UNFOLDING THE ZERO STRUCTURE OF A LINEAR CONTROL SYSTEM

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Abstract

The invariant zero structure of a linear, finite dimensional, time invariant, control system is the set of invariants, under strict equivalence transformation, of the matrix pencil known as the system matrix. Many compensation techniques depend critically on aspects of the zero structure such as the relative degree, or the nonminimum phase property. In cases where the system is parameter-dependent or uncertain, small changes in parameter values may have profound effects on the appropriate compensation strategy. This occurs because many systems of practical interest have a structurally unstable zero structure, and because these systems may bound regions corresponding to distinct qualitative behavior.

The main result of this paper is an unfolding for such systems. The unfolding imbeds the structurally unstable system in a parametrized family, chosen so that the system persists as a member of every sufficiently small perturbation of the family. The unfolding uses the fewest possible parameters necessary to achieve this property. The unfolding presented in this paper is derived from results previously reported by the authors. It is based on a canonical form that preserves the system matrix structure. Therefore all perturbed systems are easily interpreted as control systems. Several examples are discussed. The unfolding is given in full in an Appendix.
1. Introduction

This paper considers linear, time-invariant, parameter-dependent, systems,

\[ \dot{x} = A(\mu)x + B(\mu)u \]  \hspace{0.5cm} (1.1a)

\[ y = C(\mu)x + D(\mu)u \]  \hspace{0.5cm} (1.1b)

Matrix pencils are matrix-valued functions of a scalar variable of the form

\[ M(s) = M_1 + M_2s. \]  \hspace{0.5cm} Two pencils, \( M(s) \) and \( N(s) \) are said to be strictly equivalent (s.e.) if there exist constant, invertible, matrices \( G_1 \) and \( G_2 \) such that \( N_1 = G_1M_1G_2^{-1} \) and \( N_2 = G_1M_2G_2^{-1} \). The system (1.1) is associated with a parameter-dependent matrix pencil, commonly called the system matrix,

\[ \Gamma(\mu) = \begin{bmatrix}
    sI - A(\mu) & B(\mu) \\
    -C(\mu) & D(\mu)
\end{bmatrix} \]  \hspace{0.5cm} (1.2)

Many equivalence relations between system matrices are of interest in control theory. Among them are those preserving the system transfer function (Rosenbrock, 1974; Fuhrmann, 1977), and those preserving only the zero structure while destroying the pole structure (Kouvaritakis and MacFarlane, 1976; Owens, 1978). The various definitions of zeros are linked to different equivalence relations. Two surveys of interest on this topic are by MacFarlane and Karcianias (1976), and by Schrader and Sain (1989). Only strict equivalence, and the corresponding invariant zero structure of (1.1) (MacFarlane and Karcianias, 1976) are considered here. The classical canonical form under s.e. transformation is the Kronecker form (Gantmacher, 1959). The invariants include the finite and infinite zero structure, but not controllability or observability. The use of the system matrix and related pencils in control theory has been extensively studied by Van Dooren (1981).
Consider the problem of controlling a family of systems, defined by a smoothly parameter-dependent linear state equation. In such situations structurally unstable systems may arise unavoidably, in the sense that all neighboring families also contain such a member. These isolated members of a family are extremely interesting, even though they are unlikely to be observed in practice, because they organize the parameter space into distinct partitions. The partitions of the parameter space may necessitate distinct compensators to achieve a specified set of control objectives. An example of such a case is given in this paper. It also may happen that compensators designed exactly at a structurally unstable point have desirable properties. A well-known example is given by Lane and Stengel for dynamic inversion for aircraft (1988). In that work, a system is artificially forced to be structurally unstable (in this instance, to have relative degree greater than one), rather than exhibit the true, but unacceptable (vis-a-vis design by dynamic inversion), property (nonminimum phase zeros).

Once a structurally unstable system has been identified, the various structures that can arise from it after small perturbation can be explored using a miniversal deformation (Arnold, 1981; Arnold, 1983). This is a parameter-dependent normal form that uses the fewest possible parameters necessary to reach all nearby structures, in a sense to be made more clear later. A miniversal deformation of a singular point is also referred to here as an unfolding, following the nomenclature of bifurcation theory (Guckenheimer and Holmes, 1983; Golubitsky and Schaeffer, 1984). An unfolding of the Kronecker form has been presented by the authors (Berg, 1992; Berg and Kwatny, 1995). However, because the Kronecker form does not preserve the system matrix structure of a pencil, it is often not the best choice for the analysis of control systems. A more suitable canonical form, strictly equivalent to the Kronecker form, was suggested independently by Thorp (1973) and Morse (1973). The unfolding of the Kronecker form, appropriately transformed, is also an unfolding of the Thorp-Morse form. Several examples of the result are presented
here, and potential applications discussed. The actual transformation is straightforward, if somewhat labor intensive, and a full listing of the result is omitted in the interest of brevity.

Tannenbaum derives a miniversal deformation of systems under similarity transformation (1981). This paper and Tannenbaum's work are alike in their methods, but Tannenbaum is chiefly interested in examining the (structurally unstable) properties of uncontrollable and unobservable systems. The resulting miniversal deformations are unrelated to those presented here. More recently, researchers studying numerical analysis of matrix pencils have turned to versal deformations for insight into robust calculation of Kronecker invariants. Demmel and Edelman (1992) calculate the dimension of the miniversal deformation, but do not derive the deformation itself. Edelman, Elmroth, and Kagstrom (1995) compute a miniversal deformation of the Kronecker form that has desirable features for numerical analysis. Their formulation has drawbacks for application to control systems, for reasons which will be discussed below.

One important issue not discussed, is how to find the structurally unstable elements of a family. In practice such points may be known, or they may be found numerically, using techniques such as those suggested by Elmroth and Kagstrom (1993). Also, the issue of compensation is largely left for future work. Only one simple example is presented.

Section 2 of this paper presents some necessary background. Section 3 reviews canonical forms of matrix pencils. Section 4 briefly describes a miniversal parametrization for matrix pencils representing linear, finite-dimensional, time-invariant, control systems. Section 5 presents examples and applications. The unfolding itself is listed in an Appendix.
2. Background and Notation

2.1 Notation

Scalar variables are indicated with italics or Greek letters, $k$, $K$, or $\kappa$. Vectors and matrices are sans serif italic characters, lower and upper case respectively, $x$ and $A$. $0$ denotes both the zero vector and the zero matrix. $O^{(m \times n)}$ denotes an $m \times n$ matrix of zeros. $0$ is used in matrices as a “space-filling” zero. $I^{(n)}$ denotes an $n \times n$ identity matrix. $H^{(n)}$ denotes an $n \times n$ matrix with ones on the first superdiagonal and zeros elsewhere. $H_{(n)}$ denotes an $n \times n$ matrix with ones on the first subdiagonal and zeros elsewhere.

2.2 Bifurcation and Unfoldings

Structurally unstable systems have properties which may be destroyed by small perturbations. It is impossible to realize such a system if parameters are subject to perturbation. It is tempting to conclude that only structurally stable systems have any real significance. Structurally unstable systems are important, however, because they can be persistently encountered in parametrized families of systems and they form the boundaries between structurally stable regions of parameter space. In the study of parameter-dependent dynamical systems, where the partitions in parameter space correspond to topologically distinct solution trajectories, a point on a boundary is called a bifurcation point. That nomenclature is adopted here. A possible objection to this usage is that the partitions separated by these boundaries do not necessarily have different zero structures. However the results of (Kwatny, Bennett and Berg, 1990) and (Berg and Kwatny, 1994), as well as an example presented in this paper, suggest that these partitions correspond to distinct closed-loop behaviors, and that the boundaries do represent true bifurcations of the compensated system.
Bifurcation points can be classified by the codimension of their orbits with respect to the particular equivalence relation—ours being strict equivalence of matrix pencils. The importance of codimension one bifurcations is clear. The situation here is graphically represented in Fig. 1. M is a singular point, of codimension one, contained in the smooth, one parameter, family S. The orbit of M, O_M, is a two dimensional manifold of equivalent codimension one bifurcation points. M (locally) partitions S into three equivalence classes, two disjoint open sets of structurally stable systems, and a structurally unstable bifurcation point. Because O_M and S intersect transversely, a slightly perturbed family, S', will also intersect O_M, now at a slightly different point M'. Since S' and S have the same partition structure, the bifurcation behavior will persist under small perturbation.

![Figure 1. One-parameter family containing codimension one singular point.](image)

Consider now bifurcations of higher codimension. Fig. 2 shows a one-dimensional manifold O_M of equivalent codimension two bifurcations. Clearly it is not possible for any one-parameter family of systems to intersect this manifold transversely. A general two-dimensional manifold S representing the family intersects the one-dimensional manifold O_M of bifurcation points transversally, as shown in Fig. 2. But the bifurcation point contained in S does not partition the family, as it did for the codimension one case. What then is its significance? The importance arises when the orbit of the higher
codimension bifurcation point forms the boundary of several orbits of bifurcation points of codimension one. Fig. 3 shows an example. Examining a neighborhood of S near the intersection point M with OM will reveal a partition into nine sets. Four of these are open quarter-planes, four are curves consisting of codimension one bifurcation points, and one is a single bifurcation point of codimension two.

![Figure 2](image1)

Figure 2. Two-parameter family containing codimension two singular point.

![Figure 3](image2)

Figure 3. Two codimension one manifolds of singular points intersect to form a codimension two manifold of singular points. The result is a partition of S into nine pieces: the quarter-planes I, II, III, IV, the four lines forming the boundaries I-II, II-III, III-IV, and IV-I, and the point M itself.
The central tool in exploring the vicinity of a bifurcation point of high codimension is an unfolding of the bifurcation (Golubitsky and Schaeffer, 1984). An unfolding is a minimal parametrization that allows all nearby structures to be reached (it may also be thought of as a parametrization that allows the bifurcation to be recovered despite any small perturbation). The number of parameters in the unfolding of a bifurcation point is equal to the codimension of the orbit of the bifurcation point (under an appropriate equivalence relation). The partitioning of the neighborhood of the bifurcation point is explored in the parameter space of the unfolding. A sketch of these partitions, along with the corresponding structures, is called a bifurcation diagram.

The bifurcation diagram of the neighborhood of a high-dimension singular point shows precisely what structures may be encountered, what structural transitions are possible, and which singular structures will be persistent in parametrized families. This is exactly the information needed by a control designer facing the task of compensating a parametrized family of systems. The unfolding of the zero structure is given by a miniversal deformation.

**2.3 Miniversal Deformations**

Let $\mathcal{M}$ be an analytic manifold and $\mathcal{G}$ be a Lie group, with identity element $I$, that acts on $\mathcal{M}$ through $G \cdot M \rightarrow N$, where $M, N \in \mathcal{M}$, $G \in \mathcal{G}$. Recall that a smooth map $\alpha: \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$, also written $\alpha_M(G)$ or $G \cdot M$, is an action of $\mathcal{G}$ on $\mathcal{M}$ if (i) $I \cdot M = M$ and (ii) $G_2 \cdot (G_1 \cdot M) = (G_2 \cdot G_1) \cdot M$. The action defines an equivalence relation by $M \sim N$ if there exists some $G$ such that $G \cdot M = N$. Consider an element $A \in \mathcal{A}$, where $\mathcal{A}$ can be either $\mathcal{M}$ or $\mathcal{G}$, and a parameter vector $c \in \mathcal{E} \subset \mathbb{C}^k$. A mapping $A: \mathcal{E} \rightarrow \mathcal{A}$, written $A(c)$, is called a deformation of $A$ if (i) the entries of $A(c)$ are power series in the elements of $c$, convergent in some neighborhood of $c = 0$, and (ii) $A(0) = A$. The space $\mathcal{E}$ is called the base of the deformation. Now consider $\mathcal{M}$ and $\mathcal{G}$, and the action of $\mathcal{G}$ on $\mathcal{M}$. Two deformations of the
same element of $\mathcal{M}$, $\mathbf{M}(c)$ and $\mathbf{N}(c)$ say, with the same base, are equivalent if there exists a deformation of the identity element of $\mathcal{E}$, $\mathbf{G}(c)$, with that base, such that $\mathbf{M}(c) = \mathbf{G}(c) \cdot \mathbf{N}(c)$. Note that $\mathbf{M}(0) = \mathbf{N}(0) = \mathbf{M}$ and $\mathbf{G}(0) = \mathbf{I}$.

Next consider a second parameter vector $d \in \mathcal{D} \subset \mathcal{C}^I$, and a mapping $\phi: \mathcal{D} \to \mathcal{E}$. Require that (i) The elements of $\phi$ be power series in the elements of $c$ convergent in some neighborhood of $c = 0$, and (ii) $\phi(0) = 0$. Then define the composition $\mathbf{M}(\phi(d))$, the mapping induced by $\mathbf{M}(c)$ under $\phi$.

**Definition** A deformation $\mathbf{M}(c)$ is called *versal* if any arbitrary deformation of the same pencil is equivalent to a deformation induced by $\mathbf{M}(c)$. That is, any $\mathbf{N}(d)$, can be written,

$$\mathbf{N}(d) = \mathbf{G}(d) \cdot \mathbf{M}(\phi(d)) \quad (2.1)$$

with $\mathbf{M}(0) = \mathbf{N}(0) = \mathbf{M}$, $\mathbf{G}(0) = \mathbf{I}$ (that is, $\mathbf{G}(d)$ is a deformation of the identity) and $\phi(0) = 0$.

If in addition the dimension of $\mathcal{E}$ is minimal, the mapping is called *miniversal* (Arnold, 1981). A miniversal deformation at a singular point is an unfolding of that singularity. The dimension of the base is the codimension of the singularity.

A simple example is useful in understanding what a miniversal deformation is, and what it isn’t. Consider the Lie Group $SO(2)$ of rotations in two dimensions, acting on the manifold $\mathbb{R}^2$. The orbit of a non-zero element of $\mathbb{R}^2$ is the circle, centered on the origin, passing through that point. The origin is equivalent only to itself, so its orbit is a single point. The situation is as shown in Fig. 4.
Consider now a neighborhood of any non-zero element $P_1$. The neighborhood is small enough that the orbits can be treated as straight lines. Note that the orbit of $P_1$ has codimension one, and so the miniversal deformation of $x$ should require one parameter. In fact, such a deformation is given by $P_1(1+\lambda)$, with tangent space spanned by $v = P_1$. Geometrically, any nearby point $P_2$ can be reached from $P_1$ by first moving radially, to the orbit of $P_2$. Then $P_2$ is reached by moving along the orbit. Structurally, any point on the orbit of $P_2$ is equivalent to $P_2$, and so a single radial degree of freedom is both necessary and sufficient to access all nearby structures.

Now consider a miniversal deformation of the origin. The orbit has codimension two, so two parameters are required. Any linear independent choice will do, say $(w_1, w_2)$. Clearly this deformation is sufficient to reach any nearby point. However, it is easy to see that it is not necessary. As in the case of nonzero vectors, all that is required is a single radial degree of freedom. Unlike the first case, however, a large rotation may be required once
the orbit is reached. As an example, $P_3$ in Fig. 4 could be reached from the origin by moving along $w_1$ to the orbit $O_3$, then one quarter turn around the orbit to $P_3$. However, the definition of versality restricts the equivalence transformations to be in a small neighborhood of identity, which this large rotation is not. Therefore versal deformations, as constructed here, may sometimes over-parametrize a point on the manifold.

The following result is required for this paper.

**Lemma 1** Let $M(\lambda)$ be a miniversal deformation of $M$. Let $N$ be equivalent to $M$, with $N = G \cdot M$. Then $N(\lambda) = G \cdot M(\lambda)$ is a miniversal deformation of $N$.

**Proof** First show that the versality of $M(\lambda)$ implies the versality of $N(\lambda)$, and *vice versa*. Let $\tilde{N}(\xi)$ be an arbitrary deformation of $N$. Consider $G^{-1} \cdot \tilde{N}(\xi)$. The entries of $G^{-1} \cdot \tilde{N}(\xi)$ are power series in $\xi$, and $G^{-1} \cdot \tilde{N}(0) = G^{-1} \cdot N = G^{-1} \cdot G \cdot M = M$. So $G^{-1} \cdot \tilde{N}(\xi)$ is a deformation of $M$. Then, by the versality of $M(\lambda)$, there exist $H(\xi)$ and $\phi(\xi)$ such that $G^{-1} \cdot \tilde{N}(\xi) = H(\xi) \cdot M(\phi(\xi))$ with, in particular, $H(0) = I$. Write $M(\lambda) = G^{-1} \cdot N(\lambda)$ and let $G$ act on both sides of the equation to get $\tilde{N}(\xi) = G \cdot H(\xi) \cdot G^{-1} \cdot N(\phi(\xi))$. Clearly $G \cdot H(\xi) \cdot G^{-1}$ is a deformation of the identity, and so $N(\lambda)$ is versal. A similar argument works in the other direction, that is, if $N(\lambda)$ is a miniversal deformation of $N$, then $M(\lambda) = G^{-1} \cdot N(\lambda)$ is a miniversal deformation of $M$. The above argument directly proves the versality portion of the theorem. Minimality follows easily by noting that, as just shown, if $\tilde{N}(\xi)$ is a versal deformation of $N$, then $\tilde{M}(\xi) = G^{-1} \cdot \tilde{N}(\xi)$ is a versal deformation of $M$. But if the base of $\tilde{N}(\xi)$ has lower dimension than the base of $N(\lambda)$ then the base of $\tilde{M}(\xi)$ has lower dimension than the base of $M(\lambda)$. This is impossible, by the minimality of $M(\lambda)$, and so the deformation $N(\lambda)$ is miniversal.

Berg and Kwatny have calculated a miniversal parametrization of the Kronecker form (Berg, 1992; Berg and Kwatny, 1995). Since the Kronecker form and the Thorp-Morse form are both canonical under s.e. transformation, this parametrization is easily adapted
to the Thorp-Morse form via Lemma 1. All that is required is to take the s.e. transformation from Kronecker form to Thorp-Morse form, and apply it to the miniversal deformation. This is a labor-intensive, but otherwise straightforward, task.

3. Canonical Forms for the System Matrix

The classical canonical form for matrix pencils is the Kronecker form (Gantmacher, 1959). The Kronecker form clearly displays the invariants of the pencil, but it has the disadvantage, for use in systems theory, that the explicit system matrix interpretation of the pencil is destroyed. An equivalent form that preserves the system matrix structure, as well as allowing a feedback interpretation of the operations of strict equivalence transformation, has been presented by Thorp (1973) and (slightly less generally) by Morse (1973). Use of the Thorp-Morse form allows the parameter space near a structurally unstable system to be partitioned into equivalent control systems. The Thorp-Morse form is briefly discussed below.

3.1 The Kronecker Form

The Kronecker form is uniquely determined by the following set of invariants: Kronecker column (or right) indices, \( \epsilon_1 = \epsilon_2 = \ldots = \epsilon_h = 0 < \epsilon_{h+1} \leq \epsilon_{h+2} \leq \ldots \leq \epsilon_p \); Kronecker row (or left) indices, \( \eta_1 = \eta_2 = \ldots = \eta_g = 0 < \eta_{g+1} \leq \eta_{g+2} \leq \ldots \leq \eta_q \); Degree of infinite divisors, \( \rho_1 = \rho_2 = \ldots = \rho_r = \rho_{r+1} \leq \rho_{r+2} \leq \ldots \leq \rho_v \); A square matrix, \( J \), in Jordan normal form, containing the finite zero structure. Then a matrix pencil in Kronecker form has structure,

\[
M(s) = \text{diag}\{ \mathcal{O}, L_{\epsilon_{h+1}}(s), \ldots, L_{\epsilon_p}(s), L_{\eta_{g+1}}^T(s), \ldots, L_{\eta_q}^T(s), H^{(\rho_1)}(s), \ldots, H^{(\rho_v)}(s), J + sI^n \}
\]

where \( L_i(s) = \begin{bmatrix} 0 & \dot{f}^{(i)} \\ \dot{f}^{(i)} & 0 \end{bmatrix} s \).
3.2 The Thorp-Morse Form

The Thorp-Morse form is as follows:

\[ \tilde{\mathbf{F}}(s) = \begin{bmatrix} sI - \tilde{A} & \tilde{B} \\ -\tilde{C} & \tilde{D} \end{bmatrix} \]

with \( \tilde{A}, \tilde{B}, \tilde{C}, \) and \( \tilde{D} \) partitioned as,

\[
\begin{align*}
\tilde{A} &= \begin{bmatrix}
\sum_{i=1}^{\tilde{n}} a_{ii} & 0 & 0 & 0 \\
0 & A_{\mathbb{I}} & 0 & 0 \\
0 & 0 & A_{\mathbb{E}} & 0 \\
0 & 0 & 0 & A_{\mathbb{\eta}}
\end{bmatrix}, &
\tilde{B} &= \begin{bmatrix}
B_{\mathbb{I}} & 0 & 0 & 0 \\
0 & B_{\mathbb{E}} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \\
\tilde{C} &= \begin{bmatrix}
C_{\mathbb{I}} & 0 & 0 & 0 \\
0 & 0 & C_{\mathbb{\eta}} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, &
\tilde{D} &= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\end{align*}
\]

Here \( A_{\mathbb{I}} = -J \) and the other blocks have the form, \( A_{\mathbb{I}} = \text{diag}\{ H_{(\rho_i)} \} \) with \( i = 1, \ldots, \), \( A_{\mathbb{E}} = \text{diag}\{ H_{(\epsilon_i)} \} \) with \( i = 1, \ldots, \), \( A_{\mathbb{\eta}} = \text{diag}\{ H_{(\eta_i)} \} \) with \( i = 1, \ldots, \), \( \alpha_{\mathbb{\eta}} \). The \( \alpha_i \) take the values, \( \alpha_{\mathbb{I}} = v - r, \alpha_{\mathbb{E}} = p - h, \alpha_{\mathbb{\eta}} = q - g \).

The corresponding nonzero partitions of \( \tilde{B} \) and \( \tilde{C} \) have the block diagonal structure,

\[ B_{\mathbb{u}} = \begin{bmatrix}
\epsilon_1 & \epsilon_2 & 0 \\
0 & \epsilon_\alpha & \epsilon_{\alpha_i}
\end{bmatrix} \quad C_{\mathbb{u}} = \begin{bmatrix}
\hat{\epsilon}_1 & 0 \\
0 & \hat{\epsilon}_\alpha & \hat{\epsilon}_{\alpha_i}
\end{bmatrix} \]

where \( \epsilon \) is a column vector with a one in the first position, and all other elements zero, and \( \hat{\epsilon} \) is a row vector with a one in the last position, and all other elements zero. The system \( (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \) is also said to be in Thorp-Morse form.
The Thorp-Morse form is related to the Kronecker form by row and column exchanges and simple sign changes, which can be expressed as s.e. transformations. Thus the Thorp-Morse form is a canonical form for matrix pencils under strict equivalence. Thorp (1973) points out that the transformation of a system matrix structure to its Thorp-Morse form can be implemented via nonsingular coordinate transformations of the state, input, and output spaces, state feedback, and output injection.

Because the system matrix structure is preserved, the Thorp-Morse form is directly associated with a canonical control system. Techniques of control design and analysis can be applied directly to the canonical system, simplifying and clarifying the design process. Because the transformation to Thorp-Morse form has a feedback interpretation it—or some approximation to it—may be incorporated into a compensator. Thus the Thorp-Morse form has potential applications in both analysis and design.

The Thorp-Morse form preserves the zeros and relative degree of the original system, but not controllability or observability or the poles of the transfer matrix. Loosely speaking, the required transformations use state feedback and output injection to slide as many poles of the system as possible under all the finite zeros. After all the zeros have been cancelled in this way, the remaining poles are shifted to the origin. A system in Thorp-Morse form is composed of five decoupled subsystems, as follows: 1) A subsystem corresponding to the infinite zero structure, and defined by \((A_{oo}, B_{oo}, C_{oo}, D_{oo})\). It is controllable and observable. It consists of decoupled chains of integrators and has no zeros. The poles can be placed by standard means, including observer based control. The structure of this subsystem suggests that the vector relative degree of the original system be defined as the orders of these chains. 2) A subsystem corresponding to Kronecker column indices and defined by \((A_{ee}, B_{ee})\). This subsystem is controllable but unobservable. It consists of decoupled chains of integrators with inputs, but no outputs. All poles can be placed by state feedback. 3) A subsystem corresponding to Kronecker
row indices and defined by \((C_{\eta\eta}, A_{\eta\eta})\). This subsystem is observable but not controllable. It consists of decoupled chains of integrators with outputs, but no inputs. All poles can be placed by output injection. 4) A subsystem corresponding to the finite zero structure and defined totally by \(A_{lf}\). This subsystem contains the finite zeros of the original system. It is completely uncontrollable and unobservable, having no inputs or outputs, and its poles are invariant. 5) A “feedforward” subsystem defined by diag\(\{f, g^{(h)}\}\). This subsystem passes \(r\) inputs unchanged to \(r\) outputs, annihilates \(h\) inputs, and generates \(g\) identically zero outputs.

3.3 On the Equivalence and Structural Stability of Zero Structures

In the discussion so far, two systems have been considered to have equivalent zero structures if their system matrices were related by strict equivalence. This definition of equivalence in turn induces a definition for structural instability, namely that a control system has a structurally unstable zero structure if an arbitrarily small perturbation can change its Kronecker form. These are not entirely satisfactory definitions from the point of view of control system design. Consider a square system with a well-defined relative degree vector of zeros. This system has only finite zeros, in number equal to that of the poles. Small perturbations, under the constraint that the perturbed system be proper, will only slightly perturb the finite zeros. As long as these variations do not move a zero to or across the imaginary axis, they have little impact on compensator design. Leaving aside the issue of causality, which will be discussed in the next section, it seems that this system should be structurally stable. But, since two pencils are s.e. only if they have identical finite elementary divisors, it is not. For this reason, the following relaxed definition of equivalence is used.

Define two control systems to have equivalent zero structures if their system matrices have identical infinite zero structure, and right and left singular structures, and if the
number of finite zeros respectively in the left half-plane, on the imaginary axis but not at
the origin, at the origin, and in the right half-plane, all coincide. Note that under this
equivalence relation a square control system with relative degree zero and no zeros on the
imaginary axis is structurally stable. This relaxed definition of equivalence, somewhat
similar to the eigenvalue “bundles” considered by Arnold (1981), is applied to one of the
examples given in Section 5.

4. A Canonical Unfolding of the Thorp-Morse Form

This section briefly discusses some aspects of the results of applying the s.e.
transformations that take Kronecker form to Thorp-Morse form, to the miniversal
deformation presented in (Berg and Kwatny, 1995). That result was derived with these
transformations in mind. It has two features that make it suitable to control system
design. The first is that it has a simple structure, that is each element of the base vector
appears exactly once. The second is that it separates perturbations of the linear part of the
pencil, called noncausal perturbations, from perturbations to the constant part of the
pencil, called causal perturbations. Fortunately the necessary s.e. transformations do not
destroy these desirable properties. Furthermore, when transformed the parameters form
compact blocks. In particular, the noncausal perturbations are entirely characterized by
pure differentiators in the $D$ matrix. Thus, the terminology is motivated by the physical
interpretation of these terms. The separation of causal and noncausal parts allows the
analyst to consider only perturbations that result in a realizable system. The miniversal
This drawback is particularly noticeable when considering questions of genericity. If
causality is not enforced, the generic zero structures are composed entirely of singular
Kronecker structures, giving an identically zero transfer matrix. With causality enforced,
as will be seen, more practical stable structures may be obtained.
A full description of the unfolding for the Thorp-Morse form is given in the Appendix. Here, the block structure corresponding to the following natural partitioning of the Thorp-Morse form is described.

\[
\tilde{F}(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E^A & E^B \\ E^C & E^D \end{bmatrix}
\]

Most entries of the partitions are unperturbed. The noncausal terms appear only in the feedforward block of the system pencil. The structure of each partition is as follows:

\[
\delta E^A = 0, \delta E^B = 0, \delta E^C = 0,
\]

\[
\delta E^D = \begin{bmatrix}
\delta E_{\infty}^D & \delta E_{\infty}^\eta & \delta E_{\infty}^I & \delta E_{\infty}^Z \\
\delta E_{\infty}^\eta & \delta E_{\eta}^D & \delta E_{\eta}^I & \delta E_{\eta}^Z \\
\delta E_{\infty}^I & \delta E_{I}^D & \delta E_{I}^I & \delta E_{I}^Z \\
\delta E_{\infty}^Z & \delta E_{\infty}^\eta & \delta E_{\infty}^I & \delta E_{\infty}^Z
\end{bmatrix}
\]

\[
\delta A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \delta A_{\infty}^f & \delta A_{\infty}^\eta & 0 \\
0 & 0 & 0 & 0 \\
0 & \delta A_{\eta}^f & \delta A_{\eta}^\eta & 0
\end{bmatrix}
\]

\[
\delta B = \begin{bmatrix}
\delta B_{\infty}^\eta & \delta B_{\infty}^\epsilon & 0 & \delta B_{\infty}^Z \\
0 & \delta B_{\epsilon}^\eta & 0 & \delta B_{\epsilon}^Z \\
0 & 0 & \delta B_{\epsilon}^\eta & \delta B_{\epsilon}^Z \\
0 & \delta B_{\eta}^\eta & 0 & \delta B_{\eta}^Z
\end{bmatrix}
\]

\[
\delta C = \begin{bmatrix}
\delta C_{\infty}^\eta & 0 & 0 & 0 \\
\delta C_{\eta}^\eta & \delta C_{\eta}^\eta & \delta C_{\eta}^\epsilon & \delta C_{\eta}^\eta \\
0 & 0 & 0 & 0 \\
\delta C_{I}^\eta & \delta C_{I}^\eta & \delta C_{I}^\eta & \delta C_{I}^\eta
\end{bmatrix}
\]

\[
\delta D = \begin{bmatrix}
\delta D_{\infty}^\eta & \delta D_{\infty}^\epsilon & 0 & \delta D_{\infty}^Z \\
\delta D_{\eta}^\eta & \delta D_{\eta}^\epsilon & 0 & \delta D_{\eta}^Z \\
0 & 0 & 0 & 0 \\
\delta D_{I}^\eta & \delta D_{I}^\epsilon & 0 & \delta D_{I}^Z
\end{bmatrix}
\]

where each partition is itself partitioned corresponding to the diagonal blocks of the Thorp-Morse form. Two cases must be treated separately. If \( g \neq 0 \), but \( h = 0 \), that is if the system pencil has one or more Kronecker row indices equal to zero but no Kronecker column indices equal to zero then no zero block appears on the diagonal of \( D \), and the zero row indices are included in \( \delta C_{\eta0}, \delta C_{\eta1}, \delta C_{\eta\epsilon}, \delta C_{\eta\eta}, \delta C_{\etaI}, \delta D_{\eta0}, \delta D_{\eta\epsilon}, \delta D_{\eta\eta}, \delta D_{\etaI} \).
The partitions $\delta C_{\eta \ell}$ and $\delta C_{\eta \ell}$ are nonzero only under these circumstances. Likewise if $g = 0$, but $h \neq 0$, that is if the system pencil has one or more Kronecker column indices equal to zero but no Kronecker row indices equal to zero then we include the zero column indices in $\delta B_{\infty \epsilon}$, $\delta B_{\ell \epsilon}$, $\delta B_{\epsilon \ell}$, $\delta B_{\eta \ell}$, $\delta D_{\infty \epsilon}$, $\delta D_{\eta \ell}$, and $\delta D_{\ell \epsilon}$. The partition $\delta B_{\ell \epsilon}$ is nonzero only when this occurs. Note that the canonically parametrized Thorp-Morse form is partitioned like the Thorp-Morse form, but the result will not be in Thorp-Morse form.

5. Examples

The properties of miniversal parametrizations make them attractive for the study of parametrized families. By replacing the original parametrization (locally, at some nominal point) by a miniversal parametrization, the analyst obtains a system that is capable of exhibiting all possible structural behavior, with minimal complexity. This reduction in complexity is further increased by choosing a miniversal parametrization with a simple form, that is, with the fewest nonzero entries. Such a parametrization is a local canonical form for parametrized families. The analysis of a parametrized family of systems will differ depending on whether the parametrization is versal or nonversal at that point. If the original parametrization is versal, then the canonical miniversal parametrization allows the designer to explore all possible behaviors, but with the fewest number of variables. This case includes arbitrary perturbations of the coefficients of the nominal system—for example, small uncertainties in every entry of the $A$, $B$, $C$, and $D$ matrices of a state-space realization. If the original parametrization is nonversal, and the constraints that cause the nonversality are not artificial or incorrect, then corresponding restrictions must be placed on the canonical form.

5.1 Regulating a Simple Parametrized Family of Linear Systems

This example is inspired by a problem studied by Kwatny, Bennett, and Berg (1990). The regulation of flight path angle of a relaxed static stability aircraft with uncertain
parameters failed when the corresponding system pencil became singular under center of mass variation, corresponding to a saddle-node bifurcation. Examination of the system revealed that the two columns of $B$ were linearly dependent at the bifurcation point. No single linear compensator can regulate the system for parameter values in a neighborhood of the bifurcation point (Berg and Kwatny, 1994). The following related linear problem exhibits many of the same points of interest.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Gamma(s,\mu) = \begin{bmatrix} s & 0 & 1 & 0 \\ 0 & s & 0 & \mu \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

The control objective is regulation of the outputs despite step changes in the inputs (representing pilot commands) and slow variation, possibly large, in the parameter $\mu$, which represents aircraft center of mass variation. A reasonable approach to this problem is through the theory of robust regulation (Francis, 1977; Kwatny and Kalnitsky, 1978). If a solution exists, it is guaranteed to handle the step commands, as required. It is also guaranteed to handle small changes in the entries of $B$. It is not guaranteed to work if $B$ undergoes a large perturbation, as may occur in this problem. Therefore some dependence on $\mu$ may need to be built into the compensator.

By Theorem 2 of Francis (1977) a structurally stable solution exists if and only if rank $\Gamma(0,\mu) = n + p = 4$, that is, if and only if $\mu \neq 0$. So at the point $\mu \neq 0$, no solution exists.

This point presents a problem to the designer. Clearly the control objectives cannot be satisfied when $\mu = 0$. But, is it possible to design a single compensator to satisfy the control objectives everywhere else? Or perhaps this behavior is pathological, an artifact of the way the model has been constructed. Can the troublesome point be removed by
slightly perturbing the model equations? To answer these questions, consider the canonical unfolding of the system matrix \( \Gamma(0, \mu) \). The general unfolding is, for unconstrained systems,

\[
\Gamma(s, c) = \begin{bmatrix}
s & 0 & 1 & 0 \\
0 & s & 0 & c_1 \\
-1 & 0 & c_6 + c_2 & c_7 + c_3 \\
0 & -1 & c_8 + c_4 & c_9 + c_5
\end{bmatrix}
\] (5.3)

Under these general conditions, the system \( \Gamma(0, \mu) \) has codimension nine, and in fact most nearby one-parameter families will not contain this singularity. Therefore, if there are no other constraints on the system, the designer can consider a slightly perturbed model.

What happens if the system is constrained to be causal? The unfolding becomes,

\[
\Gamma(s, c) = \begin{bmatrix}
s & 0 & 1 & 0 \\
0 & s & 0 & c_1 \\
-1 & 0 & c_2 & c_3 \\
0 & -1 & c_4 & c_5
\end{bmatrix}
\] (5.4)

Now the only stable families containing the problematic structure depend on at least five parameters. So again in this case, the designer may conclude that setting the \( D \) matrix to \( 0 \) is an error in modeling, and that in practice only systems with an invertible \( D \) will occur. Assume now, and for the remainder of this problem, that it is physically realistic to enforce the condition that the perturbed system be strictly proper. The unfolding then is,

\[
\Gamma(s, c) = \begin{bmatrix}
s & 0 & 1 & 0 \\
0 & s & 0 & c_1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\] (5.5)

In this case the original parametrization coincides with the unfolding. By the properties of the unfolding, then, the unstable structure associated with the case \( \mu = c_1 = 0 \) will be persistent. Therefore, it is not an artifact of the modeling process, and must be considered in the design process. This is consistent with the results of Kwatny, Bennett, and Berg.
(1990), where it was observed in a computer simulation study involving only one parameter.

The unfolding can be used to draw a bifurcation diagram. The case $c_1 = 0$ has Kronecker indices $\varepsilon_1 = 0$, $\eta_1 = 1$, $\rho_1 = 2$. The Thorp-Morse form corresponds to two decoupled integrators, one of which is uncontrollable. For $c_1 \neq 0$, whether negative or positive,

$$
\begin{bmatrix}
    s & 0 & 1 & 0 \\
    0 & s & 0 & e_1 \\
    -1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0
\end{bmatrix} \sim
\begin{bmatrix}
    s & 0 & 1 & 0 \\
    0 & s & 0 & 1 \\
    -1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0
\end{bmatrix}
$$

with Kronecker indices $\rho_1 = 2$, $\rho_2 = 2$. The Thorp-Morse form corresponds to two decoupled integrators, both observable and controllable. Whether $c_1$ is real or complex there is a s.e. transformation to a system pencil with all real entries (the Thorp-Morse form, for example). By the same token, if $c_1$ is restricted to be real there is no loss of generality. So the parameter space is $\mathbb{R}^1$ and it can be partitioned into $\mathcal{C}^- = \{c_1 : c_1 < 0\}$, $\mathcal{C}^0 = \{c_1 : c_1 = 0\}$, $\mathcal{C}^+ = \{c_1 : c_1 > 0\}$. Since the structure on $\mathcal{C}^-$ and $\mathcal{C}^+$ is the same, and the closure of the union of these sets is the whole line, this structure is generic. Figure 5 shows the bifurcation diagram. The result is a bit different than the usual bifurcation diagram for a dynamical system. In particular, the structures on either side of the bifurcation point are identical. However, with respect to the closed-loop system a definite bifurcation does occur, as the following results make clear.

![Bifurcation diagram for system (5.1).](image)

Figure 5. Bifurcation diagram for system (5.1).
A compensator can be designed at any point \( c_1 = \bar{\mu} \neq 0 \) (Francis, 1977). It is easy to find a compensator that satisfies the requirements for regulation for every \( c \) in either \( \mathcal{C}^- \) or \( \mathcal{C}^+ \). It can be shown that no such compensator is also stabilizing on both \( \mathcal{C}^- \) and \( \mathcal{C}^+ \). Therefore the best that can be achieved is a compensator that regulates everywhere on either \( \mathcal{C}^- \) or \( \mathcal{C}^+ \). Again, it is easy to design such a compensator. So for the family of systems described by (5.1), a two member family of controllers is chosen by designing one at some \( \bar{\mu} > 0 \) and the other at some \( \bar{\mu} < 0 \). One of these two compensators will regulate the system for any value of \( c \) except zero. This is the best possible result, in that no smaller family exists with this property.

Note once again that any stabilizing compensator is a regulator. The strong regularity condition for the existence of a regulating compensator, for this example, that \( \mu \neq 0 \), is equivalent in this case to requiring stabilizability. In fact, the system is controllable if and only if \( \mu \neq 0 \). If \( \mu = 0 \) the uncontrollable subspace is neutrally stable, hence the system is not stabilizable. Typically, regulation and controllability and stabilizability are separate issues. One might think, therefore, that this example is somehow special. However it is not. Suppose that instead of the system given was the following

\[
A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (5.12)
\]

\[
C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

This singular system is completely controllable, though the strong regularity condition fails. But the Thorp-Morse form is

\[
A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (5.13)
\]

\[
C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

the same as the original system. Likewise consider
\[
A = \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

(5.14)

which is uncontrollable, but stabilizable. Again, the transformation to Thorp-Morse form recovers the original example. Transforming the (simultaneous) regulation problem to a (simultaneous) stabilization problem is a significant reduction in complexity. The catch is that the s.e. transformations to Thorp-Morse form may not be implementable.

5.2 Structural Stability for Families Containing Singular Elements

This section considers the unfolding of several other singular structures of related systems. In each case, all members of the parametrized family are required to be strictly proper. The following case is intuitively "less common" than the previous example.

\[
A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

(5.15)

The canonically perturbed system, under the restrictions that the perturbed system be causal, and that there be no feedforward term, is

\[
\Gamma(s, \mathcal{C}) = \begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s - c_1 & 0 & c_2 \\
  -1 & 0 & 0 & 0 \\
  0 & -c_3 & 0 & 0
\end{bmatrix}
\]

(5.16)

Clearly \(c_2\) and \(c_3\) can be restricted to be real without loss of generality, and \(c_1\) must be so restricted to ensure realizability, so set \(\mathcal{C} = \mathbb{R}^3\). Because \(\dim(\mathcal{C}) = 3\), parametrized families containing the singular case \(\Gamma(s, \mathcal{C})\) require at least three parameters for this structure to be persistent under perturbation.

The singularity is now unfolded. For \(c_1 = c_2 = c_3 = 0\) recover the nominal system.
For $c_1$ arbitrary, $c_2 \neq 0$, $c_3 = 0$,

$$
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s - c_1 & 0 & c_2 \\
  -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s & 0 & 1 \\
  -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
$$

That is, with Kronecker invariants $\varepsilon_1 = 1$, $\eta_1 = 0$, $\rho_1 = 2$.

For $c_1$ arbitrary, $c_2 = 0$, $c_3 \neq 0$,

$$
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s - c_1 & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & -c_3 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0
\end{bmatrix}
$$

That is, with Kronecker invariants $\varepsilon_1 = 0$, $\eta_1 = 1$, $\rho_1 = 2$.

For $c_1$ arbitrary, $c_2 \neq 0$, $c_3 \neq 0$,

$$
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s - c_1 & 0 & c_2 \\
  -1 & 0 & 0 & 0 \\
  0 & -c_3 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s & 0 & 1 \\
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0
\end{bmatrix}
$$

That is, with Kronecker invariants $\rho_1 = 2$, $\rho_2 = 2$. This case corresponds to two decoupled integrators, both observable and controllable. This structure is the generic one.

Finally, for $c_1$ arbitrary, $c_2 = 0$, $c_3 = 0$,

$$
\begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s - c_1 & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
$$

which is in Thorp-Morse form, with Kronecker invariants $\rho_1=2$, $J=c_1$, $\varepsilon_1=0$, $\eta_1=0$.

The parameter space is $\mathbb{R}^3$ and the above considerations split it into quadrants. A feature of this case not seen in Example 5.1 is the set of pencils $c_2 = 0$, $c_3 = 0$. This set forms a surface of codimension two and occurs at the intersection of two manifolds of codimension one. Those two manifolds are actually orbits under strict equivalence and
each a single partition. The set on the intersection, however, is composed of a continuum of partitions, each containing a single point. Using the relaxed definition of equivalence suggested in Section 3, this set splits into three equivalence classes, $c_1 < 0$, $c_1 = 0$, $c_1 > 0$.

The bifurcation diagram is shown in Fig. 6.

![Bifurcation Diagram](image)

**Fig. 6. Bifurcation diagram for system (5.16).**

Next consider

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$  \hspace{1cm} (5.17)

The canonical unfolding, under the restrictions that the perturbed system be strictly proper, is

$$\Gamma(s, \mathbf{c}) = \begin{bmatrix} s & 0 & 1 & 0 \\ c_1 & s & c_2 & c_3 \\ 0 & -1 & 0 & 0 \\ -c_4 & 0 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (5.18)
Because $\dim(\mathcal{E}) = 4$, persistence of the nominal structure under perturbation requires four parameter families, and so is "more rare" than the previous two. List all possible equivalence classes in a neighborhood of $\Gamma(s,0)$:

For $c_1 = c_2 = c_3 = c_4 = 0$ recover the nominal system.

For $c_1, c_2$ arbitrary, $c_3 \neq 0$, $c_4 = 0$,

$$\begin{bmatrix}
  s & 0 & 1 & 0 \\
  c_1 & s & c_2 & c_3 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s & 0 & 1 \\
  -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}$$

When $c_1, c_2$ arbitrary, $c_3 \neq 0$, $c_4 \neq 0$ the result is a generic structure,

$$\begin{bmatrix}
  s & 0 & 1 & 0 \\
  c_1 & s & c_2 & c_3 \\
  0 & -1 & 0 & 0 \\
  -c_4 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s & 0 & 1 \\
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0
\end{bmatrix}$$

For $c_1, c_2$ arbitrary, $c_3 = 0$, $c_4 \neq 0$,

$$\begin{bmatrix}
  s & 0 & 1 & 0 \\
  c_1 & s & c_2 & 0 \\
  0 & -1 & 0 & 0 \\
  -c_4 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
  s & 0 & 1 & 0 \\
  0 & s & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0
\end{bmatrix}$$

For $c_1 \neq 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$,

$$\begin{bmatrix}
  s & 0 & 1 & 0 \\
  c_1 & s & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
  s & 0 & 1 & 0 \\
  1 & s & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}$$

Finally, for $c_1$ arbitrary, $c_2 \neq 0$, $c_3 = 0$, $c_4 = 0$,

$$\begin{bmatrix}
  s & 0 & 1 & 0 \\
  c_1 & s & c_2 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
  s & -\frac{c_1}{c_2} & 0 & 0 \\
  0 & s & 1 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}$$
Here $c_1/c_2$ must be real to ensure realizability. On the other hand $c_3$ and $c_4$ can be restricted to be real without loss of generality.

The “most degenerate” system of this order is,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This system has canonically perturbed system matrix,

$$\Gamma(s, c) = \begin{bmatrix} s-c_1 & 0 & c_2 & c_3 \\ 0 & s-c_4 & c_5 & c_6 \\ -c_7 & -c_8 & 0 & 0 \\ -c_9 & -c_{10} & 0 & 0 \end{bmatrix}$$

and so requires ten parameter families for persistence. To ensure realizability, $c_1$ and $c_4$ must either be real or a complex conjugate pair.

### 6. Conclusions

This paper applies an unfolding of the zero structure of finite-dimensional, linear, time-invariant, control systems. The unfolding has been arranged so that common constraints, such as requiring the system to be proper or strictly proper, can be easily enforced.

The unfolding was applied in two ways. Several structurally unstable systems were examined, and their codimensions calculated. Based on calculations of this type, some structures can be shown to be artifacts of the modeling process. If the parametrized family containing the structure is of dimension less than the codimension of the structurally unstable member, then that member will never be observed in practice, and the designer can safely ignore it, and work with a slightly perturbed model. Otherwise, the degenerate structure may be persistent, and the designer must account for the
singularity. One such example was presented, with the unfolding used to design a family of compensators. The family of compensators was, given the basic approach, the smallest possible capable of achieving the specified control objectives at all points where necessary conditions for those objectives were satisfied.

References


**Appendix**

This appendix lists each of the blocks referred to in Section 4.

**A.1 The Partition $\delta A_{ff}$**

The perturbation of $\delta A_{ff}$ is block diagonal with two blocks, $\delta A_{ff} = \text{diag}\{ \delta A_{f1}, \delta A_{f0} \}$. The first corresponds to the diagonal matrix of distinct eigenvalues, and is itself a diagonal matrix of independent parameters.
\[
\delta A_{\Omega^k} = \begin{pmatrix}
\delta \lambda_1 & 0 \\
\delta \lambda_2 & \ddots \\
0 & \ddots & \delta \lambda_k \\
\end{pmatrix}^k
\]

The second corresponds to \( J_0 \) and has the following form,

\[
\delta A_{\Omega^0} = \begin{array}{c|c|c|c|c}
\hline
n_{k+1} & n_{k+2} & n_{k+\gamma} \\
\hline
\hline
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

where each \( \ast \) represents an independent parameter. This is exactly the Jordan-Arnold form for \( J_0 \) (Arnold, 1981). In fact, since the Jordan-Arnold deformation of a diagonal matrix with distinct entries is a diagonal matrix of independent parameters, \( \delta A_{\Omega^k} \) is the Jordan-Arnold deformation of \( J \).

A.2 The Partition \( \delta A_{fe} \)

Every block of \( \delta A_{fe} \) has its last column filled with independent parameters.
A.3 The Partition $\delta A_{\eta f}$

The first row of each block of $\delta A_{\eta f}$ is filled with independent parameters.

A.4 The Partition $\delta A_{\eta e}$

The first row of each block of $\delta A_{\eta e}$ is filled with independent parameters.
\[ \delta A_{\eta \epsilon} = \begin{bmatrix} \vdots & \vdots & \eta_{g+1} \\ \vdots & \vdots & \eta_{g+2} \\ \vdots & \vdots & \eta_{q} \end{bmatrix} \]

### A.5 The Partition $\delta B_{\infty \infty}$

Consider the partition $[\delta B_{\infty \infty}]_{ij}$ (which is a column vector). If $\rho_i \geq \rho_j$ then $[\delta B_{\infty \infty}]_{ij} = 0$. If $\rho_i < \rho_j$ then the first element of $[\delta B_{\infty \infty}]_{ij}$ is 0 and the rest are independent parameters. The infinite elementary divisors are arranged from lowest multiplicity to highest, so the partitions of $\delta B_{\infty \infty}$ on and below the "partition diagonal" are zero.

\[ \delta B_{\infty \epsilon} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \rho_{r+1} \\ \rho_{v-1} \\ \rho_v \end{bmatrix} \]

### A.6 The Partition $\delta B_{\infty \epsilon}$

The first element of each partition of $\delta B_{\infty \epsilon}$ is zero, and the rest of the entries are independent parameters.
\[
\delta B_{\infty} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
* & * & \cdots & * \\
* & * & \cdots & * \\
0 & 0 & \cdots & 0 \\
* & * & \cdots & * \\
* & * & \cdots & * \\
\end{bmatrix}
\begin{array}{c}
\rho_{n+1} \\
\rho_{n-1} \\
\rho_n \\
\end{array}
\]

If \( g = 0 \), but \( h \neq 0 \) then the form of this partition does not change, but the first \( h \) columns that would have otherwise been part of \( \delta B_{\infty} \) must also be included. So this partition then has \( p \) columns.

**A.7 The Partition \( \delta B_{\infty Z} \)**

The first row of each block of \( \delta B_{\infty Z} \) is zero, and the remaining entries are independent parameters. \( \delta B_{\infty Z} \) has the form,

\[
\delta B_{\infty Z} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
* & * & \cdots & * \\
* & * & \cdots & * \\
\end{bmatrix}
\begin{array}{c}
\rho_{r+1} \\
\rho_{r+2} \\
\rho_n \\
\end{array}
\]
A.8 The Partition $\delta B_{fe}$

If $g = 0$, but $h \neq 0$ then the following nonzero piece of $B_{fe}$ must be included. Every entry of $\delta B_{fe}$ contains an independent parameter. This is just the partition $\delta B_{fz}$, but in this case there will be no $D_{zz}$, and therefore no $B_{fz}$.

\[
\delta B_{fe} = \begin{bmatrix}
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
\vdots \\
* * * * * \\
\end{bmatrix}
\]

A.9 The Partition $\delta B_{fz}$

The deformation $\delta B_{fz}$ is filled with independent parameters.

\[
\delta B_{fz} = \begin{bmatrix}
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
* * * * * \\
\vdots \\
* * * * * \\
\end{bmatrix}
\]
A.10 The Partition $\delta B_{ee}$

Consider the partition $[\delta B_{ee}]_{ij}$. If $\varepsilon_j + 1 \geq \varepsilon_i$ then $[\delta B_{ee}]_{ij} = 0$. If $\varepsilon_j + 1 < \varepsilon_i$ then the first entry of $[\delta B_{ee}]_{ij}$ is zero, and the next $\varepsilon_i - \varepsilon_j - 1$ entries are independent parameters. The Kronecker column indices are arranged from smallest to largest, so the partitions of $\delta B_{ee}$ on and above the “partition diagonal” are zero. The exact form of $\delta B_{ee}$ depends on the values of the column indices, but typically looks something like the following:

$$
\delta B_{ee} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & * & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & * \\
* & * & \cdots & 0 \\
* & * & \cdots & 0
\end{bmatrix}
\begin{array}{c}
\varepsilon_{h+1} \\
\varepsilon_{h+2} \\
\varepsilon_p
\end{array}
$$

If $g = 0$, but $h \neq 0$ then the form of this partition does not change, but the first $h$ columns that would have other wise been part of $\delta B_{ez}$ must also be included. This partition then has $p$ columns. The number of rows, however, does not increase. The additional $h$ columns, concatenated to the front of $\delta B_{ee}$, are,
A.11 The Partition $\delta B_{ez}$

The first row in each block of $\delta B_{ez}$ is zero, and the remaining entries are independent parameters.

A.12 The Partition $\delta B_{\eta\epsilon}$

Every entry of $\delta B_{\eta\epsilon}$ contains an independent parameter.
If $g = 0$, but $h \neq 0$ then the form of this partition does not change, but the first $h$ columns that would have otherwise been part of $\delta B_{\eta z}$ must also be included. This partition then has $p$ columns. The number of rows, however, does not increase. The additional $h$ columns, concatenated to the front of $\delta B_{\eta z}$, are,

A.13 The Partition $\delta B_{\eta z}$

Every entry of $\delta B_{\eta z}$ contains an independent parameter.
A.14 The Partition $\delta C_{\infty \infty}$

Consider the partition $[\delta C_{\infty \infty}]_{ij}$ (a row vector). If $\rho_i < \rho_j$ then $[\delta C_{\infty \infty}]_{ij} = 0$. If $\rho_i \geq \rho_j$ then the last element of $[\delta C_{\infty \infty}]_{ij}$ is 0 and the rest are independent parameters. The Kronecker indices are arranged from smallest to largest, but they may be equal, so it is not sure that any part of this perturbation will be zero.

$$\delta B_{\eta\kappa} = \begin{pmatrix}
* * * * * & \eta_{r+1} \\
* * * * * & \eta_{r+2} \\
* * * * * & \vdots \\
* * * * * & \eta_q
\end{pmatrix}$$

A.15 The Partition $\delta C_{\eta \infty}$

The last column of each row vector partition of $\delta C_{\eta \infty}$ is zero. The remaining entries are filled with independent parameters.
\[ \delta C_{\eta\infty} = \begin{bmatrix} * & 0 & * & 0 & \cdots & * & * \cr * & 0 & * & 0 & \cdots & * & * \cr \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \cr * & 0 & * & 0 & \cdots & * & * \end{bmatrix} \]

If \( g \neq 0 \), but \( h = 0 \), then add the \( g \) rows that would otherwise appear in \( \delta C_{z\infty} \). Concatenate the following rows to the top of \( \delta C_{\eta\infty} \).

\[ \begin{bmatrix} * & 0 & * & 0 & \cdots & * & * \cr * & 0 & * & 0 & \cdots & * & * \cr \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \cr * & 0 & * & 0 & \cdots & * & * \end{bmatrix} \]

**A.16 The Partition \( \delta C_{\eta\ell} \)**

This partition is nonzero only if \( g \neq 0 \), but \( h = 0 \). In this case the first \( g \) rows of this partition take the place of \( \delta C_{z\ell} \), which does not appear. Like \( \delta C_{z\ell} \), \( \delta C_{\eta\ell} \) is filled with independent parameters.

\[ \delta C_{\eta\ell} = \begin{bmatrix} * & * & * & * & \cdots & * & * \cr * & * & * & * & \cdots & * & * \cr * & * & * & * & \cdots & * & * \cr \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \cr * & * & * & * & \cdots & * & * \end{bmatrix} \]

**A.17 The Partition \( \delta C_{\eta\epsilon} \)**

This partition is nonzero only if \( g \neq 0 \), but \( h = 0 \). In this case the first \( g \) rows of this partition take the place of \( \delta C_{z\epsilon} \), which does not appear. Like \( \delta C_{z\epsilon} \), the first \( g \) rows of \( \delta C_{\eta\epsilon} \)
are completely filled, with an independent parameter in each entry. The remainder of the partition is zero.

\[
\delta C_{\eta \epsilon} = \begin{bmatrix}
\varepsilon_{b+1} & \varepsilon_{b+2} & \varepsilon_p \\
*** & *** & *** \\
*** & *** & *** \\
*** & *** & *** \\
\end{bmatrix} G
\]

A.18 The Partition \(\delta C_{\eta \eta}\)

Consider the partition \(\delta C_{\eta \eta}_{ij}\). If \(\eta_{i+1} \geq \eta_j\) then \(\delta C_{\eta \eta}_{ij} = 0\). If \(\eta_{i+1} < \eta_j\) then the last entry of \(\delta C_{\eta \eta}_{ij}\) is zero, and the preceding \(\eta_j - \eta_i - 1\) entries are independent parameters. The Kronecker row indices are arranged from smallest to largest, so the partitions of \(\delta C_{\eta \eta}\) on and below the “partition diagonal” are zero. This partition might resemble the following:

\[
\delta C_{\eta \eta} = \begin{bmatrix}
\eta_{g+1} & \eta_{g+2} & \eta_q \\
0 & *0 & \cdots & ***0 \\
0 & 0 & \cdots & **0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix} \begin{array}{c} q-g \\
\end{array}
\]

If \(g \neq 0\), but \(h = 0\), then add the \(g\) rows that would otherwise appear in \(\delta C_{\eta \eta}\). Concatenate the following rows to the top of \(\delta C_{\eta \eta}\):

\[
\begin{bmatrix}
\eta_{g+1} & \eta_{g+2} & \eta_q \\
*0 & *0 & \cdots & ***0 \\
*0 & *0 & \cdots & ***0 \\
*0 & *0 & \cdots & ***0 \\
*0 & *0 & \cdots & ***0 \\
*0 & *0 & \cdots & ***0 \\
\end{bmatrix} G
\]
A.19 The Partition $\delta C_{z^\infty}$

The last column of each block of $\delta C_{z^\infty}$ is zero, and the remaining entries are independent parameters.

$$
\delta C_{z^\infty} = \left[ \begin{array}{cccc}
\rho_{r+1} & \rho_{r+2} & \rho_e \\
0 & *00 & *00 & *00 \\
0 & *00 & *00 & *00 \\
0 & *00 & *00 & *00 \\
0 & *00 & *00 & *00
\end{array} \right]_g
$$

A.20 The Partition $\delta C_{zf}$

The deformation $\delta C_{zf}$ is filled with independent parameters.

$$
\delta C_{zf} = \left[ \begin{array}{cccc}
k & n_{h+1} & n_{h+2} & n_{h+\tau} \\
*000 & *000 & *000 & *000 \\
*000 & *000 & *000 & *000 \\
*000 & *000 & *000 & *000 \\
*000 & *000 & *000 & *000
\end{array} \right]_g
$$

A.21 The Partition $\delta C_{zl}$

$\delta C_{zl}$ is completely filled, with an independent parameter in each entry.

$$
\delta C_{zl} = \left[ \begin{array}{cccc}
\varepsilon_{h+1} & \varepsilon_{h+2} & \varepsilon_p \\
*000 & *000 & *000 \\
*000 & *000 & *000 \\
*000 & *000 & *000 \\
*000 & *000 & *000
\end{array} \right]_g
$$
A.22 The Partition $\delta C_{\eta}$

The last column of each block of $\delta C_{\eta}$ is zero, and all other entries are filled with independent parameters.

$$\delta C_{\eta} = \left[ \begin{array}{ccc} \eta_{\delta+1} & \eta_{\delta+2} & \eta_{\eta} \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{array} \right]_g$$

A.23 The Feedforward Blocks $\delta D$ and $\delta E^D$

Every nonzero partition of $\delta D$ and $\delta E^D$ is completely filled with independent parameters. Note that in the special circumstance $g = 0$, but $h \neq 0$, that is if the system pencil has one or more Kronecker column indices equal to zero but no Kronecker row indices equal to zero, the number of columns of $\delta D_{\eta\epsilon}$, $\delta D_{\eta\epsilon}$, $\delta D_{I\epsilon}$, $\delta E^D_{\eta\epsilon}$, $\delta E^D_{I\epsilon}$, and $\delta E^D_{\eta\epsilon}$ increases from $p-h$ to $p$, but the partitions do not change form. In the special circumstance $g \neq 0$, but $h = 0$, that is if the system pencil has one or more Kronecker row indices equal to zero but no Kronecker column indices equal to zero, the number of rows of $\delta D_{\eta\epsilon}$, $\delta D_{\eta\epsilon}$, $\delta D_{I\epsilon}$, $\delta E^D_{\eta\epsilon}$, $\delta E^D_{I\epsilon}$, and $\delta E^D_{\eta\epsilon}$ increase from $q-g$ to $q$, but the partitions do not change form.
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