GENERATION AND PROPAGATION OF THE INTERFACE
FOR REACTION–DIFFUSION EQUATIONS

By

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Generation and Propagation of the Interface for Reaction–Diffusion Equations

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Abstract. This paper is concerned with the asymptotic behavior as $\epsilon \searrow 0$ of solutions $u^\epsilon$ of the reaction–diffusion equation in $\mathbb{R}^N \times \mathbb{R}^+$: $u_t - \Delta u + \frac{1}{\epsilon^2} \phi(u) = 0$, where $\phi$ is the derivative of a bistable potential. We show that if the initial data $u(\cdot, 0)$ has values in both domains of attraction of the potential, then an interface will develop in a short time $O(\epsilon^2 |\ln \epsilon|)$. We also show that if the wells of the potential are of equal depth, then this interface will propagate with normal velocity equal to the mean curvature of this interface. In case the depths of the wells are not equal then, at the slower time scale $s = t/\epsilon$, the interface moves with a constant speed proportional to the difference of the depths of the two wells, along the normal, towards the domain of the deeper well.

Finally, we consider the homogeneous Neumann problem ($\partial u/\partial n = 0$ on the lateral boundary) and prove that the motion of the interface is determined by its normal velocity being either its mean curvature (in case of equal depth of wells) or a constant (in case of unequal depth of wells and slower time scale), and the orthogonality of the interface to the boundary of the domain (at their intersection).

Key words. Reaction–diffusion equation, motion by mean curvature, internal layers.

1. Introduction.

In this paper, we consider the solution $u^\epsilon$ of the reaction–diffusion equation:

\begin{equation}
\mathcal{L}u \equiv u_t - \Delta u + \frac{1}{\epsilon^2} \phi(u) = 0, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}^+
\end{equation}

with the initial condition:

\begin{equation}
u(x, 0) = g(x), \quad x \in \mathbb{R}^N,
\end{equation}

where $\epsilon$ is a small positive parameter, $g(\cdot)$ is a bounded and continuous function in $\mathbb{R}^N$, and $\phi(\cdot)$ is the derivative of a bistable potential $V(\cdot)$. Here “bistable” means that the potential has exactly two local minima; the exact conditions for $\phi$ are:

\begin{equation}
\begin{cases}
\phi \in C^2(\mathbb{R}^1), \phi \text{ has exactly three zeros: } u_- < u_0 < u_+; \\
\phi(u) < 0, \forall u \in (-\infty, u_-) \cup (u_0, u_+); \\
\phi(u) > 0, \forall u \in (u_-, u_0) \cup (u_+, +\infty); \\
\phi'(u_-) > 0, \phi'(u_+) > 0, \phi'(u_0) < 0.
\end{cases}
\end{equation}

We shall derive the asymptotic behavior of $u^\epsilon$ as $\epsilon$ tends to 0.

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Equation (1.1) or its slower time scaled \((s = t/\epsilon)\) equation:

\[
(1.4) \quad u_s - \epsilon \Delta u + \frac{1}{\epsilon} \phi(u) = 0
\]

gives a dynamical theory which is well understood (see [12], [21] and the references given there). For not too large time, the effect of the diffusion term \(\Delta u\) is negligible, so that the solution \(u^\epsilon(x, t)\) of (1.1) behaves as the function \(w(g(x), t/\epsilon^2)\), where \(w(\xi, \tau)\) is the solution of the ordinary differential equation

\[
(1.5) \quad \begin{cases} 
  w_\tau(\xi, \tau) + \phi(w(\xi, \tau)) = 0, & (\xi, \tau) \in \mathbb{R}^1 \times \mathbb{R}^+, \\
  w(\xi, 0) = \xi, & \xi \in \mathbb{R}^1.
\end{cases}
\]

Since the potential \(V(\cdot)\) is bistable, the ordinary differential equation (1.5) has two stable solutions \(w = u_-\) and \(w = u_+\), and one unstable solution \(w = u_0\). Accordingly, \(u^\epsilon(x, t)\) will approach \(u_+\) if the value \(g(x)\) is in the attraction domain \((u_0, +\infty)\) of the solution \(w = u_+\) or will approach \(u_-\) if the value \(g(x)\) is in the attraction domain \((-\infty, u_0)\) of the solution \(w = u_-\). Hence, an interface develops near the set \(\Gamma_0 := \{ x \in \mathbb{R}^N \mid g(x) = u_0 \}\) in a short time period. Subsequently, the diffusion term \(\Delta u\) near the interface will become large enough to balance the kinetic effects and therefore the interface will begin to propagate.

Allen and Cahn [1] have observed the following phenomena: If the wells of the potential are not of equal depth, i.e., \(V(u_-) \neq V(u_+)\), then in the slower time scale of equation (1.4), the interface will propagate with a constant speed proportional to \(V(u_-) - V(u_+)\), along its normal, towards the domain of the deeper well. If the wells are of equal depth, the interface is almost still in the slower time scale of equation (1.4), but, in the faster time scale of equation (1.1), it will propagate with normal velocity equal to the mean curvature \(K\) of the interface.

A formal proof for the above phenomena was given by Rubinstein, Sternberg, and Keller [21]; see also Fife’s book [12] for more general cases such as system of equations.

Some rigorous proofs were recently given under various restrictions. In one dimensional case, rigorous proofs were given by Fife and Hsiao [14] and Fife [11] (notice that curvature plays no role in this case). For high dimensions \((N \geq 2)\), Bronsard and Kohn [3] considered the radially symmetric case and proved (for wells of equal depth) that the normal velocity of the motion of the interface is the mean curvature \(\frac{N-1}{R} R\), where \(R\) is the radius of the interface. Finally, DeMottoni and Schatzman ([7] and [8]) considered the general \(N\)-dimensional Cauchy problem (1.4) and (1.2). In [7], they proved that an interface develops in time \(O(\epsilon |\epsilon|)\) and the solution stays close to a certain profile of travelling wave for all time \(\leq C\sqrt{\epsilon}\). In [8], they derived an asymptotic expansion, in \(\epsilon\), of arbitrarily high order for both the solution and the interface with initial value being the profile which they obtained in [7], and obtained the error estimates for the expansion. In [7], they assume that \(\phi\) is odd and convex in \((0, +\infty)\). In [8], they need to assume that

\[
\phi \in C^k, \quad \Gamma_0 \equiv \{ x \mid g(x) = u_0 \} \in C^k, \quad k \geq \max\{6, N/2\}
\]

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to show rigorously the mean curvature motion of the interface.

In this paper, we present a new proof for the dynamical theory described above. It is based on the construction of two subsolutions and has the advantage of being quite simple, and at the same time, requiring less regularity on $\phi$ and $g$, and weaker assumption on the potential function than in [7] and [8].

Our method does not seem, however, capable of establishing the fine profile of the solution (established in [7]) when the interface is generating, although we can prove the preservation of the profile when the interface is propagating; see Remark 6.5.

Our method also works for the Neumann boundary condition problem. For this problem, we show that the motion of the interface is determined by its normal velocity being either its mean curvature (in case of equal depth of wells) or a constant (in case of unequal depth of wells and slower time scale), and the orthogonality of the interface to the boundary of the domain (at their intersection).

This paper is organized as follows. In section 2 we describe the motion of the interface and state our main results for the Cauchy problem (1.1), (1.2). In section 3 we show that the interface will develop in time $O(\epsilon^2|\ln \epsilon|)$ and in section 4 we show that the interface will propagate with normal velocity equal to its mean curvature (in case of equal depth of wells). In section 5 we prove that if the depths of the wells are equal and the initial value is uniformly below (or above) $u_0$ at far distance, then the interface will shrink and then disappear in finite time. The Neumann problem is considered in section 6.

2. Statement of the Main Result.

Let $\Gamma(t)$ be a smooth hypersurface which is the boundary of an open set $U(t) \in \mathbb{R}^N$, where either $U(t)$ or $\mathbb{R}^N - U(t)$ is bounded. Let $\Lambda(x, t)$ be the signed distance from $x \in \mathbb{R}^N$ to the hypersurface $\Gamma(t)$; namely, $\Lambda$ is the distance function in $U(t)$ and minus the distance function in $\mathbb{R}^N - U(t)$. Clearly,

$$|\nabla \Lambda| = 1.$$  

Recall that the normal velocity of the motion of the surface $\Gamma(t)$ at a point $x$ is given by $-\Lambda_t(x, t)$ (if consider $\vec{n} = \nabla \Lambda$ as the positive normal direction) and the mean curvature $K$ of that hypersurface is given by $-\Delta \Lambda(x, t)$. Hence if the normal velocity of the motion of the surface $\Gamma(t)$ is $K + \nu$ for some constant $\nu$ (positive or negative), then the equation describing that surface motion is

$$\Lambda_t(x, t) = \Delta \Lambda(x, t) - \nu, \quad \forall x \in \Gamma(t) \equiv \{ x \in \mathbb{R}^N \mid \Lambda(x, t) = 0 \}, \quad t > 0.$$  

Lemma 2.1. (1) If $\Gamma(0) \in C^{k+\alpha}$ ($k \geq 2, 0 < \alpha < 1$) is the boundary of a bounded open set, then there exists a unique solution of the motion problem (2.1) and (2.2) in time
interval \((0, T)\) for some \(T > 0\); moreover the solution \(\Lambda(x, t)\) is a \(C^{k+\alpha,(k+\alpha)/2}\) function in a small neighborhood of \(\bigcup_{0 < t < T} \Gamma(t)\), where \(\Gamma(t) \equiv \{ x \in \mathbb{R}^N | \Lambda(x, t) = 0 \}\). (2) If \(v = 0\), then either \(N = 2\) or the convexity of \(\Gamma(0)\) will imply that the solution can be extended up to \(T = T^{\text{max}}\), where \(T^{\text{max}}\) is the time at which \(\Gamma(t)\) shrinks to a single point.

Proof. The assertion (1) was proved by Hamilton [18] and by Evans and Spruck [10] in case \(v = 0\); the method in [10] applies also in case \(v \neq 0\).

The assertion (2) was proved by Gage and Hamilton [16] under the assumption that \(N = 2\) and \(\Gamma(0)\) is convex, by Grayson [17] under the assumption that \(N = 2\), and by Huisken [19] under the assumption that \(N \geq 2\) and \(\Gamma(0)\) is convex. \(\square\) (Weak global solutions have been established recently by Evans and Spruck [10], and Chen, Giga, and Goto [5].)

In this paper, we assume that there exist positive constants \(c_0\) and \(C_0\) such that:

\[
\sup_{\mathbb{R}^N} |g(x)| \leq C_0, \quad \limsup_{|x| \to \infty} g(x) < u_0 - c_0 \quad \text{(or \(\liminf_{|x| \to \infty} g(x) > u_0 + c_0\))},
\]

\[
\Gamma_0 \equiv \{ x \in \mathbb{R}^N | g(x) = u_0 \} \in C^{3+\alpha},
\]

\[
c_0 \leq |\nabla g(x)| \leq C_0, \quad \forall x \in \{ x \in \mathbb{R}^N | |g(x) - u_0| \leq c_0 \},
\]

where \(u_0\) is the intermediate zero of \(\phi\).

For convenience, we shall always assume that the sign of the distance from \(x\) to \(\Gamma_0\) is the same as that of \(g(x) - u_0\).

Note that (2.4) implies that the signed distance function from \(x\) to \(\Gamma_0\) is \(C^{3+\alpha}\); therefore if \(|\eta|\) is small enough (positive or negative), then

\[
\Gamma_{\eta} \equiv \{ x \in \mathbb{R}^N | \text{signed distance from} \ x \ \text{to} \ \Gamma_0 = \eta \} \in C^{3+\alpha}.
\]

Hence, by Lemma 2.1, there exists a unique local \(C^{3+\alpha,(3+\alpha)/2}\) solution of the surface motion problem (2.1) and (2.2) with the initial condition

\[
\Lambda(x, 0) = \text{signed distance from} \ x \ \text{to} \ \Gamma_{\eta} \left( \equiv -\eta + \text{signed distance from} \ x \ \text{to} \ \Gamma_0 \right).
\]

Thus, there exist positive constants \(v^*, \eta^*, T^*, c_1,\) and \(C_1\) such that for any \(v \in [-v^*, v^*]\) and \(\eta \in [-\eta^*, \eta^*]\), the solution of the problem (2.1), (2.2), and (2.6) exists in time interval \((0, T^*)\) and satisfies

\[
\sup_{0 \leq t \leq T^*} \left( |D_t D_x \Lambda(x, t)| + |D_{xx} \Lambda(x, t)| \right) \leq C_1.
\]

From now on, we shall denote by \(\Lambda^{v,\eta}\), or simply by \(\Lambda\), the solution of (2.1), (2.2), and (2.6), and denote by \(\Gamma^{v,\eta}(t)\), or simply by \(\Gamma(t)\), the set \(\{ x \in \mathbb{R}^N | \Lambda^{v,\eta}(x, t) = 0 \}\).
THEOREM 2.2. Assume that \( g \) satisfies (2.3)-(2.5) and \( \phi \) satisfies (1.3) and

\[
\int_{u_-}^{u_+} \phi(u) \, du = 0 \quad \text{(equal depths of the two wells)}.
\]

Let \( u^\varepsilon(x, t) \) be the solution of (1.1) with initial condition (1.2). Then for any \( k > 0 \), there exists a positive constant \( C = C(k) \) independent of \( \varepsilon \), such that for \( \nu = C\varepsilon, \eta = C\varepsilon|\ln\varepsilon| \), and \( C\varepsilon^2|\ln\varepsilon| \leq t \leq T^* \),

\[
\begin{align*}
(2.9) & \quad u^\varepsilon(x, t) > u_+ - C\varepsilon^k, & & \forall x \in \{x \in \mathbb{R}^N \mid \Lambda''(x, t) > C\varepsilon|\ln\varepsilon|\}, \\
(2.10) & \quad u^\varepsilon(x, t) < u_- + C\varepsilon^k, & & \forall x \in \{x \in \mathbb{R}^N \mid \Lambda''(x, t) < -C\varepsilon|\ln\varepsilon|\}, \\
(2.11) & \quad u_- - C\varepsilon^k \leq u^\varepsilon(x, t) \leq u_+ + C\varepsilon^k, & & \forall x \in \mathbb{R}^N.
\end{align*}
\]

The proof will be given in the following two sections.

Theorem 2.2 has a direct consequence:

COROLLARY 2.3. Under the same assumptions as in Theorem 2.2, for all \( 0 < t \leq T^* \) and \( x \notin \{x \in \mathbb{R}^N \mid \Lambda^{0,0}(x, t) = 0\} \),

\[
\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = \begin{cases} 
  u_+ & \text{if } \Lambda^{0,0}(x, t) > 0, \\
  u_- & \text{if } \Lambda^{0,0}(x, t) < 0.
\end{cases}
\]

THEOREM 2.4. Assume that \( N \geq 2 \) and that (1.3), (2.3), and (2.8) hold. Then there exists a finite time \( T^{**} > T^* \) such that

\[
\begin{align*}
(2.12) & \quad |u^\varepsilon - u_-| \leq C \exp\left(-c \frac{t-T^{**}}{\varepsilon^2}\right), & & \forall t > T^{**} \\
& \quad \text{(or } |u^\varepsilon - u_+| \leq C \exp\left(-c \frac{t-T^{**}}{\varepsilon^2}\right), & & \forall t > T^{**}\text{)}
\end{align*}
\]

for some constant \( c > 0 \). Furthermore, if \( \Gamma(0) \) is convex or if \( N = 2 \), then \( T^{**} \) and \( T^* \) can be taken arbitrarily close to \( T^{max} \) defined in Lemma 2.1.

This theorem will be proved in section 5.

REMARK 2.5. In case \( N = 1 \), Theorem 2.4 was proved by Fife [13].

In the sequel, we assume that \( \varepsilon \) is a positive constant which can be arbitrarily small and denote by \( c \) and \( C \) the various positive constants which are independent of \( \varepsilon \).


Motivated by the dynamical theory explained in section 1, we shall show that a modification \( \bar{w} \) of \( w(g(x), t/\varepsilon^2) \) (cf. (1.5)) is a subsolution of (1.1).
We begin by modifying the function $\phi$. Define $C_2 = \sup_{R^1} |\phi'|$ and

$$
(3.1) \quad \overline{\phi}(s) = (1 - \zeta(s)) \phi(s) + \zeta(s) \frac{u_0 + \varepsilon|\ln \varepsilon| - s}{|\ln \varepsilon|}, \quad \forall s \in R^1,
$$

where $\zeta \geq 0$ is a $C^0_0(R^1)$ cut-off function satisfying

$$
(3.2) \quad \begin{cases}
\zeta(s) = 0, & \forall s \in (-\infty, u_0 - \varepsilon/C_2] \cup [u_0 + 3\varepsilon|\ln \varepsilon|, +\infty), \\
\zeta(s) = 1, & \forall s \in [u_0, u_0 + 2\varepsilon|\ln \varepsilon|], \\
0 \leq \zeta'(s) \leq 2C_2\varepsilon^{-1}, & \forall s < u_0, \\
-2(\varepsilon|\ln \varepsilon|)^{-1} \leq \zeta'(s) \leq 0, & \forall s > u_0.
\end{cases}
$$

Clearly, the above properties of $\zeta$ imply that

$$
(3.3) \quad \overline{\phi} = \phi(s), \quad \forall s \in (-\infty, u_0 - \varepsilon/C_2] \cup [u_0 + 3\varepsilon|\ln \varepsilon|, +\infty),
$$

$$
(3.4) \quad \overline{\phi} = \frac{u_0 + \varepsilon|\ln \varepsilon| - s}{|\ln \varepsilon|}, \quad \forall s \in [u_0, u_0 + 2\varepsilon|\ln \varepsilon|],
$$

and

$$
(3.5) \quad |\overline{\phi}'| = \left| (1 - \zeta)\phi' - \zeta' \phi - \frac{1}{\ln \varepsilon} + \zeta' \frac{u_0 + \varepsilon|\ln \varepsilon| - s}{\ln \varepsilon} \right| \leq C
$$

since $|\phi(s)| \leq \varepsilon$ for $s \in [u_0 - \varepsilon/C_2, u_0]$ and $|\phi(s)| \leq 3C_2\varepsilon|\ln \varepsilon|$ for $s \in [u_0, u_0 + 3\varepsilon|\ln \varepsilon|]$. By the definition of $C_2$, the assumption that $\phi'(u_0) < 0$ yields

$$
(3.6) \quad \overline{\phi}(s) = \phi(s) + \zeta(s) \left[ \varepsilon + \left( \frac{1}{\ln \varepsilon} + \frac{\phi(s) - \phi(u_0)}{s - u_0} \right)(u_0 - s) \right] \geq \phi(s), \quad \forall s \in R^1
$$

and, in conjunction with (3.2),

$$
(3.7) \quad \overline{\phi}(s) \geq (1 - \zeta(s))\phi(s) + \varepsilon \zeta(s) \geq c\varepsilon, \quad \forall s \in [u_0 - \varepsilon/C_2, u_0],
$$

$$
(3.8) \quad \overline{\phi}(s) \leq (1 - \zeta(s))\phi(s) - \varepsilon \zeta(s) \leq -c\varepsilon, \quad \forall s \in [u_0 + 2\varepsilon|\ln \varepsilon|, u_0 + 3\varepsilon|\ln \varepsilon|]
$$

for some positive constant $c$ independent of $\varepsilon$.

In view of (3.3), (3.4), (3.7), and (3.8), we conclude that $\overline{\phi}$ has exactly three zeros: $u_-, u_0 + \varepsilon|\ln \varepsilon|$, and $u_+$.

We now define $\overline{w}(\xi, \tau)$ as the solution of the ordinary differential equation:

$$
(3.9) \quad \begin{cases}
\overline{w}_0(\xi, \tau) + \overline{\phi}(\overline{w}(\xi, \tau)) = 0, & (\xi, \tau) \in R^1 \times R^+, \\
\overline{w}(\xi, 0) = \xi, & \xi \in R^1.
\end{cases}
$$

Some properties of the function $\overline{w}(\xi, \tau)$ are listed in the following lemma:
Lemma 3.1. Let $\bar{w}(\xi, \tau)$ be the solution of (3.9). Then

(1) as $\tau \nearrow +\infty$,

\[
\frac{\bar{w}(\xi, \tau) - u_+}{\epsilon} \leq 0 \quad \text{for} \quad \xi > u_0 + \epsilon|\ln \epsilon|, \\
\frac{\bar{w}(\xi, \tau) - u_-}{\epsilon} \leq 0 \quad \text{for} \quad \xi < u_0 + \epsilon|\ln \epsilon|; \tag{3.10}
\]

(2) if $\tau > 0$ and $|\xi - u_0 - \epsilon|\ln \epsilon|e^{\frac{\tau}{|\ln \epsilon|}} \leq \epsilon|\ln \epsilon|$, then

\[
\bar{w}(\xi, \tau) = u_0 + \epsilon|\ln \epsilon| + (\xi - u_0 - \epsilon|\ln \epsilon|)e^{\frac{\tau}{|\ln \epsilon|}}; \tag{3.11}
\]

(3) for any $k > 0$, there exists a constant $T = T(k)$ independent of $\epsilon$, such that

\[
\bar{w}(\xi, \tau) \geq u_+ - \epsilon^k \quad \text{if} \quad \tau \geq T|\ln \epsilon| \quad \text{and} \quad \xi \geq u_0 + 3\epsilon|\ln \epsilon|, \tag{3.12}
\]

\[
\bar{w}(\xi, \tau) \geq u_- - \epsilon^k \quad \text{if} \quad \tau \geq T|\ln \epsilon| \quad \text{and} \quad \xi \geq -C_0 \left(-\sup_{x \in \mathbb{R}^N}|g(x)|\right); \tag{3.13}
\]

(4) $\bar{w}(\xi, \tau) \in C^2(\mathbb{R}^1 \times [0, +\infty))$,

\[
0 < \bar{w}_\xi(\xi, \tau) = \begin{cases} \frac{\phi(\bar{w}(\xi, \tau))}{\phi(\xi)} & \text{if} \quad \xi \neq u_-, \ u_0 + \epsilon|\ln \epsilon|, \ \text{and} \ u_+ , \\ \frac{\phi(\xi)}{e^{\frac{\tau}{|\ln \epsilon|}}} & \text{if} \quad \xi = u_0 + \epsilon|\ln \epsilon| , \\ e^{-\phi'(u_\pm)\tau} & \text{if} \quad \xi = u_\pm , \end{cases} \tag{3.14}
\]

and

\[
\bar{w}_{\xi\xi}(\xi, \tau) = \begin{cases} \frac{\phi'(\bar{w}(\xi, \tau)) - \phi'(\xi)}{\phi(\xi)} & \text{if} \quad \xi \neq u_\pm, \ u_0 + \epsilon|\ln \epsilon|, \\ 0 & \text{if} \quad \xi = u_0 + \epsilon|\ln \epsilon|, \\ e^{-\phi'(u_\pm)\tau} \left(e^{-\phi'(u_\pm)\tau} - 1\right)\phi''(u_\pm) & \text{if} \quad \xi = u_\pm; \end{cases} \tag{3.15}
\]

(5) there exists a continuous and monotone increasing function $M_1(a)$ defined on $\mathbb{R}^+$ and independent of $\epsilon$, such that for any $a > 0$,

\[
\left|\frac{\bar{w}_{\xi\xi}(\xi, \tau)}{\bar{w}_\xi(\xi, \tau)}\right| \leq \frac{M_1(a)}{\epsilon} \quad \text{if} \quad \tau \leq a|\ln \epsilon| \quad \text{and} \quad |\xi| \leq C_0. \tag{3.16}
\]
Proof. The assertion (1) is immediate since \( \overline{\phi}(s) > 0 \) for \( s \in (u_-, u_0 + \epsilon \ln \epsilon) \cup (u_+, +\infty) \) and \( \overline{\phi}(s) < 0 \) for \( s \in (-\infty, u_-) \cup (u_0 + \epsilon \ln \epsilon, u_+) \). The assertion (2) follows from (3.4) by solving the ODE explicitly. The assertion (3) follows from (3.3) and the assumption on \( \phi \) in (1.3).

Since \( \overline{w}_\xi \) satisfies \( (\overline{w}_\xi)_\tau = \overline{\phi}'(\overline{w}) \overline{w}_\xi \) and \( \overline{w}_\xi(\xi, 0) = 1 \),

\[
\overline{w}_\xi(\xi, \tau) = \exp \left( -\int_0^\tau \overline{\phi}'(\overline{w}(\xi, t)) \, dt \right).
\]

Hence if \( \overline{\phi}(\xi) \neq 0 \), then

\[
\overline{w}_\xi(\xi, \tau) = \exp \left( -\int_0^\tau \overline{\phi}'(\overline{w}(\xi, t)) \left[ \frac{\overline{w}_\tau(\xi, t)}{-\overline{\phi}(\overline{w}(\xi, t))} \right] \, dt \right)
= \exp \left( \ln \left| \overline{\phi}(\overline{w}(\xi, t)) \right| \right)_{t=0}^{t=\tau} = \frac{\overline{\phi}(\overline{w}(\xi, \tau))}{\overline{\phi}(\xi)}.
\]

On the other hand, if \( \overline{\phi}(\xi) = 0 \), then \( \overline{w}(\xi, \tau) = \xi \) and therefore \( \overline{w}_\xi(\xi, \tau) = \exp(-\overline{\phi}'(\xi)\tau) \).

The representation (3.14) thus follows.

Differentiating (3.14) with respect to \( \xi \) and noting that \( \overline{w}_\xi \) is continuous, we obtain (3.15).

Note that (3.15) implies that

\[
(3.17) \quad \left| \frac{\overline{w}(\xi, \tau)}{\overline{w}_\xi(\xi, \tau)} \right| = \left| \frac{\overline{\phi}'(\overline{w}(\xi, \tau)) - \overline{\phi}'(\xi)}{\overline{\phi}(\xi)} \right|
\]

and (3.5) shows that \( \overline{\phi}' \) is bounded; hence if \( \overline{\phi} \) stays away from zero, then \( \overline{w}_\xi \) is bounded. Thus, to prove (3.16), it suffices to consider the case where \( \xi \in \{\xi \in \mathbb{R}^1 \mid |\overline{\phi}(\xi)| < c\epsilon \} \). In view of (3.3), (3.7), and (3.8), we know that this set is contained in \( I_- \cup I_0 \cup I_+ \), where \( I_- = (u_- - C\epsilon, u_- + C\epsilon) \), \( I_0 = (u_0, u_0 + 2\epsilon \ln \epsilon) \), and \( I_+ = (u_+ - C\epsilon, u_+ + C\epsilon) \). In case \( \xi \in I_- \),

\[
|\overline{\phi}(\overline{w}(\xi, \tau)) - \overline{\phi}'(\xi)| = |\overline{\phi}''(\theta)(\overline{w}(\xi, \tau) - \xi)| \leq C|\xi - u_-| \leq C'|\phi(\xi)|
\]

by (3.3), (3.10), and the assumption that \( \phi \in C^2 \) and \( \phi'(u_-) \neq 0 \). Hence, in view of (3.17), (3.16) holds for this case. Similarly, (3.16) holds for case \( \xi \in I_+ \). It finally remains to consider the case \( \xi \in I_0 \). We divide it into three subcases:

(i) \( u_0 < \overline{w}(\xi, \tau) < u_0 + 2\epsilon \ln \epsilon \), \( \forall \tau \leq a|\ln \epsilon| \);

(ii) there exists a \( 0 < \tau_0 \leq a|\ln \epsilon| \), such that \( \overline{w}(\xi, \tau_0) = u_0 + 2\epsilon |\ln \epsilon| \);

(iii) there exists a \( 0 < \tau_0 \leq a|\ln \epsilon| \), such that \( \overline{w}(\xi, \tau_0) = u_0 \).
In the first subcase, \( \bar{w}_{\xi\xi} = 0 \) since \( w \) is explicitly expressed by (3.11). In the second subcase, (3.11) implies that

\[
\bar{w}(\xi, \tau) = u_0 + \varepsilon |\ln \varepsilon| + \varepsilon |\ln \varepsilon| e^{\frac{\tau - \tau_0}{\ln \varepsilon}} \quad \text{when} \quad \tau < \tau_0;
\]

so that \( \xi = \bar{w}(\xi, 0) = u_0 + \varepsilon |\ln \varepsilon| + \varepsilon |\ln \varepsilon| e^{-\frac{\tau_0}{\ln \varepsilon}} \). Noting that \( \tau_0 \leq a|\ln \varepsilon| \), we conclude that \( \xi \geq u_0 + \varepsilon |\ln \varepsilon| + \varepsilon |\ln \varepsilon| e^{-a} \). It follows that

\[
\bar{\phi}(\xi) = \frac{u_0 + \varepsilon |\ln \varepsilon| - \xi}{|\ln \varepsilon|} \leq -\varepsilon e^{-a}.
\]

Hence by (3.17), the inequality (3.16) holds if we choose \( M_1(a) \) large enough. The third subcase is similar to the second one. The estimate (3.16) thus follows. \( \square \)

**Theorem 3.2.** Let \( u^\varepsilon \) be the solution of (1.1) with initial condition (1.2). Assume that \( g \) satisfies

\begin{equation}
\sup_{x \in \mathbb{R}^N} |g(x)| + \sup_{\{x \in \mathbb{R}^N \mid g(x) - u_0 \leq \varepsilon_0\}} |\nabla g(x)| \leq C_0 < +\infty.
\end{equation}

(3.18)

Then for any \( k > 0 \), there exists a positive constant \( C = C(k) \) independent of \( \varepsilon \), such that

\begin{align}
(3.19) \quad & u^\varepsilon(x, t \varepsilon^2 |\ln \varepsilon|) \geq u_+ - \varepsilon^k, \quad \forall x \in \{x \in \mathbb{R}^N \mid g(x) \geq u_0 + C\varepsilon |\ln \varepsilon|\}, \\
(3.20) \quad & u^\varepsilon(x, t \varepsilon^2 |\ln \varepsilon|) \leq u_- + \varepsilon^k, \quad \forall x \in \{x \in \mathbb{R}^N \mid g(x) \leq u_0 - C\varepsilon |\ln \varepsilon|\},
\end{align}

and

\begin{equation}
(3.21) \quad u_- - \varepsilon^k \leq u^\varepsilon(x, t) \leq u_+ + \varepsilon^k, \quad \forall x \in \mathbb{R}^N, \forall t \geq T \varepsilon^2 |\ln \varepsilon|,
\end{equation}

where \( T = T(k) \) is the constant appearing in the third case of Lemma 3.1.

**Proof.** Since, by (1.5), \( w(-C_0, t/\varepsilon^2) \) is a solution of (1.1), a parabolic comparison theorem yields \( w(-C_0, t/\varepsilon^2) \leq u^\varepsilon(x, t), \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}^+ \). The left inequality in (3.21) thus follows from the asymptotic behavior of the solution of the ODE (1.5) (cf. (3.13)). Similarly, we can prove the right inequality in (3.21).

We now proceed to establish (3.19). First we assume that \( g \) satisfies

\begin{equation}
(3.22) \quad \sup_{x \in \mathbb{R}^N} \left( \varepsilon |\Delta g| + |\nabla g|^2 \right) \leq \widetilde{C}_0 < +\infty
\end{equation}

for some constant \( \widetilde{C}_0 \) independent of \( \varepsilon \).
We claim that in this case there exists a continuous and monotone increasing function $M_2(a)$ defined on $\mathbb{R}^+$ and independent of $\epsilon$ such that for all $a > 0$, \begin{equation}
abla v(x, t) \equiv \nabla w \left( g(x) - \frac{M_2(a)}{\epsilon} t, \frac{1}{\epsilon^2} t \right) \end{equation}
is a subsolution of (1.1) for $t \in [0, a \epsilon^2 |\ln \epsilon|]$. Indeed, if we take $M_2(a) = (1 + M_1(a))C_0$, then

\begin{equation}
\mathcal{L}v \equiv v_t - \Delta v + \frac{1}{\epsilon^2} \phi(v)
= -\frac{M_2(a)}{\epsilon} w_{\xi} + \frac{1}{\epsilon^2} w_{\eta} - \left[ \frac{M_2(a)}{\epsilon} |\nabla g|^2 + \nabla g \Delta g \right] + \frac{1}{\epsilon^2} \phi(w)
= -\nabla w \left\{ \frac{M_2(a)}{\epsilon} + \frac{\nabla g}{w_{\xi}} |\nabla g|^2 \right\} + \frac{1}{\epsilon^2} \left\{ \phi(w) - \frac{\phi(w)}{\nabla w} \right\}
\leq 0,
\end{equation}

where the last step follow by (3.14), (3.16), and (3.6).

Noting that $v(\cdot, 0) = \nabla (g(\cdot), 0) = g(\cdot)$, we conclude, by a parabolic comparison theorem, that

$$ u^e(x, t) \geq v(x, t) = \nabla \left( g(x) - \frac{M_2(T)}{\epsilon} t, \frac{1}{\epsilon^2} t \right), \quad \forall x \in \mathbb{R}^N, 0 < t \leq T \epsilon^2 |\ln \epsilon|. $$

Taking $t = T \epsilon^2 |\ln \epsilon|$ in the last relation and using (3.12), we get (3.19) with $C = (TM_2(T) + 3)$.

So far we have assumed that $g$ satisfies (2.22). For general $g$, we can replace it in (3.23) by $
abla$ which satisfies (2.22) and

$$ g(x) \geq \nabla(x) \begin{cases} 
\geq g(x) - C \epsilon & \text{if } |g(x) - u_0| \leq c_0/2; \\
= u_0 + c_0/2 & \text{if } g(x) \geq u_0 + c_0; \\
\inf_{x \in \mathbb{R}^N} g(x) & \text{if } g(x) < u_0 - c_0.
\end{cases} $$

Similarly, we can prove (3.20) and Theorem 3.2 follows. \[ \]


In this section we assume that the two wells of the potential are of equal depth, that is, equation (2.8) holds.

The first lemma is concerned with travelling wave solutions $U(x - ct)$ of equations of the form (1.1) for $\epsilon = 1$.  

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Lemma 4.1. Suppose that \( f(u) \in C^1(\mathbb{R}^1) \) has three zeros: \( u_0^f < u_+^f < u_-^f \). Assume that \( f(u) > 0 \) for \( u \in (u_-^f, u_0^f) \), \( f(u) < 0 \) for \( u \in (u_0^f, u_+^f) \), \( f'(u_-^f) > 0 \), \( f'(u_0^f) < 0 \), and \( f'(u_+^f) > 0 \). Then there exists a unique constant \( c = c(f) \), such that the problem

\begin{align*}
(4.1) & -U_{zz}(z) - cU_z(z) + f(U(z)) = 0, \quad -\infty < z < +\infty, \\
(4.2) & U(-\infty) = u_-^f, \quad U(+\infty) = u_+^f, \quad U(0) = u_0^f
\end{align*}

has a unique solution. Furthermore:

1. \( U_z(z) > 0 \); this implies that

\begin{equation}
(4.3) \quad u_-^f < U(z) < u_+^f, \quad \forall z \in \mathbb{R}^1,
\end{equation}

that the inverse function \( z = z(U) \) of \( U = U(z) \) exists, and that \( P \equiv U_z \) can be written as a function \( P = P(U) \), \( \forall U \in (u_-^f, u_+^f) \);

2. there exist constants \( \alpha > 0 \) and \( C > 0 \) such that

\begin{equation}
(4.4) \quad \begin{cases}
|U - u_+^f| + |U_z(z)| + |U_{zz}(z)| \leq C e^{-\alpha z}, & \forall z \in \mathbb{R}^+, \\
|U - u_-^f| + |U_z(z)| + |U_{zz}(z)| \leq C e^{\alpha z}, & \forall z \in \mathbb{R}^-;
\end{cases}
\end{equation}

3. the constant \( c \), called the speed of the travelling wave, satisfies

\begin{equation}
(4.5) \quad c(f) = \int_{u_-^f}^{u_+^f} f(u) \, du \left/ \int_{u_-^f}^{u_+^f} P(u) \, du; \right.
\end{equation}

4. if \( \int_{u_-^f}^{u_+^f} f(s) \, ds > 0 \), then \( c(f) > 0 \) and

\begin{equation}
(4.6) \quad \left(2 \int_{u_+^f}^{U} f(s) \, ds\right)^{1/2} < P(U), \quad \forall U \in (u_0^f, u_+^f).
\end{equation}

Proof. The existence and uniqueness of the system (4.1), (4.2) was proved in [20] and [2, theorem 4.1]. The assertion (1) follows from [15].

The differential equation (4.1) is equivalent to the autonomous system

\begin{equation}
(4.7) \quad \begin{cases}
U' = P, \\
P' = f(U) - cP.
\end{cases}
\end{equation}

This system has only three singular points: \((u_-^f, 0)\), \((u_+^f, 0)\), and \((u_0^f, 0)\); the first two are saddle points and the third one is a stable point.
By the linearization theory for ordinary differential equations ([6, chapter 13]), we know that in a small neighborhood of the singular point \((u^f_+, 0)\), the autonomous system (4.7) is diffeomorphic to the linear autonomous system

\[
\begin{align*}
U' &= P, \\
P' &= f'(u^f_+)(U - u^f_+) - cP.
\end{align*}
\]

Hence, as \(z \to \infty\), any stable solution of (4.7) in this neighborhood satisfies

\[
\begin{align*}
U &= u^f_+ + ae^{(\lambda^f_+ + o(1))z}, \\
P &= a(\lambda^f_+ + o(1))e^{(\lambda^f_+ + o(0))z}
\end{align*}
\]  

(4.8)

for some \(a \in \mathbb{R}^1\), where

\[
\lambda^f_+ = \frac{1}{2} \left[ -c - (c^2 + 4f'(u^f_+))^{1/2} \right].
\]  

(4.9)

Similarly, in a small neighborhood of the singular point \((u^f_-, 0)\), any stable solution of (4.7) (as \(z \to -\infty\)) satisfies

\[
\begin{align*}
U &= u^f_- + be^{(\lambda^f_- + o(1))z}, \\
P &= b(\lambda^f_- + o(1))e^{(\lambda^f_- + o(1))z}
\end{align*}
\]  

(4.10)

for some \(b \in \mathbb{R}^1\), where \(\lambda^f_- = \frac{1}{2} \left[ -c + (c^2 + 4f'(u^f_-))^{1/2} \right].
\)

By (4.2), \((U, P)\) has to have the form given by (4.8) and (4.10) for some constants \(a < 0\) and \(b > 0\). Hence, the assertion (2) follows by taking any \(\alpha \in (0, \min\{\lambda^f_-, -\lambda^f_+\})\) and \(C\) large enough.

Multiplying equation (4.1) by \(U_z\) and integrating with respect to \(z\) in \(\mathbb{R}^1\), we get

\[
0 = - \int_{-\infty}^{+\infty} U_z U_z \, dz - c \int_{-\infty}^{+\infty} U_z U_z \, dz + \int_{-\infty}^{+\infty} f(U)U_z \, dz = 0 - c \int_{u^f_-}^{u^f_+} P(U)dU + \int_{u^f_-}^{u^f_+} f(U)dU,
\]

and the relation (4.5) thus follows. Consequently, \(c(f) > 0\) if \(\int_{u^f_-}^{u^f_+} f(u) \, du > 0\).

Now consider the function \(P_1(U) \equiv \left(2 \int_{u^f_+}^{U} f(s)ds\right)^{1/2}, \quad u \in (u^f_0, u^f_+).\) Note first that as \(U \nearrow u^f_+\), \(P_1(U) = (f'(u^f_+) + o(1))^{1/2}(u^f_+ - U)\) and, by (4.8), \(P = (-\lambda^f_+ + o(1))(u^f_+ - U).\)
It follows that \( P_1(U) \leq P(U) \) when \( 0 < u^J_+ - U \ll 1 \) because \( \sqrt{f'(u^J_+)} < -\lambda^J_+ \). Since \( P_1(U) \) satisfies
\[
\frac{dP_1}{dU} - \frac{f(U)}{P_1} = 0, \quad \forall U \in (u^J_0, u^J_+) 
\]
and \( P(U) \) satisfies, by (4.7),
\[
\frac{dP}{dU} - \frac{f(U)}{P} = -c(f) < 0, \quad U \in (u_-, u_+),
\]
the inequality (4.6) follows by applying an ODE comparison theorem to the functions \( P = P(U) \) and \( P = P_1(U) \). \( \square \)

Let \( M \) be a large constant to be determined and
\[
\phi^M(u) \equiv \phi(u) + M \epsilon^2. \tag{4.11}
\]

Since \( \phi'(u_-) \neq 0, \phi'(u_0) \neq 0, \) and \( \phi'(u_+) \neq 0, \) if \( \epsilon \) is small enough, then \( \phi^M \) has exactly three zeros: \( u^-_M, u^M_0, \) and \( u^+_M \). Hence by Lemma 4.1, for \( f = \phi^M \), the system (4.1), (4.2) has a unique solution \( (c(\phi^M, U \phi^M)). \)

Since \( \phi'(u_-) > 0 \) and \( \phi'(u_+) > 0, \) \( u^-_M \) and \( u^+_M \) satisfy
\[
\tag{4.12}
u_\pm - CM \epsilon^2 \leq u^M_\pm \leq u_\pm - cM \epsilon^2.
\]

The assumption (2.8) that the potential has equal depth of wells then yields
\[
cM \epsilon^2 \leq \int_{u^-_M}^{u^+_M} \phi^M(u) \, du \leq C M \epsilon^2.
\]

Using this and (4.6) in (4.5), we get
\[
\tag{4.13}
cM \epsilon^2 \leq c(\phi^M) \leq C M \epsilon^2.
\]

Let \( v \in [-v^*, v^*] \) and \( \eta \in [-\eta^*, \eta^*] \) be any constants, and let \( \Lambda^{v, \eta} \) be the solution of (2.1), (2.2), and (2.6). Set
\[
d(x, t) = \frac{c_1}{3} h\left( \frac{3 \Lambda^{v, \eta}}{c_1} \right),
\]
where \( c_1 \) is the constant appearing in (2.7) and \( h \) is a \( C^\infty \) function satisfying
\[
\tag{4.14}
\begin{cases}
h(s) = s, & \forall s \in [-1, +1]; \\
h(s) = 2, & \forall s \in [3, +\infty); \quad h(s) = -2, \quad \forall s \in (-\infty, -3]; \\
0 \leq h'(s) \leq 1, & \forall s \in (-\infty, +\infty); \quad |h''(s)| \leq 1, \quad \forall s \in (-\infty, \infty).
\end{cases}
\]

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**Lemma 4.2.** There exist positive constants $\epsilon_0$, $M_0$, and $V_0$ such that for any $\epsilon \in (0, \epsilon_0]$, $M \in [M_0, \frac{V_0}{\epsilon}]$, $v = V_0 M \epsilon$, and $\eta \in [-\eta^*, \eta^*]$, the function

\begin{equation}
(4.15) \quad u^{\epsilon, \eta, M}(x,t) \equiv U^{\phi M} \left( \frac{d(x,t)}{\epsilon} \right), \quad (x,t) \in \mathbb{R}^N \times (0, T^*)
\end{equation}

is a subsolution of (1.1) in $\mathbb{R}^N \times (0, T^*)$, that is

\begin{equation}
(4.15) \quad \mathcal{L}u^{\epsilon, \eta, M} \equiv u_t^{\epsilon, \eta, M}(x,t) - \Delta u^{\epsilon, \eta, M}(x,t) + \frac{1}{\epsilon^2} \phi(u^{\epsilon, \eta, M}(x,t)) \leq 0, \quad \forall (x,t) \in \mathbb{R}^N \times (0, T^*].
\end{equation}

**Proof.** Set, for brevity, $U = U^{\phi M}$. Then,

\begin{equation}
(4.16) \quad \mathcal{L}u^{\epsilon, \eta, M} = \frac{d_t}{\epsilon} - \left( U'' \frac{|\nabla d|^2}{\epsilon^2} + U' \frac{\Delta d}{\epsilon} \right) + \frac{\phi(U)}{\epsilon^2}
\end{equation}

\begin{align*}
&= \frac{d_t}{\epsilon} - \Delta d + v \\
&\quad + \frac{U'}{\epsilon} \left[ c(\phi^M) \frac{\phi(U)}{\epsilon} - v \right] \\
&\quad + \frac{1}{\epsilon^2} \left[ -|\nabla d|^2 U'' - c(\phi^M) U' + \phi^M(U) \right] \\
&\quad + \frac{1}{\epsilon^2} \left[ \phi(U) - \phi^M(U) \right].
\end{align*}

By (2.2) and the first equation in (4.14), $d_t - \Delta d + v = 0$ on $\Gamma^{\nu, \eta}(t) \equiv \{ x \in \mathbb{R}^N \mid \Lambda^{\nu, \eta}(x,t) = 0 \} = \{ x \in \mathbb{R}^N \mid d(x,t) = 0 \}$. Hence, applying the mean value theorem and the assumption (2.7), we get

\begin{equation}
(4.16a) \quad |d_t(x,t) - \Delta d(x,t) + v| \leq C |d(x,t)|
\end{equation}

provided that $|v| \leq v^*$ and $|\eta| \leq \eta^*$. By (4.13),

\begin{equation}
(4.16b) \quad c(\phi^M)/\epsilon - v \leq 0
\end{equation}

if $v = V_0 M \epsilon$ for some $V_0$ large enough. In this case, $|v| \leq v^*$ (i.e. (4.16a) holds) if

\begin{equation}
(4.17) \quad M \leq \frac{v^*}{V_0 \epsilon}.
\end{equation}

Estimating the first bracket in (4.16) by (4.16a) and the second bracket by (4.16b) (noting that $U' > 0$), and substituting (4.1) into the third bracket and (4.11) into the last bracket, we then obtain

\begin{equation}
(4.17) \quad \mathcal{L}u^{\epsilon, \eta, M} \leq C \frac{|d|}{\epsilon} U' \left( \frac{d}{\epsilon} \right) + \frac{1}{\epsilon^2} \left( 1 - |\nabla d|^2 \right) U'' \left( \frac{d}{\epsilon} \right) - M.
\end{equation}
By (4.4), (2.1), and (4.14), it follows that

\[ \mathcal{L} u^{\epsilon, \eta, M} \leq C \sup_{z \in \mathbb{R}^1} (|z| e^{-\alpha|z|}) + C e^{-\alpha \frac{\epsilon^2}{\epsilon^2}} - M \leq 0 \]

provided \( M \) satisfies (4.17) and

\[ M \geq C \sup_{z \in \mathbb{R}^1} (|z| e^{-\alpha|z|}) + C \sup_{\epsilon > 0} \left( \frac{e^{-\alpha \frac{\epsilon^2}{\epsilon^2}}}{\epsilon^2} \right) \equiv M_0. \]

This completes the proof of the lemma. \( \square \)

**Proof of theorem 2.2.** By (4.12), there exists a positive constant \( M_0 \) independent of \( \epsilon \), such that if \( M \geq M_0 \),

\[ u_{-}^{M} \leq u_{-} - 2 \epsilon^2 \quad \text{and} \quad u_{+}^{M} \leq u_{-} - \epsilon^2. \]

We shall henceforth take \( M = \max\{ M_0, M_0 \} \).

For any \( \eta \in [-\eta^*, \eta^*] \) and \( x \in \{ x \in \mathbb{R}^N \mid g(x) \geq u_0 + C\epsilon|\ln \epsilon| \} \),

\[ u^{\epsilon, \eta, M}(x, 0) = U^{M} \left( \frac{d(x, 0)}{\epsilon} \right) < u_{+}^{M} \leq u_{+} - \epsilon^2 \leq u'(x, T \epsilon^2|\ln \epsilon|), \]

where the three inequalities follow respectively from (4.3), (4.18), and (3.19).

On the other hand, if \( x \notin \{ x \in \mathbb{R}^N \mid g(x) \geq u_0 + C\epsilon|\ln \epsilon| \} \), i.e., \( g(x) < u_0 + C\epsilon|\ln \epsilon| \), then, by (2.5), the signed distance from \( x \) to \( \Gamma_0 \) is less than \( C\epsilon|\ln \epsilon| \). Hence, by (2.6),

\[ \lambda^{v, \eta}(x, 0) = -\eta + \text{signed distance from } x \text{ to } \Gamma_0 \leq -C\epsilon|\ln \epsilon| \]

if \( \eta = \overline{C}\epsilon|\ln \epsilon| \) for some constant \( \overline{C} \) large enough. In this case,

\[ u^{\epsilon, \eta, M}(x, 0) \equiv U^{M} \left( \frac{d(x, 0)}{\epsilon} \right) \leq U^{M} (-C|\ln \epsilon|) \leq u_{-}^{M} + \epsilon^2 \leq u_{-} - \epsilon^2 \leq u'(x, T \epsilon^2|\ln \epsilon|), \]

where the last three inequalities follow respectively from (4.4), (4.18), and (3.21).

In summary, if \( M = \max\{ M_0, M_0 \}, \ v = V_0 \epsilon, \) and \( \eta = \overline{C}\epsilon|\ln \epsilon| \), then

\[ \mathcal{L} u^{v, \eta, M} \leq 0, \quad \forall (x, t) \in \mathbb{R}^N \times (0, T^*), \]

\[ u^{\epsilon, \eta, M}(x, 0) \leq u'(x, T \epsilon^2|\ln \epsilon|), \quad \forall x \in \mathbb{R}^N. \]

Applying a parabolic comparison theorem to \( u'(x, t + T \epsilon^2|\ln \epsilon|) \) and \( u^{\epsilon, \eta, M}(x, t) \) on \( \mathbb{R}^N \times (0, T^*) \), we conclude that

\[ u^{\epsilon, \eta, M}(x, t) \leq u'(x, t + T \epsilon^2|\ln \epsilon|), \quad \forall (x, t) \in \mathbb{R}^N \times [0, T^*]. \]
Hence, by (4.4) and (4.12), (2.9) holds for \(k = 2\).

We now proceed to prove (2.9) for arbitrary \(k > 0\), taking the advantage of the fact that it is already true for \(k = 2\).

For any \(T\epsilon^2|\ln \epsilon| \leq t_0 < T^*\), set

\[\Omega \equiv \left\{ x \in \mathbb{R}^N \mid \inf_{0 \leq t \leq T\epsilon^2|\ln \epsilon|} \Lambda^{v,\eta}(x, t + t_0) > C\epsilon|\ln \epsilon| \right\} \]

Since (2.9) holds for \(k = 2\),

\[(4.19) \quad u^\epsilon(x, t_0 + t) \geq u_+ - C\epsilon^2, \quad \forall (x, t) \in \overline{\Omega} \times [0, T\epsilon^2|\ln \epsilon|].\]

Let \(\overline{g}\) be a function satisfying (3.18) and

\[\begin{cases} 
\overline{g} \leq u_0 + C\epsilon|\ln \epsilon|, & \forall x \in \mathbb{R}^N, \\
\overline{g} = u_0, & \forall x \in \mathbb{R}^N \setminus \Omega, \\
\overline{g} = u_0 + C\epsilon|\ln \epsilon|, & \forall x \in \Omega_1 \equiv \{ x \in \mathbb{R}^N \mid \inf_{0 \leq t \leq T\epsilon^2|\ln \epsilon|} \Lambda^{v,\eta} > 2C\epsilon|\ln \epsilon| \}. 
\end{cases}\]

Consider the function \(v(x, t) \equiv \overline{w}(\overline{g}(x) - M_2(T)t/\epsilon, t/\epsilon^2)\) (cf. (3.23)). It satisfies (cf. (3.24)), \(L v(x, t) \leq 0, \forall x \in \mathbb{R}^N, 0 < t \leq T\epsilon^2|\ln \epsilon|\). Also, by (3.9) and (3.10),

\[v(x, t) \leq u_0 + C\epsilon|\ln \epsilon|, \quad \forall (x, t) \in (\Omega \times \{0\}) \cup (\partial \Omega \times [0, +\infty)).\]

Hence, by (4.19),

\[u^\epsilon(x, t_0 + t) \geq v(x, t) \quad \text{on} \quad (\Omega \times \{0\}) \cup (\partial \Omega \times [0, T\epsilon^2|\ln \epsilon|]).\]

Applying a parabolic comparison theorem to the functions \(u^\epsilon(x, t_0 + t)\) and \(v(x, t)\) in the domain \(\Omega \times (0, T\epsilon^2|\ln \epsilon|)\), we conclude that

\[u^\epsilon(x, t_0 + t) \geq v(x, t), \quad \forall (x, t) \in \Omega \times [0, T\epsilon^2|\ln \epsilon|].\]

By (3.12), we know that \(u^\epsilon(x, t_0 + T\epsilon^2|\ln \epsilon|) \geq u_+ - \epsilon^k\) if \(\overline{g}(x) \geq (3 + TM_2(T))\epsilon|\ln \epsilon|\) or, in view of (4.20), if \(x \in \Omega_1\) (note that the constant \(C_0\) in (3.18) for \(\overline{g}\) can be made to be independent of \(C\), and therefore \(M_2(T)\) is also independent of \(C\)). Since, by continuity,

\[\inf_{0 < t < T\epsilon^2|\ln \epsilon|} \Lambda^{v,\eta}(x, t_0 + t) \geq \Lambda^{v,\eta}(x, t_0 + T\epsilon^2|\ln \epsilon|) - C\epsilon^2|\ln \epsilon|,\]

if \(\overline{C}\) is large enough, we have that the set \(\{ x \in \mathbb{R}^N \mid \Lambda^{v,\eta}(x, t_0 + T\epsilon^2|\ln \epsilon|) \geq \overline{C}\epsilon|\ln \epsilon| \}\) is contained in \(\Omega_1\), so that (2.9) holds for \(t = t_0 + T\epsilon^2|\ln \epsilon|\). Since \(t_0\) can be taken arbitrarily in \([T\epsilon^2|\ln \epsilon|, T^*]\), (2.9) holds for all \(t \in [2T\epsilon^2|\ln \epsilon|, T^*]\).
The proof of (2.10) can be obtained in a similar way. In combination with (3.21), Theorem 2.2 thus follows. □

Since $\Lambda^{n, \eta}$ is continuous with respect to $v$ and $\eta$, corollary 2.3 follows from Theorem 2.2.

5. Proof of Theorem 2.4.

In this section we assume that $N \geq 2$ and that (2.8) holds.

If $\Gamma(0)$ is the boundary of a ball $B_{r_{0}}(0)$, then the solution of (2.1), (2.2) (if it exists) is obviously radially symmetric and is given by $\Lambda(x, t) = r(t) - |x|$ (or $= |x| - r(t)$) with $r(t)$ satisfying

$$\frac{dr(t)}{dt} = -\frac{N - 1}{r(t)} - v \quad \text{(or} \quad \frac{dr(t)}{dt} = -\frac{N - 1}{r(t)} + v\text{)}.$$  

Hence,

$$-\frac{N - 0.5}{r(t)} \leq \frac{dr(t)}{dt} \leq -\frac{(N - 1.5)}{r(t)} \quad \text{if} \quad |vr(t)| \leq 0.5 \quad \text{and} \quad r(t) > 0.$$ 

Consequently, $r(t)$ exists in $(0, T^{\text{max}})$ for some $T^{\text{max}}$ satisfying

$$r(t) \xrightarrow{t \to T^{\text{max}}} 0 \quad \text{as} \quad t \to T^{\text{max}} \quad \text{and} \quad \frac{r_{0}^{2}}{2(N - 0.5)} \leq T^{\text{max}} \leq \frac{r_{0}^{2}}{2(N - 1.5)} \quad \text{if} \quad |vr| \leq 0.5.$$ 

**Lemma 5.1.** Assume that, for some $p \in \mathbb{R}^{N}$, $r_{0} > 0$, and $c_{0} > 0,$

$$\sup_{x \in \mathbb{R}^{N}} g(x) \leq C_{0} \quad \text{and} \quad \sup_{x \in \mathbb{R}^{N} \setminus B_{\frac{1}{2}r_{0}}(p)} g(x) \leq u_{0} - c_{0}.$$ 

Then if $\epsilon$ is small enough, there exists a time $t(r_{0})$, satisfying

$$t(r_{0}) \leq \frac{r_{0}^{2}}{2(N - 1.5)},$$ 

such that

$$u^{\epsilon}(x, t(r_{0})) \leq u_{-} + C\epsilon, \quad \forall x \in \mathbb{R}^{N} \setminus B_{\frac{1}{2}r_{0}}(p);$$

where $\epsilon$ need not decrease to 0 as $r_{0} \to +\infty$. Similarly, if

$$\inf_{x \in \mathbb{R}^{N}} g(x) \geq -C_{0} \quad \text{and} \quad \inf_{x \in \mathbb{R}^{N} \setminus B_{\frac{1}{2}r_{0}}(p)} g(x) \geq u_{0} + c_{0},$$

then

$$u^{\epsilon}(x, t(r_{0})) \geq u_{+} - C\epsilon, \quad \forall x \in \mathbb{R}^{N} \setminus B_{\frac{1}{2}r_{0}}(p).$$
Proof. It is enough to prove (5.5). Let \( \bar{g} \geq g \) be a radially symmetric function satisfies (2.3)--(2.5) and

\[
\{ x \in \mathbb{R}^N \mid \bar{g} = u_0 \} = \partial B_{r_0}(p).
\]

Denote by \( \bar{u}^- \) the solution of (1.1) with initial condition \( \bar{u}^-(x, 0) = \bar{g}(x) \). By Theorem 2.2, \( \bar{u}^-(x,t) \leq u_- + Ce^k \) if \( |x-p| \geq r(t) + C\epsilon \ln \epsilon \) and \( t \in \{ t \in \mathbb{R}^+ \mid r(t) > r_0/16 \} \), where \( r(t) \) is the solution of (5.1) with \( v = -C\epsilon \). Hence, if

\[
\epsilon \leq 0.5/(Cr_0),
\]

then \( |v_{r_0}| \leq 0.5 \) and (5.2) holds, and consequently, (5.5) holds for \( \bar{u}^- \). Since, by comparison, \( u^- \leq \bar{u}^- \), (5.5) holds also for \( u^- \).

From (5.8) it seems that the smallness of \( \epsilon \) depends on the upper bound of \( r_0 \). Actually, it does not. In fact, by (5.1), for \( \Gamma(0) = \partial B_{r_0}(0) \), the solution \( \Lambda(x,t) \) of (2.1), (2.2) satisfies

\[
|\Lambda - \Delta \Lambda - v| = \left| \frac{N-1}{r(t)} - \frac{N-1}{|x|} \right| = \left| \frac{(N-1)|\Lambda(x,t)|}{r(t)|x|} \right|, \quad \forall (x,t) \in (\mathbb{R}^N \setminus \{0\}) \times (0, T^{\text{max}}).
\]

Now checking the proof of Lemma 4.2, we find that the constant \( M \) in Theorem 4.2 can be taken to be smaller than \( \frac{C\epsilon}{r_0^2} \) when \( r(t) \geq r_0/16 \), and consequently, \( v \) can be taken to be smaller than \( \frac{C\epsilon}{r_0^2} \). Thus |v_{r_0}| \leq 0.5 is satisfied as long as \( r(t) \geq r_0/16 \) and \( r_0 \) is not too small. 

Proof of Theorem 2.4. By the inequalities in (2.3), we can find two balls \( B_{r_0}(p_1) \) and \( B_{r_0}(p_2) \) with \( |p_1 - p_2| \geq r_0/4 \) such that (5.3) (or (5.6)) is true for each of these two balls. Since \( (\mathbb{R}^N \setminus B_{r_0/8}(p_1)) \cup (\mathbb{R}^N \setminus B_{r_0/8}(p_2)) = \mathbb{R}^N \), Lemma 5.1 yields

\[
(5.9) \quad u^-(x, t(r_0)) \leq u_- + C\epsilon \quad \text{or} \quad u^-(x, t) \geq u_+ - C\epsilon, \quad \forall x \in \mathbb{R}^N.
\]

Let \( T^{**} = t(r_0) \). Since \( w(u_- + C\epsilon, t/e^2) \) (or \( w(u_+ - C\epsilon, t/e^2) \)) is a solution of (1.1),

\[
u^-(x, T^{**} + t) \leq w(u_- + C\epsilon, t/e^2) \quad \text{or} \quad u^-(x, T^{**} + t) \geq w(u_+ - C\epsilon, t/e^2), \quad \forall t > 0,
\]

by comparison. Therefore, by the asymptotic behavior of the solution of the ODE (1.5), (2.12) follows.

In case either \( \Gamma(0) \) is convex or \( N = 2 \), \( \Gamma_{\epsilon,0}^0(t) \) exists in \( (0, T^{\text{max}}) \) and it shrinks into a single point as \( t \searrow T^{\text{max}} \) by Lemma 2.1. Thus, when \( \epsilon \searrow 0 \), \( T^* \) can be arbitrarily close to \( T^{\text{max}} \) since \( \Gamma^v, \eta \) is continuous with respect to \( v \) and \( \eta \), \( |v| \leq C\epsilon \), and \( |\eta| \leq C\epsilon \ln \epsilon \). Moreover, \( \Gamma^{v,\eta}(t_1) \) will be contained in a ball whose radius \( r_0 \) can be taken arbitrarily small when \( t_1 \) is arbitrarily close to \( T^{\text{max}} \). By the proof of (5.9), \( T^{**} \leq t_1 + t(4r_0) \). Thus, by (5.4), \( T^{**} \) can also be arbitrarily close to \( T^{\text{max}} \). 

\( \square \)
6. The Neumann Problem.

Now we consider the initial and Neumann boundary condition problem:

\begin{align}
(6.1) \quad u_t - \Delta u + \frac{1}{\varepsilon^2} \phi(u) &= 0, \quad \forall (x, t) \in \Omega \times (0, +\infty), \\
(6.2) \quad u(x, 0) &= g(x), \quad \forall x \in \partial \Omega, \\
(6.3) \quad \partial_n u &= 0, \quad \forall (x, t) \in \partial \Omega \times [0, +\infty),
\end{align}

where \( \Omega \) is a bounded smooth \((C^2)\) domain in \( \mathbb{R}^N \) and \( \partial_n \) is the derivative normal to \( \partial \Omega \). We assume that \( g \in C(\overline{\Omega}) \) and that for some positive constants \( c_0 \) and \( C_0 \),

\begin{align}
(6.4) \quad c_0 \leq |\nabla g| \leq C_0, \quad \forall x \in \{ x \in \overline{\Omega} \mid |g(x) - u_0| \leq c_0 \}.
\end{align}

**Theorem 6.1.** If the \( u_0 \)-level set \( \Gamma_0(0) \equiv \{ x \in \overline{\Omega} | g(x) = u_0 \} \) is a \( C^{3+\alpha} \) hypersurface contained in \( \Omega \), then the results in Theorem 2.2 and Corollary 2.3 (replacing \( \mathbb{R}^N \) by \( \Omega \)) hold for all time \( t < \tilde{T}^* \equiv \min \{ T^*, T^{\text{touch}} \} \), where \( T^{\text{touch}} \) is the time at which \( \Gamma^{u, \eta} \) touches \( \partial \Omega \). If in addition \( \Omega \) is convex, then \( \tilde{T}^* = T^* \) and the assertion of Theorem 2.4 holds.

**Proof.** Noting that the function \( v \) in (3.23) (if necessary, replace \( g \) by \( \overline{g} \) which satisfies \( \partial_n \overline{g}|_{\partial \Omega} = 0 \)) satisfies

\[ \partial_n v(x, t) = \overline{w}_\varepsilon \left( g(x) - \frac{M_2(T)}{\varepsilon} t, \frac{1}{\varepsilon^2} \right) \partial_n g(x) = 0, \quad \forall (x, t) \in \partial \Omega \times (0, T \varepsilon^2 |\ln \varepsilon|), \]

we know that \( v \) is a subsolution of (6.1)-(6.3) and hence Theorem 3.2 regarding the generation of the interface holds for the present case.

Similarly, the function \( u^{\varepsilon, \eta, M} \) in (4.15) (take \( c_1 \) smaller than the distance of the two hypersurfaces \( \partial \Omega \) and \( \Gamma^{u, \eta}(t) \)) also satisfies (6.3) and thus is a subsolution of (6.1) and (6.3). The same proof as the one for Theorem 2.2 then establishes the first half of the theorem.

In case \( \Omega \) is convex, the hypersurface \( \Gamma_\Omega(t) \) \((0 < t < T^{\text{max}}_\Omega)\), the solution of the mean curvature motion problem (2.1) and (2.2) with initial hypersurface \( \Gamma_\Omega(0) = \partial \Omega \), shrinks. Since \( \Gamma(t) \) is contained in \( \Gamma_\Omega(t) \), it will not touch \( \Omega \), that is, \( \tilde{T}^* = T^* \).

We now discuss the case that \( \Gamma_0(0) \) intersects \( \partial \Omega \). Assume that \( g \in C^1(\overline{\Omega}) \) is compatible with the boundary condition (6.3) in the sense that

\begin{align}
(6.5) \quad \partial_n g(x) &= 0, \quad \forall x \in \partial \Omega.
\end{align}

**Lemma 6.2.** Let \( u^\varepsilon \) be the solution of (6.1)-(6.3) and \( a \) be any real number. Denote by \( \Gamma_a(t) \) the set \( \{ x \in \overline{\Omega} \mid u^\varepsilon(x, t) = a \} \). If \( \partial \Omega \cap \Gamma_a(t) \neq \emptyset \) and \( |\nabla u^\varepsilon(x, t)| \neq 0 \) on \( \partial \Omega \cap \Gamma_a(t) \), then \( \Gamma_a(t) \) is orthogonal to \( \partial \Omega \).
Proof. Denote by \( n_{\Gamma_a(t)} \) and \( n_{\partial \Omega} \) the unit normal vectors of \( \Gamma_a(t) \) and \( \partial \Omega \) respectively. Since \( n_{\Gamma_a(t)} = \frac{\nabla u^\epsilon}{|\nabla u^\epsilon|} \), it follows by (6.3) that

\[
n_{\Gamma_a(t)} \cdot n_{\partial \Omega} = \frac{\nabla u^\epsilon}{|\nabla u^\epsilon|} \cdot n_{\partial \Omega} = \frac{1}{|\nabla u^\epsilon|} \partial_n u^\epsilon = 0, \quad \forall x \in \partial \Omega \cap \Gamma_a(t).
\]

Thus, \( \Gamma_a(t) \) is orthogonal to \( \partial \Omega \) at \( \partial \Omega \cap \Gamma_a(t) \). \( \square \)

The above lemma motivates the following assumption:

(*) There exist positive constants \( v^*, \eta^*, T^*, c_1, \) and \( C_1 \), such that for any \( v \in [-v^*, v^*] \) and \( \eta \in [-\eta^*, \eta^*] \), there exist hypersurfaces \( \Gamma^{v, \eta}(t) \in \overline{\Omega}, 0 \leq t \leq T^* \), for which the signed distance function, \( \Lambda^{v, \eta}(x, t) \), satisfies

(6.6) \[
\Lambda^{v, \eta}(x, t) = \Delta \Lambda^{v, \eta}(x, t) - v, \quad \forall x \in \Omega \cap \Gamma^{v, \eta}(t), \quad 0 < t \leq T^*,
\]

(6.7) \[
n_{\Gamma^{v, \eta}}(x, t) \cdot n_{\partial \Omega}(x) = 0, \quad \forall x \in \Gamma^{v, \eta}(t) \cap \partial \Omega,
\]

\[
\{ x \in \overline{\Omega} \mid \Lambda^{v, \eta}(x, 0) = 0 \} = \{ x \in \overline{\Omega} \mid g(x) = u_0 + \eta \},
\]

and

\[
\sup_{x \in \Omega, 0 \leq t \leq T^*, \ |\Lambda^{v, \eta}(x, t)| \leq c_1} \left( |D_t D_x \Lambda^{v, \eta}(x, t)| + |D^3_x \Lambda^{v, \eta}(x, t)| \right) \leq C_1.
\]

Remark 6.3. We do not know whether in general there exist solutions of the mean curvature flow problem (6.6) satisfying the boundary condition (6.7).

Theorem 6.4. Assume that (*) (2.8), (6.4), and (6.5) hold. Then for any \( k > 0 \) and \( 0 < \alpha < \frac{1}{2} \), there exists a constant \( C = C(k, \alpha) \) such that for \( v = C \epsilon^\alpha, \eta = C \epsilon |\ln \epsilon| \), and \( C \epsilon^2 |\ln \epsilon| \leq t \leq T^* \),

(6.8) \[
u^\epsilon(x, t) \geq u_+ - C \epsilon^k, \quad \forall x \in \{ x \in \overline{\Omega} \mid \Lambda^{v, \eta}(x, t) \geq C \epsilon |\ln \epsilon| \},
\]

(6.9) \[
u^\epsilon(x, t) \leq u_- + C \epsilon^k, \quad \forall x \in \{ x \in \overline{\Omega} \mid \Lambda^{v, -\eta}(x, t) \leq -C \epsilon |\ln \epsilon| \},
\]

(6.10) \[
u_- - C \epsilon^k \leq u^\epsilon \leq u_+ + C \epsilon^k, \quad \forall x \in \overline{\Omega}.
\]

Proof. The proof of (6.10) is exactly the same as that for (2.11). We proceed to prove (6.8); the proof of (6.9) is similar. From the proof of Theorem 6.1, we know that (6.8) holds for \( t = T \epsilon^2 |\ln \epsilon| \). Thus it remains to construct a subsolution for the propagation of the interface because the rest is the same as for Theorem 2.2.

As we did in section 4, let \( M \) be a large constant to be determined and let

\[
\phi^M = \phi + M \epsilon^{1+\alpha}.
\]
Denote by \((C(\phi^M), \overline{U})\) the solution of (4.1) and (4.2) with \(f = \phi^M\). Analogously to (4.13), we have

\[ cM\varepsilon^{\alpha+1} \leq C(\phi^M) \leq CM\varepsilon^{\alpha+1}. \]

For brevity, we shall denote by \(A\) the function \(\Lambda^{\nu, \eta}\) and by \(\Gamma\) the hypersurface \(\Gamma^{\nu, \eta}\) with \(\nu = CM\varepsilon^{\alpha}\) and \(\eta = Ce^{-\ln \varepsilon}\).

Let \(h(\cdot)\) be the function satisfying (4.14) and \(\sigma(x)\) be the distance from \(x \in \Omega\) to \(\partial \Omega\). Evidently, \(|\nabla \sigma| = 1\) and \(n_{\partial \Omega} = -\nabla \sigma|_{\partial \Omega}\). Define

\[ d(x, t) = \varepsilon^{\alpha+1/2} h \left( \frac{\Lambda(x, t)}{\varepsilon^{\alpha+1/2}} \right) + C\varepsilon^{\alpha+1} \frac{\sigma(x)}{\sqrt{\varepsilon}}. \]

Since \(\nabla \sigma \cdot \nabla A|_{\sigma=0, \Lambda=0} = \nabla \sigma \cdot \nabla A|_{\partial \Omega \cap \Gamma(t)} = -\partial_n \Lambda|_{\partial \Omega \cap \Gamma(t)} = 0\), the mean value theorem gives

\[ |\nabla \sigma(x, t) \cdot \nabla A(x, t)| \leq C_1(|\sigma| + |\Lambda|), \quad \forall (x, t) \in \Omega \times [0, T^*]. \]

Denote by \(\chi_A\) the characteristic function of a set \(A\). Noting (4.14), we get the estimate

\begin{align*}
\left| |\nabla d|^2 - 1 \right| & = \left| h' \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right) \nabla \Lambda + C\varepsilon^{\alpha+1/2} h' \left( \frac{\sigma}{\sqrt{\varepsilon}} \right) \nabla \sigma \right|^2 - 1 \\
& = \left| h'^2 \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right) - 1 \right| + 2C\varepsilon^{\alpha+1/2} h' \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right) h' \left( \frac{\sigma}{\sqrt{\varepsilon}} \right) \nabla \Lambda \cdot \nabla \sigma + C^2\varepsilon^{1+2\alpha} h'^2 \left( \frac{\sigma}{\sqrt{\varepsilon}} \right) \\
& \leq \chi_{(|\Lambda| > \varepsilon^{\alpha+1/2})} + 2CC_1\varepsilon^{\alpha+1/2} \frac{h'}{\varepsilon^{\alpha+1/2}} h' \left( \frac{\sigma}{\sqrt{\varepsilon}} \right) [||\Lambda| + |\sigma|] + C^2\varepsilon^{1+2\alpha} \\
& \leq \chi_{(|\Lambda| > \varepsilon^{\alpha+1/2})} + 2CC_1\varepsilon^{\alpha+1/2} \left[ 3\varepsilon^{\alpha+1/2} + 3\varepsilon^{1/2} \right] + C^2\varepsilon^{1+2\alpha} \\
& \leq \chi_{(|\Lambda| > \varepsilon^{\alpha+1/2})} + C\varepsilon^{1+\alpha} \\
& \leq \chi_{(|d| > \frac{1}{2}\varepsilon^{\alpha+1/2})} + C\varepsilon^{1+\alpha},
\end{align*}

where the last step is obtained by the fact that

\[ |d(x, t)| \geq \varepsilon^{\alpha+1/2} - C\varepsilon^{1+\alpha} \geq \frac{1}{2}\varepsilon^{\alpha+1/2}. \]

whenever \(|\Lambda| > \varepsilon^{\alpha+1/2}\). Also,

\begin{align*}
\left| d_t - \Delta d + v \right| & = \left| h' \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right) [\Lambda_t - \Delta \Lambda + v] + (1 - h' \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right)) v - \varepsilon^{-\alpha-1/2} h'' \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right) \Delta \sigma - C\varepsilon^{\alpha+1/2} h' \left( \frac{\sigma}{\sqrt{\varepsilon}} \right) \nabla \sigma - C\varepsilon^{\alpha} h'' \left( \frac{\sigma}{\sqrt{\varepsilon}} \right) \right| \\
& \leq \left| h' \left( \frac{\Lambda}{\varepsilon^{\alpha+1/2}} \right) C\Lambda \right| + (|v| + C\varepsilon^{-\alpha-1/2}) \chi_{(|\Lambda| > \varepsilon^{\alpha+1/2})} + C\varepsilon^{\alpha} \\
& \leq C\varepsilon^{-\alpha-1/2} \chi_{(|d| > \frac{1}{2}\varepsilon^{\alpha+1/2})} + C\varepsilon^{\alpha}. \]
Using the above two estimates, we deduce, as in the proof of Lemma 4.2, that the function \( v(x, t) \equiv \overline{U}(d(x, t)/\epsilon) \) satisfies \( \mathcal{L}v \leq 0 \) if

\[
M \geq C \left(1 + \epsilon^{-2\alpha - 3/2}e^{-c\epsilon^{\alpha - 1/2}}\right),
\]

which is certainly true for \( M \) large enough since \( \alpha < \frac{1}{2} \).

Since \( \overline{U}' > 0 \) and \( h'(s) = 0 \) when \( |s| > 3 \), we also have

\[
\partial_n v|_{\partial \Omega} = \frac{1}{\epsilon} \overline{U}' \partial_n d(x, t)|_{\sigma=0} = \frac{1}{\epsilon} \overline{U}' \left(h' \left(\frac{\Lambda}{\epsilon^{\alpha+1/2}}\right) \partial_n \Lambda + C \epsilon^{\alpha+1/2} \partial_n \sigma\right)|_{\sigma=0} \leq \frac{1}{\epsilon} \overline{U}' \left(C_1 |\Lambda| \chi_{\{|\Lambda| \leq 3\epsilon^{\alpha+1/2}\}} - C \epsilon^{\alpha+1/2}\right) \leq 0
\]

if \( C \geq 3C_1 \). Thus \( v(x, t) \) is a subsolution of (6.1) and (6.3). Similarly to the proof of Theorem 2.2, we conclude that if \( \eta \geq C(\epsilon|\ln \epsilon| + \epsilon^{\alpha+1}) \), then \( v(x, 0) \leq u^\epsilon(x, T\epsilon^2|\ln \epsilon|) \) and consequently, by a comparison theorem,

\[
v(x, t) \equiv \overline{U}(d(x, t)/\epsilon) \leq u^\epsilon(x, t + T\epsilon^2|\ln \epsilon|), \quad \forall x \in \Omega, 0 \leq t \leq T^*.
\]

The inequality (6.8) for \( k = 1 + \alpha \) thus follows since \( \Lambda > C\epsilon|\ln \epsilon| \) implies that \( d > \frac{1}{2}C\epsilon|\ln \epsilon| \) and therefore implies that \( v(x, t) \geq \overline{U}(\frac{1}{2}C|\ln \epsilon|) \geq u_+ - C\epsilon^k \). The proof of (6.8) for general \( k \) is analogous to that of Theorem 2.2. \( \square \)

**Remark 6.4.** In case the depths of the wells of the potential are not equal, Theorem 2.2 and Corollary 2.3 are still valid provided that we use the slower time scaled equation (1.4) and replace equation (2.2) by

\[
\Lambda_s = \epsilon \Delta \Lambda - c(\phi),
\]

or simply by

\[
\Lambda_s = -c(\phi) \left(\text{the solution } \Lambda(x, s) \text{ of this equation is } \Lambda(x, 0) - c(\phi)s\right),
\]

where \( c(\phi) \) is the speed of the travelling wave (cf. Lemma 3.1).

**Remark 6.5.**

(1) Although the method for the generation of the interface (cf. Theorem 3.2) is rather simple, it has its limitations. In [7], it is proved that if \( \phi \) is odd and convex, then for any smooth initial function \( g \) satisfying (2.3)–(2.5),

\[
(6.11) \quad u^\epsilon(x, t) = U_0 \left(\Lambda(x, 0)/\epsilon\right) + o(1), \quad \forall C\epsilon^2|\ln \epsilon| \leq t \leq c\epsilon \sqrt{\epsilon},
\]

\[
\overline{U}''(\Lambda(x, 0)/\epsilon) \leq \frac{\Lambda(x, 0)}{\epsilon} - C \epsilon^{\alpha} |\Lambda(x, 0)|^{-\frac{1}{2}},
\]

\[
\overline{U}''(\Lambda(x, 0)/\epsilon) \leq C_1 |\Lambda(x, 0)|^{-\frac{1}{2}},
\]

\[
\overline{U}''(\Lambda(x, 0)/\epsilon) \leq C_2 |\Lambda(x, 0)|^{\frac{1}{2}},
\]

\[
\overline{U}''(\Lambda(x, 0)/\epsilon) \leq C_3 |\Lambda(x, 0)|^{\frac{1}{2}}.
\]

\[
\overline{U}''(\Lambda(x, 0)/\epsilon) \leq C_4 |\Lambda(x, 0)|^{\frac{1}{2}}.
\]

\[
\overline{U}''(\Lambda(x, 0)/\epsilon) \leq C_5 |\Lambda(x, 0)|^{\frac{1}{2}}.
\]
where $U_0$ is the solution of (4.1) and (4.2) with $f = \phi$. Our subsolution is too simple to take the advantage of the attraction properties of the equilibrium solution $(U_0(X \cdot \xi - ct), \xi \in \mathbb{R}^N, |\xi| = 1)$ of (1.1). Thus our method does not seem capable of establishing the more refined asymptotic behavior (6.11).

(2) For the propagation of the interface, our result can be improved; namely, we can get the profile of the solution near the interface under the assumption that the initial value $g$ is a certain profile near the interface (the same assumption as in [8]).

Let $\alpha(x, t) = \left[ \Lambda_{t}^{0,0}(x, t) - \Delta \Lambda^{0,0}(x, t) \right]/\Lambda^{0,0}(x, t), \ z_0 = \int_{\mathbb{R}^1} x U_0^{2}(z)dz/\int_{\mathbb{R}^1} U_0^{2}(z)dz, \ Z = Z(U)$ be the inverse function of $U = U_0(z+z_0)$, and $\zeta(s)$ be a nonnegative cut-off function taking values 1 when $|s| \leq C|\ln \epsilon|$ and 0 when $|s| > 2C|\ln \epsilon|$. Let $(c^M(x, t), U^M(\cdot, x, t))$ be the solution of

\[
\begin{cases}
- U_{zz} - \left[ c + \epsilon \alpha(\zeta(Z(U))Z(U))U_z + [\phi(U) + Me^3] \right] = 0, \quad \forall \ z \in \mathbb{R}^1, \\
U(-\infty, x, t) = u^-_M, \quad U(+\infty, x, t) = u^+_M, \quad \int_{\mathbb{R}^1} z U^2(z, x, t)dz = 0, 
\end{cases}
\]

where $u^-_M$ and $u^+_M$ are respectively the largest and the smallest zero points of $\phi(u) + Me^3$, and $(x, t)$ are merely parameters. One can show that $|U^M(\cdot, x, t) - U_0(\cdot + z_0, x, t)| \leq CMe|\ln \epsilon|$, $c^M = c^0(x, t) + O(\epsilon M^3)$ for some $|c^0(x, t)| \leq C\epsilon^{-2}$, and

(6.11) $|U^M(z, x, t) - U^-M(z, x, t)| \leq CMe^3$.

Let $\overline{\Lambda}$ and $\underline{\Lambda}$ be respectively the solutions of

\[
\overline{\Lambda}_t = \Delta \overline{\Lambda} - \frac{c^0(x, t)}{\epsilon} - CMe^2, \quad \forall x \in \{ x \in \mathbb{R}^N | \overline{\Lambda}(x, t) = 0 \},
\]

\[
\underline{\Lambda}_t = \Delta \underline{\Lambda} - \frac{c^0(x, t)}{\epsilon} + CMe^2, \quad \forall x \in \{ x \in \mathbb{R}^N | \underline{\Lambda}(x, t) = 0 \}.
\]

One also can show that

(6.12) $|\overline{\Lambda}(x, t) - \Delta(x, t)| \leq C \left[ \sup_{y \in \mathbb{R}^N} \left| \overline{\Lambda}(y, 0) - \Delta(y, 0) \right| + Me^2 \right], \quad \forall (x, t) \in \mathbb{R}^N \times [0, T^*]$.

Similarly to the proof of Lemma 4.2, we can show that if $M$ is large enough, then $U^M \left( ch(\overline{\Lambda}/c)/\epsilon \right)$ is a subsolution and $U^-M \left( ch(\underline{\Lambda}/c)/\epsilon \right)$ is a supersolution of (1.1), where $h$ is the function satisfying (4.14).

The profile of the solution near the interface can thus be derived from (6.11), (6.12), by a suitable choice of the initial value $g$.

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