eta''(x) = \lambda \eta(x) - \psi \left( \frac{w(x + \delta)}{w(x)} w(x) |\eta(x)| \right) \eta(x) \\
\leq \lambda \eta(x) - \psi \left( \frac{w(\beta)}{w(b)} w(x) |\eta(x)| \right) \eta(x),
\]

5
the last inequality follows from the assumptions (w2) and (ψ2). Let ζ(x) be the unique positive solution of

\[ -ζ'' = λζ - ψ \left( \frac{w(β)}{w(β)} w(x)|ζ| \right) ζ , \]
\[ ζ(b) = 0 , ζ \in L^2[b, \infty) . \]

It follows from [2, Lemma 1.8] that

\[ |η'(b)| ≤ |ζ'(b)| . \quad (10) \]

Invoking [1, Lemma 2.2], we know that \( u'(b) = \frac{w(β)}{w(β)} ζ'(b) \). This together with (10) yields

\[ |u'(b)| ≥ \frac{w(β)}{w(β)} |η'(b)| = \frac{w(β)}{w(β)} |v'(β)| \]

which completes the proof.

We are now in the position to prove Lemma 1 and Lemma 2. We prove Lemma 2 first.

**Proof of Lemma 2.** Let \( δ = β - b \) and \( η(x) = V_+(a, b, x - δ) \) for \( x \in [a + δ, β] \). Arguing like in obtaining (10), we have

\[ -η'' ≤ λη - ψ \left( \frac{w(β)}{w(β)} w(x)|η| \right) η \]

and hence

\[ |η'(β)| ≤ |ζ'(β)| \quad (11) \]

by making use of [2, Lemma 1.8], where ζ is the unique positive solution of

\[ -ζ'' = λζ - ψ \left( \frac{w(β)}{w(β)} w(x)|ζ| \right) ζ , \]
\[ ζ(a + δ) = ζ(β) = 0 . \]
Invoking [2, Lemma 1.30] and [1, Lemma 2.2], we get

\[ |V'_+(a, \beta, \beta)| > |V'_+(a + \delta, \beta, \beta)| = \frac{w(b)}{w(\beta)} |\zeta'(\beta)| . \] (12)

This together with (11) and the fact that \( \eta'(\beta) = V'_+(a, b, b) \) yields (5).

**Remark 3.** In applying [2, Lemma 1.30], the inequality appeared in (12) should be in the strict sense since the function \( \psi \) is assumed to be locally Lipschitz continuous. Also the part of results we invoked in this lemma need not the assumption (F2) imposed there.

In the proof of Lemma 1, the uniqueness of positive or negative solution of \((I)_a\) or \((I)_{a,b}\), and the fact just mentioned in Remark 3 will be used without further comment. We let \( V_+(a, \infty, \cdot) \) denote the unique positive solution of \((I)_a\). The fact \( V_- = -V_+ \) will be used in the proof only for the sake of simplicity but making no essential difference in case \( \psi_1 \neq \psi_2 \), where \( V_- \) is the unique negative solution of \((I)_a\) or \((I)_{a,b}\).

**Proof of Lemma 1.** We proceed the proof by induction. For \( n = 1 \), (4) is void and (3) is true by Lemma 3. We now prove (4) for \( n = 2 \). We first prove \( \tilde{z}_1 > z_1 \) by arguing indirectly. Suppose that \( z_1 \geq \tilde{z}_1 \). It follows from Lemma 3 that

\[ |V'_+(\tilde{z}_1, \infty, \tilde{z}_1)| \geq \frac{w(z_1)}{w(\tilde{z}_1)} |V'_+(z_1, \infty, z_1)| . \] (13)

Invoking [2, Lemma 1.30] and Lemma 2 yields

\[ |V'_+(\beta, \tilde{z}_1, \tilde{z}_1)| < |V'_+(b, \tilde{z}_1, \tilde{z}_1)| \leq \frac{w(z_1)}{w(\tilde{z}_1)} |V'_+(b, z_1, z_1)| . \] (14)

Since \( u \in S^{+}_{b,2} \), we know that \( |V'_+(b, z_1, z_1)| = |u'(z_1)| = |V'_-(z_1, \infty, z_1)| = |V'_+(z_1, \infty, z_1)| \). This together with (13) and (14) yields

\[ |V'_+(\beta, \tilde{z}_1, \tilde{z}_1)| < |V'_+(\tilde{z}_1, \infty, \tilde{z}_1)| = |V'_-(\tilde{z}_1, \infty, \tilde{z}_1)| \]

which contradicts the fact that \( v \in S^{+}_{\beta,2} \). Thus we have \( \tilde{z}_1 > z_1 \).
Suppose that \( z_1 - z_0 < \tilde{z}_1 - \tilde{z}_0 \). Let \( \delta = \tilde{z}_1 - z_1 \). It follows that \( \delta > \tilde{z}_0 - z_0 = \beta - b \). Let \( \eta(x) = V_+(b, z_1, x - \delta) \) for \( x \in [\beta, \tilde{z}_1] \). Then it follows from the argument used in proving Lemma 2 that

\[
|\eta'(\tilde{z}_1)| \leq \frac{w(\tilde{z}_1)}{w(z_1)} |V'_+(b + \delta, \tilde{z}_1, \tilde{z}_1)| .
\]

(15)

Since \( b + \delta > \beta \), we know from [2, Lemma 1.30] that

\[
|V'_+(b + \delta, \tilde{z}_1, \tilde{z}_1)| < |V'_+(\beta, \tilde{z}_1, \tilde{z}_1)| .
\]

(16)

Combining (15) with (16) and using the fact that \( \eta'(\tilde{z}_1) = V'_+(b, z_1, z_1) \) yield

\[
|V'_+(b, z_1, z_1)| < \frac{w(\tilde{z}_1)}{w(z_1)} |V'_+(\beta, \tilde{z}_1, \tilde{z}_1)| .
\]

(17)

On the other hand, since \( \tilde{z}_1 > z_1 \), it follows from Lemma 3 that

\[
|V'_+(z_1, \infty, z_1)| \geq \frac{w(\tilde{z}_1)}{w(z_1)} |V'_+(\tilde{z}_1, \infty, \tilde{z}_1)| .
\]

This together with \( |V'_+(b, z_1, z_1)| = |V'_+(z_1, \infty, z_1)| \) and \( |V'_+(\beta, \tilde{z}_1, \tilde{z}_1)| = |V'_+(\tilde{z}_1, \infty, \tilde{z}_1)| \)
leads to

\[
|V'_+(b, z_1, z_1)| \geq \frac{w(\tilde{z}_1)}{w(z_1)} |V'_+(\beta, \tilde{z}_1, \tilde{z}_1)| ,
\]

which is incompatible with (17). Therefore \( z_1 - z_0 \geq \tilde{z}_1 - \tilde{z}_0 \) must be true and the proof of (4) for \( n = 2 \) is complete.

To proceed with the proof by induction, we need to show the following:

If (3) is true for \( n = j - 1 \) and (4) is true for \( n = j \) then (3) is true for \( n = j \)
and (4) is true for \( n = j + 1 \).
Let \( u \in S_{b,j}^+ \), \( v \in S_{\beta,j}^+ \) and \( \beta > b \geq 0 \). By the induction hypothesis, we know that \( z_1 - b \geq \tilde{z}_1 - \beta \). Let \( \delta = \beta - b \) and \( \eta(x) = v(x + \delta) \) for \( x \in [b, \tilde{z}_1 - \delta] \). Arguing like in the beginning of the proof of Lemma 2, we get

\[
\frac{w(\beta)}{w(b)} \eta'(b) \leq |V'_+(b, \tilde{z}_1 - \delta, b)|. \tag{18}
\]

Since \( \tilde{z}_1 - \delta \leq z_1 \), it follows from [2, Lemma 1.30] that

\[
|V'_+(b, \tilde{z}_1 - \delta, b)| \leq |V'_+(b, z_1, b)|. \tag{19}
\]

Combining (18) with (19) and using the facts that \( \eta'(b) = v'(\beta) \) and \(|u'(b)| = |V'_+(b, z_1, b)|\), we conclude that (3) is true for \( n = j \).

It remains to show that (4) is true for \( n = j + 1 \). This only requires slight modifications in the proof of \( n = 2 \). Let \( u \in S_{b,j+1}^+ \), \( v \in S_{\beta,j+1}^+ \) and \( \beta > b \geq 0 \). We first prove \( \tilde{z}_1 > z_1 \).

Suppose that \( z_1 \geq \tilde{z}_1 \). Letting \( u_1(x) = u(x) \) for \( x \in (z_1, \infty) \) and \( v_1(x) = v(x) \) for \( x \in (\tilde{z}_1, \infty) \), we conclude from (3) for \( n = j \) that

\[
|v'_1(\tilde{z}_1)| \geq \frac{w(z_1)}{w(\tilde{z}_1)} |u'_1(z_1)|,
\]

since \( u_1 \in S_{z_1,j}^+ \) and \( v_1 \in S_{\tilde{z}_1,j}^+ \). By the same reasoning we have (14). Then an argument similar to that of \( n = 2 \) shows that the same kind of contradiction occurs. Therefore \( z_1 < \tilde{z}_1 \). From the induction hypothesis, we know that

\[
z_k - z_{k-1} \geq \tilde{z}_k - \tilde{z}_{k-1}
\]

for \( 2 \leq k \leq j \). It remains to show that \( z_1 - b \geq \tilde{z}_1 - \beta \). This follows from the same lines of reasoning as that of \( n = 2 \) with only \( V_+(z_1, \infty, \cdot) \) and \( V_+(\tilde{z}_1, \infty, \cdot) \) changed to \( u_1 \) and \( v_1 \) respectively. Thus the proof is complete.
Remark 4. In view of [2, Lemma 1.8] and the proof of Lemma 1, it is not difficult to show that inequalities (3) and (4) are strict if the function \( g_b \) is strictly increasing.

§3 Final remarks. The assumption \( \psi(0) = 0 \) is not required in Theorems 1 and 2. In fact, the uniqueness results stated in the previous section is applicable to the solutions of the problem

\[
-u'' = \phi(w(x)|u|)u, \quad a < x < \infty, \tag{20}
\]

\[
u(a) \cos \theta - u'(a) \sin \theta = 0, \quad u \in L^2[a, \infty).
\]

Here the function \( w \) satisfies the same hypothesis as in section 2 and the function \( \phi \) is assumed to be locally Lipschitz continuous and strictly decreasing on \([0, \infty)\). In particular, letting \( \phi = \lambda - \psi \) in Eq. (20) leads to Eq. (1). However, it is clear that the uniqueness results proved in [1] do not cover equation like

\[-u'' = \lambda u + e^{-x}|u|^{-\tau}u\]

with \( \tau \in (0, 1] \) and \( \lambda \in \mathbb{R} \). The generalization noted above also holds for the bounded interval case.

In closing, we look at a differential equation

\[-u'' = G(x, u)u \tag{21}\]

on the real line. Here the function \( G \) is defined by

\[
G(x, y) = \begin{cases} 
\phi_1(w_1(x)|y|) & \text{if } y \geq 0 \\
\phi_2(w_2(-x)|y|) & \text{if } y < 0.
\end{cases}
\]

Assumed that \( \phi_1(0) = \phi_2(0) \) and \( \phi_1, \phi_2 \) are locally Lipschitz continuous and strictly decreasing on \([0, \infty)\), and \( w_1, w_2 \) satisfy \((w2)\). We have the following uniqueness result.
Theorem 3. Suppose that \( w_1(x)w_2(-x) \neq w_1(x + \delta)w_2(-x - \delta) \) for all \( \delta > 0 \) and \( x \in \mathbb{R} \). Then there is at most one function \( u \in L^2(\mathbb{R}) \) such that \( u \) satisfies (21) and changes sign exactly once, \( u > 0 \) for \( x \) near \( \infty \) and \( u < 0 \) for \( x \) near \( -\infty \).

Remark 5. The hypothesis of Theorem 3 is satisfied if \( w_1 \) is strictly increasing and \( w_2 \) is strictly decreasing.

Proof. Suppose that there is another function \( v \) which satisfies the same properties as \( u \) does. Applying a result of Wintner and Hartman [9], we know that \( u, v \in H^1(\mathbb{R}) \) and

\[
\lim_{|x| \to \infty} |u(x)| + |u'(x)| = \lim_{|x| \to \infty} |v(x)| + |v'(x)| = 0.
\]

Letting \( b \) be the zero of \( u \) and \( \beta \) be the zero of \( v \), we see that \( b \neq \beta \) by the uniqueness of positive solutions of Eq. (21) under the boundary conditions \( u(b) = 0 \) and \( u \in L^2[b, \infty) \). Without loss of generality, we may assume \( \beta - b = \delta > 0 \). It follows from a modified version of Lemma 3 that

\[
|u'(b)| \geq \frac{w_1(\beta)}{w_1(b)} |v'(\beta)| \quad (22)
\]

and

\[
|v'(\beta)| \geq \frac{w_2(-b)}{w_2(-\beta)} |u'(b)| \quad (23)
\]

However (22) is incompatible with (23) unless

\[
\frac{w_1(\beta)}{w_1(b)} \cdot \frac{w_2(-b)}{w_2(-\beta)} = 1. \quad (24)
\]

This completes the proof of the theorem since (24) violates our hypothesis.
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By

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ON THE UNIQUENESS OF SOLUTIONS OF SOME SECOND ORDER DIFFERENTIAL EQUATIONS

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§1 Introduction. In [1] we proved a uniqueness result for the solutions of

\[ (I)_a \begin{align*}
-u'' &= \lambda u - \psi(w(x)|u|)u, \quad a < x < \infty, \\
u(a)\cos \theta - u'(a)\sin \theta &= 0, \quad u \in L^2[a, \infty),
\end{align*} \tag{1}
\]

where \( \lambda > 0 \) and \( \theta \in [0, \frac{\pi}{2}] \). For notational convenience, let \( S^+_a, n \) denote the set of \( u \in C^2[a, \infty) \cap H^1[a, \infty) \) such that \( u \) satisfies \((I)_a\), \( u > 0 \) in a deleted neighborhood of \( x = a \) and \( u \) has exactly \( n - 1 \) simple zeros in \((a, \infty)\), where \( a \geq 0 \) and \( n \in \mathbb{N} \). Similarly \( S^-_{a, n} \) denotes the set \( u < 0 \) in a deleted neighborhood of \( x = a \). It has been shown in [1] that problem \((I)_a\) has at most one solution in each nodal class \( S^\pm_{a, n} \) provided that the functions \( \psi \) and \( w \) are continuously differentiable and satisfy the following conditions:

\((\psi 1)\) \( \psi(0) = 0 \) and for \( t > 0 \), \( \psi'(t) > 0 \) and \( \psi(t) \geq p \cdot t^q \) for some \( p, q > 0 \).

\((w 1)\) \( w > 0, w'(0) \geq 0 \) and \( \frac{w'}{w} \) is nondecreasing on \([0, \infty)\).

The method we used in [1] was shooting with the aid of comparison. Our aim in this note is to describe a new proof which makes use of comparison arguments only. This proof is more concise and requires less smoothness on the functions \( \psi \) and \( w \); we can treat problem \((I)_a\) when \( \psi \) and \( w \) are continuous, but not \( C^1 \) as in the preceding proof. Moreover, the new proof reveals more information about the locations of the nodes of the solutions (see Lemma 1 and Remark 4). For the existence of solutions of \((I)_a\), we refer to [2–6] and the references therein.

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There are some generalizations about utilizing such comparison arguments for showing
uniqueness of solutions of certain boundary value problems. They will be discussed in the
final section.

§2 A new proof for uniqueness. Throughout this section, the functions $\psi$ and $w$
are assumed to satisfy the following conditions.

$(\psi 2)$ $\psi(0) = 0$, $\psi$ is locally Lipschitz continuous and strictly increasing on $[0, \infty)$.

$(w 2)$ $w$ is a positive continuous function and for any $\delta > 0$, the function $g_\delta(x) = \frac{w(x+\delta)}{w(x)}$
is a nondecreasing function of $x$ for $x \in [0, \infty)$.

Remark 1. If $w$ is positive and $\frac{w'}{w}$ is nondecreasing on $[0, \infty)$, it is easy to check that
$(w 2)$ is satisfied by invoking the identity $\frac{w(x+\delta)}{w(x)} = \exp \left( \int_0^\delta \frac{w'(t)}{w(t)} \, dt \right)$.

Theorem 1. Let $\theta = 0$ and $\lambda$ be a fixed positive number. If $(\psi 2)$ and $(w 2)$ are
satisfied, then for every $a \geq 0$ and $n \in \mathbb{N}$, $S^\pm_{a,n}$ contains at most one element.

Theorem 2. Let $\theta \in (0, \frac{\pi}{2})$ and $\lambda > 0$ be fixed. In addition to $(\psi 2)$ and $(w 2)$, if $w$
is a nondecreasing function on $[0, \infty)$, then for every $a \geq 0$ and $n \in \mathbb{N}$, $S^\pm_{a,n}$ contains at
most one element.

Remark 2. (a) It is known that $u \equiv 0$ is the only solution of $(I)_a$ if $\lambda \leq 0$.
(b) In case $0 < \theta \leq \frac{\pi}{2}$, the need for the extra assumption has been indicated in [1, Remark 1.2.(a)]. The key point is that Lemma 1.30(iii) of [2] rests on the same sort of assumption and will be used in the proof.

(c) In fact, Theorems 1 and 2 are valid for an equation having a more general nonlinear
term than that of Eq (1); that is, instead of \( \psi(w(x)|u|)u \), we can consider

\[
F(x, u)u = \begin{cases} 
\psi_1(w(x)|u|)u & \text{if } u \geq 0 \\
\psi_2(w(x)|u|)u & \text{if } u < 0 
\end{cases}
\]

with both \( \psi_1 \) and \( \psi_2 \) satisfying \((\psi2)\). For simplicity we will carry out the proof for the case \( \psi_1 = \psi_2 \) only since there is no essential difference in the proof when \( \psi_1 \neq \psi_2 \).

(d) In [1] the nonlinear term of Eq (1) was written in the form \( \psi(w(x)|u|^\sigma)u \), where \( \sigma \) is a positive number. However, one can rewrite \( \psi(w(x)|u|^\sigma) \) in a simpler form \( \tilde{\psi}(\tilde{w}(x)|u|) \) with \( \tilde{w}(x) = (w(x))^{1/\sigma} \) and \( \tilde{\psi} = \psi \circ h \), where \( h(t) = t^\sigma \), since \( \psi \) and \( w \) satisfy \((\psi2)\) and \((w2)\) if and only if \( \tilde{\psi} \) and \( \tilde{w} \) do.

(e) With slight modifications in the proofs of Theorems 1 and 2, one can show the same sort of uniqueness results for the solutions of Eq.(1) on the bounded intervals under the boundary conditions \( u(a) \cos \theta - u'(a) \sin \theta = 0 \) and \( u(b) = 0 \).

(f) The new proof covers the case \( n \geq 2 \) only. We recall that the uniqueness of positive and negative solutions of \((I)_a\) as well as its counterpart on the bounded interval case has already been established in [2, Theorem 1.5].

Since the proofs of Theorem 1 and Theorem 2 are similar, only the former will be carried out. The proof of Theorem 1 rests on the following two preliminary lemmas.

**Lemma 1.** Let \( n \) be a fixed positive integer. Suppose for \( 1 \leq l \leq n \) that \( S^\pm_{a,l} \) contains at most one element for every \( a \geq 0 \). For any \( \beta > b \geq 0 \), if \( u \in S^+_b,n \) and \( v \in S^+_\beta,n \) then

\[
|u'(b)| \geq \frac{w(\beta)}{w(b)} |v'(\beta)| .
\]

(3)
Moreover, if \( z_k \) is the \( k \)-th zero of \( u \) in \((b, \infty)\) and \( \tilde{z}_k \) is the \( k \)-th zero of \( v \) in \((\beta, \infty)\) then
\[
z_k - z_{k-1} \geq \tilde{z}_k - \tilde{z}_{k-1}
\]  
(4)

for \( 1 \leq k \leq n - 1 \), where \( z_0 = b \) and \( \tilde{z}_0 = \beta \). The same result holds if \( u \in S_{b,n}^- \) and \( v \in S_{\beta,n}^- \).

For what follows in this section, we let \( V_+(a, b, \cdot) \) denote the unique positive solution of
\[
(I)_{a,b} \quad -u'' = \lambda u - \psi(w(x)|u|)u \quad , \quad a < x < b ,
\]
\[
u(a) = u(b) = 0 ,
\]
if the positive solution exists.

**Lemma 2.** Let \( \beta > b > 0 \). If \( V_+(a, b, \cdot) \) and \( V_+(a, \beta, \cdot) \) exist then
\[
|V_+'(a, b, b)| < \frac{w(\beta)}{w(b)} |V_+'(a, \beta, \beta)| .
\]  
(5)

**Proof of Theorem 1.** We proceed the proof by induction. Since \( n = 1 \) is true by Remark 2 (f), it suffices to show that

if for every \( a > 0 \) and \( 1 \leq l \leq n \), \( S_{a,l}^\pm \) contains at most one element

then so does \( S_{a,n+1}^\pm \).  

(6)

We prove (6) indirectly. Suppose that \( u, \tilde{u} \in S_{a,n+1}^+ \). Let \( b \) and \( \beta \) be the first zero of \( u \) and \( \tilde{u} \) in \((a, \infty)\) respectively. Let \( u_1(x) = u(x) \) for \( x \in [b, \infty) \) and \( \tilde{u}_1(x) = \tilde{u}(x) \) for \( x \in [\beta, \infty) \). It is clear that \( u_1 \in S_{b,n}^- \) and \( \tilde{u}_1 \in S_{\beta,n}^- \). By the induction hypothesis we know that \( b \neq \beta \). Without loss of generality, we may assume that \( \beta > b \). It follows from Lemma 1 that
\[
|u_1'(b)| \geq \frac{w(\beta)}{w(b)} |\tilde{u}_1'(\beta)| .
\]  
(7)
By the uniqueness of positive solution of \((I)_{a,b}\) we know that \(u(x) = V_+(a, b, x)\) if \(x \in [a, b]\).

Similarly \(\tilde{u}(x) = V_+(a, \beta, x)\) for \(x \in [a, \beta]\). Invoking Lemma 2 yields

\[
\frac{w(\beta)}{w(b)} |V_+'(a, \beta, \beta)| > |V_+'(a, b, b)| .
\] (8)

Combining (7) with (8) gives

\[
|u'(b)| = |u_1'(b)| \geq \frac{w(\beta)}{w(b)} |\tilde{u}'_1(\beta)| = \frac{w(\beta)}{w(b)} |\tilde{u}'(\beta)| = \frac{w(\beta)}{w(b)} |V_+''(a, \beta, \beta)|
\]

\[
> |V_+(a, b, b)| = |u'(b)| ,
\]

which is absurd. Thus \(S_{a,n+1}^+\) contains at most one element.

The proof of \(S_{a,n+1}^-\) is similar.

We are now going to prove Lemma 1 and Lemma 2. We first prove the following lemma which contains a partial result of Lemma 1.

**Lemma 3.** For any \(\beta > b \geq 0\), if \(u \in S_{b,1}^+\) and \(v \in S_{\beta,1}^+\) then

\[
|u'(b)| \geq \frac{w(\beta)}{w(b)} |v'(\beta)| .
\] (9)

The same result holds if \(u \in S_{b,1}^-\) and \(v \in S_{\beta,1}^-\).

**Proof of Lemma 3.** We prove the case \(u \in S_{b,1}^+\) and \(v \in S_{\beta,1}^+\) only. Let \(\delta = \beta - b\) and \(\eta(x) = v(x + \delta)\) for \(x \in [b, \infty)\). Then by direct computation, we get

\[
-\eta''(x) = \lambda \eta(x) - \psi \left( \frac{w(x + \delta)}{w(x)} w(x)|\eta(x)| \right) \eta(x)
\]

\[
\leq \lambda \eta(x) - \psi \left( \frac{w(\beta)}{w(b)} w(x)|\eta(x)| \right) \eta(x) ,
\]
the last inequality follows from the assumptions \((w2)\) and \((\psi2)\). Let \(\zeta(x)\) be the unique positive solution of

\[
-\zeta'' = \lambda \zeta - \psi \left( \frac{w(\beta)}{w(b)} w(x)|\zeta| \right) \zeta , \\
\zeta(b) = 0 , \ z \in L^2[b, \infty) .
\]

It follows from [2, Lemma 1.8] that

\[
|\eta'(b)| \leq |\zeta'(b)| .
\]  \hspace{1cm} (10)

Invoking [1, Lemma 2.2], we know that \(u'(b) = \frac{w(\beta)}{w(b)} \zeta'(b)\). This together with (10) yields

\[
|u'(b)| \geq \frac{w(\beta)}{w(b)} |\eta'(b)| = \frac{w(\beta)}{w(b)} |u'(\beta)|
\]

which completes the proof.

We are now in the position to prove Lemma 1 and Lemma 2. We prove Lemma 2 first.

**Proof of Lemma 2.** Let \(\delta = \beta - b\) and \(\eta(x) = V_+(a, b, x - \delta)\) for \(x \in [a + \delta, \beta]\). Arguing like in obtaining (10), we have

\[
-\eta'' \leq \lambda \eta - \psi \left( \frac{w(b)}{w(\beta)} w(x)|\eta| \right) \eta
\]

and hence

\[
|\eta'(\beta)| \leq |\zeta'(\beta)|
\]  \hspace{1cm} (11)

by making use of [2, Lemma 1.8], where \(\zeta\) is the unique positive solution of

\[
-\zeta'' = \lambda \zeta - \psi \left( \frac{w(b)}{w(\beta)} w(x)|\zeta| \right) \zeta , \\
\zeta(a + \delta) = \zeta(\beta) = 0 .
\]
Invoking [2, Lemma 1.30] and [1, Lemma 2.2], we get

\[ |V_+^\prime(a, \beta, \beta)| > |V_+^\prime(a + \delta, \beta, \beta)| = \frac{w(b)}{w(\beta)} |\zeta^\prime(\beta)|. \quad (12) \]

This together with (11) and the fact that \( \eta^\prime(\beta) = V_+^\prime(a, b, b) \) yields (5).

**Remark 3.** In applying [2, Lemma 1.30], the inequality appeared in (12) should be in the strict sense since the function \( \psi \) is assumed to be locally Lipschitz continuous. Also the part of results we invoked in this lemma need not the assumption (F2) imposed there.

In the proof of Lemma 1, the uniqueness of positive or negative solution of \((I)_a\) or \((I)_{a,b}\), and the fact just mentioned in Remark 3 will be used without further comment.

We let \( V_+(a, \infty, \cdot) \) denote the unique positive solution of \((I)_a\). The fact \( V_- = -V_+ \) will be used in the proof only for the sake of simplicity but making no essential difference in case \( \psi_1 \neq \psi_2 \), where \( V_- \) is the unique negative solution of \((I)_a\) or \((I)_{a,b}\).

**Proof of Lemma 1.** We proceed the proof by induction. For \( n = 1 \), (4) is void and (3) is true by Lemma 3. We now prove (4) for \( n = 2 \). We first prove \( \tilde{z}_1 > z_1 \) by arguing indirectly. Suppose that \( z_1 \geq \tilde{z}_1 \). It follows from Lemma 3 that

\[ |V_+^\prime(\tilde{z}_1, \infty, \tilde{z}_1)| \geq \frac{w(z_1)}{w(\tilde{z}_1)} |V_+^\prime(z_1, \infty, z_1)|. \quad (13) \]

Invoking [2, Lemma 1.30] and Lemma 2 yields

\[ |V_+^\prime(\beta, \tilde{z}_1, \tilde{z}_1)| < |V_+^\prime(b, \tilde{z}_1, \tilde{z}_1)| \leq \frac{w(z_1)}{w(\tilde{z}_1)} |V_+^\prime(b, z_1, z_1)|. \quad (14) \]

Since \( u \in S_{b,2}^+ \), we know that \( |V_+(b, z_1, z_1)| = |u'(z_1)| = |V_-(z_1, \infty, z_1)| = |V_+(z_1, \infty, z_1)|. \)

This together with (13) and (14) yields \( |V_+^\prime(\beta, \tilde{z}_1, \tilde{z}_1)| < |V_+^\prime(\tilde{z}_1, \infty, \tilde{z}_1)| = |V_-(\tilde{z}_1, \infty, \tilde{z}_1)| \) which contradicts the fact that \( v \in S_{\beta,2}^+ \). Thus we have \( \tilde{z}_1 > z_1 \).
Suppose that $z_1 - z_0 < \tilde{z}_1 - \tilde{z}_0$. Let $\delta = \tilde{z}_1 - z_1$. It follows that $\delta > \tilde{z}_0 - z_0 = \beta - b$. Let $\eta(x) = V_+(b, z_1, x - \delta)$ for $x \in [\beta, \tilde{z}_1]$. Then it follows from the argument used in proving Lemma 2 that

$$|\eta'(\tilde{z}_1)| \leq \frac{w(\tilde{z}_1)}{w(z_1)} |V_+'(b + \delta, \tilde{z}_1, \tilde{z}_1)|. \quad (15)$$

Since $b + \delta > \beta$, we know from [2, Lemma 1.30] that

$$|V_+'(b + \delta, \tilde{z}_1, \tilde{z}_1)| < |V_+'(\beta, \tilde{z}_1, \tilde{z}_1)|. \quad (16)$$

Combining (15) with (16) and using the fact that $\eta'(\tilde{z}_1) = V_+'(b, z_1, z_1)$ yield

$$|V_+'(b, z_1, z_1)| < \frac{w(\tilde{z}_1)}{w(z_1)} |V_+'(\beta, \tilde{z}_1, \tilde{z}_1)|. \quad (17)$$

On the other hand, since $\tilde{z}_1 > z_1$, it follows from Lemma 3 that

$$|V_+'(z_1, \infty, z_1)| \geq \frac{w(\tilde{z}_1)}{w(z_1)} |V_+'(\tilde{z}_1, \infty, \tilde{z}_1)|.$$

This together with $|V_+'(b, z_1, z_1)| = |V_+'(z_1, \infty, z_1)|$ and $|V_+'(\beta, \tilde{z}_1, \tilde{z}_1)| = |V_+'(\tilde{z}_1, \infty, \tilde{z}_1)|$ leads to

$$|V_+'(b, z_1, z_1)| \geq \frac{w(\tilde{z}_1)}{w(z_1)} |V_+'(\beta, \tilde{z}_1, \tilde{z}_1)|,$$

which is incompatible with (17). Therefore $z_1 - z_0 \geq \tilde{z}_1 - \tilde{z}_0$ must be true and the proof of (4) for $n = 2$ is complete.

To proceed with the proof by induction, we need to show the following:

If (3) is true for $n = j - 1$ and (4) is true for $n = j$ then (3) is true for $n = j$ and (4) is true for $n = j + 1$.  

8
Let \( u \in S_{b,j}^+ \), \( v \in S_{\beta,j}^+ \) and \( \beta > b \geq 0 \). By the induction hypothesis, we know that \( z_1 - b \geq \tilde{z}_1 - \beta \). Let \( \delta = \beta - b \) and \( \eta(x) = v(x + \delta) \) for \( x \in [b, \tilde{z}_1 - \delta] \). Arguing like in the beginning of the proof of Lemma 2, we get

\[
\frac{w(\beta)}{w(b)} |\eta'(b)| \leq |V_+^I(b, \tilde{z}_1 - \delta, b)|. \tag{18}
\]

Since \( \tilde{z}_1 - \delta \leq z_1 \), it follows from [2, Lemma 1.30] that

\[
|V_+^I(b, \tilde{z}_1 - \delta, b)| \leq |V_+^I(b, z_1, b)|. \tag{19}
\]

Combining (18) with (19) and using the facts that \( \eta'(b) = v'(\beta) \) and \( |u'(b)| = |V_+^I(b, z_1, b)| \), we conclude that (3) is true for \( n = j \).

It remains to show that (4) is true for \( n = j + 1 \). This only requires slight modifications in the proof of \( n = 2 \). Let \( u \in S_{b,j+1}^+ \), \( v \in S_{\beta,j+1}^+ \) and \( \beta > b \geq 0 \). We first prove \( \tilde{z}_1 > z_1 \).

Suppose that \( z_1 \geq \tilde{z}_1 \). Letting \( u_1(x) = u(x) \) for \( x \in (z_1, \infty) \) and \( v_1(x) = v(x) \) for \( x \in (\tilde{z}_1, \infty) \), we conclude from (3) for \( n = j \) that

\[
|v'_1(\tilde{z}_1)| \geq \frac{w(z_1)}{w(\tilde{z}_1)} |u'_1(z_1)|,
\]

since \( u_1 \in S_{z_1,j}^+ \) and \( v_1 \in S_{\tilde{z}_1,j}^+ \). By the same reasoning we have (14). Then an argument similar to that of \( n = 2 \) shows that the same kind of contradiction occurs. Therefore \( z_1 < \tilde{z}_1 \). From the induction hypothesis, we know that

\[
z_k - z_{k-1} \geq \tilde{z}_k - \tilde{z}_{k-1}
\]

for \( 2 \leq k \leq j \). It remains to show that \( z_1 - b \geq \tilde{z}_1 - \beta \). This follows from the same lines of reasoning as that of \( n = 2 \) with only \( V_+(z_1, \infty, \cdot) \) and \( V_+(\tilde{z}_1, \infty, \cdot) \) changed to \( u_1 \) and \( v_1 \) respectively. Thus the proof is complete.
Remark 4. In view of [2, Lemma 1.8] and the proof of Lemma 1, it is not difficult to show that inequalities (3) and (4) are strict if the function $g_\delta$ is strictly increasing.

§3 Final remarks. The assumption $\psi(0) = 0$ is not required in Theorems 1 and 2. In fact, the uniqueness results stated in the previous section is applicable to the solutions of the problem

$$-u'' = \phi(w(x)|u|)u , \quad a < x < \infty ,$$

$$u(a) \cos \theta - u'(a) \sin \theta = 0 , \quad u \in L^2[a, \infty) .$$

Here the function $w$ satisfies the same hypothesis as in section 2 and the function $\phi$ is assumed to be locally Lipschitz continuous and strictly decreasing on $[0, \infty)$. In particular, letting $\phi = \lambda - \psi$ in Eq. (20) leads to Eq. (1). However, it is clear that the uniqueness results proved in [1] do not cover equation like

$$-u'' = \lambda u + e^{-\tau |u|^\tau} u$$

with $\tau \in (0,1]$ and $\lambda \in \mathbb{R}$. The generalization noted above also holds for the bounded interval case.

In closing, we look at a differential equation

$$-u'' = G(x, u)u$$

on the real line. Here the function $G$ is defined by

$$G(x, y) = \begin{cases} 
\phi_1(w_1(x)|y|) & \text{if } y \geq 0 \\
\phi_2(w_2(-x)|y|) & \text{if } y < 0 .
\end{cases}$$

Assumed that $\phi_1(0) = \phi_2(0)$ and $\phi_1, \phi_2$ are locally Lipschitz continuous and strictly decreasing on $[0, \infty)$, and $w_1, w_2$ satisfy (w2). We have the following uniqueness result.
**Theorem 3.** Suppose that \( w_1(x)w_2(-x) \neq w_1(x + \delta)w_2(-x - \delta) \) for all \( \delta > 0 \) and \( x \in \mathbb{R} \). Then there is at most one function \( u \in L^2(\mathbb{R}) \) such that \( u \) satisfies (21) and changes sign exactly once, \( u > 0 \) for \( x \) near \( \infty \) and \( u < 0 \) for \( x \) near \( -\infty \).

**Remark 5.** The hypothesis of Theorem 3 is satisfied if \( w_1 \) is strictly increasing and \( w_2 \) is strictly decreasing.

**Proof.** Suppose that there is another function \( v \) which satisfies the same properties as \( u \) does. Applying a result of Wintner and Hartman [9], we know that \( u, v \in H^1(\mathbb{R}) \) and

\[
\lim_{|x| \to \infty} |u(x)| + |u'(x)| = \lim_{|x| \to \infty} |v(x)| + |v'(x)| = 0 .
\]

Letting \( b \) be the zero of \( u \) and \( \beta \) be the zero of \( v \), we see that \( b \neq \beta \) by the uniqueness of positive solutions of Eq. (21) under the boundary conditions \( u(b) = 0 \) and \( u \in L^2[b, \infty) \).

Without loss of generality, we may assume \( \beta - b = \delta > 0 \). It follows from a modified version of Lemma 3 that

\[
|u'(b)| \geq \frac{w_1(\beta)}{w_1(b)} |v'(\beta)| .
\]

and

\[
|v'(\beta)| \geq \frac{w_2(-b)}{w_2(-\beta)} |u'(b)| .
\]

However (22) is incompatible with (23) unless

\[
\frac{w_1(\beta)}{w_1(b)} \cdot \frac{w_2(-b)}{w_2(-\beta)} = 1 .
\]

This completes the proof of the theorem since (24) violates our hypothesis.
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