CONCERNING THE WELL-POSEDNESS OF A NONLINEARLY COUPLED SEMILINEAR WAVE AND BEAM-LIKE EQUATION

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Concerning the Well–posedness of a Nonlinearly Coupled Semilinear Wave and Beam–like Equation

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Abstract

In this work, we show the local existence and uniqueness of a coupled hyperbolic/parabolic system, where the coupling is partially through a strongly nonlinear term of polynomial growth. We show ultimately that the degree of the nonlinearity allowed depends upon the smoothness of a “piece” of the initial data and the geometry where the equations take place, and under a relatively mild imposition of smoothness, one can solve the system for arbitrary polynomially bounded nonlinearities.

1 Introduction

1.1 Statement of Problem

Let Ω be a bounded domain in \( \mathbb{R}^2 \) with Lipschitz boundary \( \Gamma \), with \( \Gamma_0 \) a nonempty, smooth and simply connected segment of \( \Gamma \). In this paper, we consider the problem of finding functions \( z(t, x) \) and \( v(t, x) \) which solve the following coupled system consisting of a coupled semilinear wave and elastic “beam-like” equation on finite time \( T \): 

\[
\begin{align*}
    z_{tt} &= \Delta z + f_1(z) + g_1 & \text{on } \Omega \times (0, T) \\
    \frac{\partial z}{\partial \nu} &= \begin{cases} 
        h(v_t) & \text{on } \Gamma_0 \times (0, T) \\
        0 & \text{on } \Gamma \setminus \Gamma_0 \times (0, T) 
    \end{cases} \\
    z^0(t = 0) &= z_0 \in H^1(\Omega) \times L^2(\Omega); \\
    v_{tt} &= -\Delta^2 v - \Delta v_t - z_t + f_2(v) + f_3(v_t) + g_2 & \text{on } \Gamma_0 \times (0, T) \\
    v(t = 0) &= v^0 \in H^2_0(\Gamma_0), v_t(t = 0) = v^1 \in H^1(\Gamma_0) \\
    v &= \frac{\partial v}{\partial \nu} = 0 \text{ on } (0, T) \times \partial \Gamma_0.
\end{align*}
\]

(1)

where the parameter \( \eta \in [0, \frac{1}{2}) \), \( z_0 \) and \( [v^0, v^1] \) given initial data, and we will make here the following assumptions concerning the functions \( f_i \) and \( g_i \) and \( h \):

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(H0) $h$ is a $C^2(\mathbb{R})$ function which satisfies the growth condition $|h''(\xi)| \leq M \left(1 + |\xi|^{p-2}\right)$ for $|\xi| > M > 1$ and $p \in [2, p_\eta)$, with $\eta$ as above, and where

$$
\rho_\eta \equiv \begin{cases} 
\frac{3-3\eta}{1-2\eta} & \text{if } \Omega \text{ has a smooth boundary } \Gamma \\
\frac{11-10\eta}{3(1-2\eta)} & \text{if } \Omega \text{ is a rectangular domain;}
\end{cases}
$$

(H1) The function $f_1 : \mathbb{R} \rightarrow R$ is globally Lipshitz continuous, viz. $\exists C_{f_1} > 0 \ni |f_1(\xi_1) - f_1(\xi_2)| \leq C_{f_1} |\xi_1 - \xi_2|$ for all $\xi_1, \xi_2 \in \mathbb{R}$;

(H2) $f_2$ is a $C^1(\mathbb{R})$ function which satisfies the growth condition $|f_2'(\xi)| \leq N_0 \left(1 + |\xi|^{r-1}\right)$ for $|\xi| > N_0 > 1$ and $r > 1$;

(H3) $f_3$ is a $C^1(\mathbb{R})$ function which satisfies the growth condition $|f_3'(\xi)| \leq N_1 \left(1 + |\xi|^{s-1}\right)$ for $|\xi| > N_1 > 1$ and $s \in \left[1, \frac{s-4\eta}{1-2\eta}\right]$;

(H4) The forcing function $g_1$ is an element of $L^2(0, T \times \Omega)$;

(H5) The forcing function $g_2$ is an element of $L^2(0, T; H^{-\frac{3}{2}-\epsilon}(\Gamma_0)) \forall \epsilon > 0$.

The coupled system (1)–(2) is a nonlinear version of the “structural acoustics” model derived by H.T. Banks et al in [4] and [5] to describe the active control of acoustic pressure in a 2-D chamber through the placement and implementation of piezoelectric ceramic patches on one of the chamber walls; in this work, we consider a situation in which the coupling at the Neumann boundary condition of the wave equation has a general nonlinearity majorized by polynomial growth of degree $p$, where $p$ falls into a certain range governed by the parameter $\eta$ (see (H0)). In addition, a semlinearity induced by $f$ is also introduced into the wave component, as are polynomially bounded nonlinearities into the beam component. We will be concerned here with finding local (in time) solutions $[z, \bar{v}]$ to (1)–(2), with the regularity of the solutions (actually just the component $v_t$) depending on $\eta$, and to some extent on the geometry of $\Omega$.

Even in the linear case, it is only very recently that issues of well–posedness and mathematical control have been settled for this particular model. Banks and R. C. Smith in [4] ascertained the wellposedness of the linear version

$$
\begin{cases}
  z_{tt} = \Delta z & \text{on } \Omega \times (0, T) \\
  \frac{\partial z}{\partial \nu} = \begin{cases} 
v_t & \text{on } \Gamma_0 \times (0, T) \\
 0 & \text{on } \Gamma \setminus \Gamma_0 \times (0, T) \end{cases} \\
  \bar{z}(t = 0) = z_0 \in H^1(\Omega) \times L^2(\Omega); \\
  v_{tt} = -\Delta^2 v - \Delta^2 v_t - z_t + g_2(\tau) & \text{on } \Gamma_0 \times (0, T) \\
  v(t = 0) = v_0 \in H^2(\Gamma_0), v_1(t = 0) = v^1 \in L^2(\Gamma_0) \\
  v = \frac{\partial v}{\partial \nu} = 0 & \text{on } (0, T) \times \partial \Gamma_0,
\end{cases}
$$

(4)
establishing with a semigroup argument that there exists a pair \([\mathcal{Z}, \mathcal{V}]\), living in a specified negative Sobolev space, which uniquely solves (4)–(5) in a weak sense; in the applications \(g_2\) is a linear combination of derivatives of delta functions, and hence the assumption in (H5) that \(g_2\) be so “rough”. Later, Avalos and I. Lasiecka in [1] and [3] demonstrated that the solution \([\mathcal{Z}, \mathcal{V}]\) of (4)–(5) could actually be taken to be of finite energy in the relatively benign space \(C([0,T]; L^2(\Omega) \times H^1(\Omega)) \times C([0,T]; H^1_0(\Gamma_0) \times L^2(\Gamma_0)) \cap L^2(0,T; [H^2_0(\Gamma_0)]^2)\); this improvement of the regularity was requisite for the subsequent work in [1], which deals with handling the associated optimal control problem, including the characterization of the optimal control by an differential Riccati equation.

In this work regarding the nonlinear model (1)–(2), we take as a guideline the recently derived linear theory in [1], but the polynomially bounded nonlinearities in the Neumann boundary condition and in the beam component introduce complications not seen in the linear case, even at the level of establishing the well–definition of terms. Indeed, in the absence of the forcing term \(g_2\), the linear version (4)–(5) is essentially a dissipative system (there is a cancellation between \(v_t\) in the Neumann boundary condition of (4) and \(-z_t\) in the component (5)), and so well–posedness can be attained in this case through straightforward semigroup arguments. With a nonzero \(g_2\) in the beam component (5), the work in [1] equates the system (4)–(5) into a coupled pair of integral equations, and it is from this milieu that the question of well–posedness is addressed. Herein, we likewise employ an analogue of the integral equation approach of [1], attracted thereto by the fact that the dissipativity of the system (4)–(5) is irrelevant to the success of the method in establishing well–posedness; because of the presence of the nonlinearity in the Neumann boundary condition, the system (1)–(2) is not dissipative, and so we must eschew classical arguments in favor of this more accommodating technique. Moreover, the coupled integral equation method in our earlier paper makes use of sharp regularity results for wave equations with given input in the Neumann boundary condition (see [2]), and reconciling the nonlinearity \(h(v_t)\) in (1) to these needed regularity results is an added difficulty in our present problem; the extent of the nonlinearity \(f_3\) must also be considered in the analysis ahead. Consequently, restrictions on the polynomial growth bound on \(h\) and \(f_3\) have to be enforced in (H0) and (H2) for the extraction of meaningful results; these restrictions translate into forcing the component \(v^1\) to be smoother than \(L^2(\Gamma_0)\) if one wishes to solve the system for large \(p\) and \(s\) (of (H0) and (H1) respectively). On the other hand, the extra smoothness required is not too demanding, and indeed, the result in Theorem 1 below yields that if the initial data \(v^1\) is in \(H^\frac{1}{2}\(\Gamma_0)\), then one can solve the problem for \(h\) and \(f_3\) of arbitrary polynomial growth. The result here is sharp with respect to the regularity imposed upon \(g_2\) in the sense that if one is interested in solving (1)–(2) for arbitrary \(p\) and \(s\) (taking initial data \(v^1 \in H^\frac{1}{2}\(\Gamma_0)\)), then the forcing term \(g_2\) can be no rougher than \(L^2(0,T; H^{-\frac{3}{2}}(\Gamma_0))\) in order that a solution \([\mathcal{Z}, \mathcal{V}]\) exist.

1.2 Preliminaries

We wish to recast (1)–(2) within an operator theoretic framework for which we will need the following facts and definitions:

- Let the operator \(A : L^2(\Omega) \supset D(A) \rightarrow L^2(\Omega)\) be defined by

\[
Az = (-\Delta + I)z, \quad D(A) = \left\{ z \in H^1(\Omega) : \Delta z \in L^2(\Omega) \text{ and } \frac{\partial z}{\partial v} = 0 \right\}.
\]

Note that \(A\) is self-adjoint, positive definite, and hence the fractional powers of \(A\) are well defined.

- By [9], we moreover have the following characterization:

\[
D(A^{\frac{1}{2}}) = H^1(\Omega)
\]

with \(\left\| A^{\frac{1}{2}} z \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \left[ |\nabla z|^2 + |z|^2 \right] d\Omega = \left\| z \right\|^2_{H^1(\Omega)} \quad \forall z \in D(A^{\frac{1}{2}}).
\]
• We define the map \( N \) by
\[
  z = N g \iff \begin{cases} \, \, \, \, \, \, \, \, \, \, \, \,-\Delta + I) z = 0 \quad \text{on } \Omega \\ \frac{\partial z}{\partial \nu} \bigg|_{\Gamma \setminus \Gamma_0} = 0 \quad \text{on } \Gamma \setminus \Gamma_0 \\ \frac{\partial z}{\partial \nu} \bigg|_{\Gamma_0} = g \quad \text{on } \Gamma_0; \end{cases}
\]
(8)

elliptic theory will then yield that
\[
  N \in \mathcal{L}(L^2(\Gamma_0), D(A^{1/2})) \quad \forall \varepsilon > 0.
\]
(9)

• Let \( \gamma : H^1(\Omega) \to H^{1/2}(\Gamma_0) \) be the restriction to \( \Gamma_0 \) of the familiar Sobolev trace map; viz.
\[
  \forall z \in H^1(\Omega), \gamma(z) = z|_{\Gamma_0};
\]
(10)

one can then show straightforwardly that
\[
  N^{\ast} Az = \gamma(z) \quad \forall z \in D(A^{1/2}).
\]
(11)

• We set \( \tilde{A} : L^2(\Gamma_0) \supset D(\tilde{A}) \to L^2(\Gamma_0) \) to be
\[
  \tilde{A} = \Delta^2 \text{ with } D(\tilde{A}) = H^4(\Gamma_0) \cap H_0^2(\Gamma_0);
\]
(12)

\( \tilde{A} \) is also self-adjoint, positive definite, and again by [9] we have the characterization
\[
  D(\tilde{A}^\theta) = H_0^{4\theta}(\Gamma_0), \quad 0 < \theta < 1,
\]
(13)

and in particular,
\[
  \left\| \tilde{A}^{1/2} v \right\|_{L^2(\Gamma_0)}^2 = \int_{\Gamma_0} |\Delta v|^2 \, d\Gamma_0 = \|v\|^2_{H_0^2(\Gamma_0)} \quad \forall v \in D(\tilde{A}^{1/2}).
\]
(14)

• We define the energy spaces
\[
  H_1 \equiv D(A^{1/2}) \times L^2(\Omega);
\]
(15)
\[
  H_0 \equiv D(\tilde{A}^{1/2}) \times L^2(\Gamma_0).
\]
(16)

• We define \( A_1 : H_1 \supset D(A_1) \to H_1 \) and \( A_0 : H_0 \supset D(A_0) \to H_0 \) to be
\[
  A_1 \equiv \begin{bmatrix} 0 & I \\ -\tilde{A} + I & 0 \end{bmatrix} \quad \text{with}
\]
(17)
\[
  D(A_1) = \left\{ [z_1, z_2]^T \in D(A) \times D(A^{1/2}) \right\};
\]
(18)
\[
  A_0 \equiv \begin{bmatrix} 0 & I \\ -\tilde{A} & -\tilde{A} \end{bmatrix} \quad \text{with}
\]
(19)
\[
  D(A_0) = \left\{ [v_1, v_2]^T \in \left[ D(\tilde{A}^{1/2}) \right]^2 \, : \, v_1 + v_2 \in D(\tilde{A}) \right\}.
\]
(20)
Through the use of Lumer–Phillips, we have easily,

(A0) $A_1$ generates a $C_0$–semigroup $\{e^{A_1 t}\}_{t \geq 0}$ on the energy space $H_1$;

(A0) $A_0$ generates a $C_0$–semigroup $\{e^{A_0 t}\}_{t \geq 0}$ on the energy space $H_0$.

With $\gamma$ as defined in (10), we set

$$ C = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^* \end{bmatrix} \in \mathcal{L} \left( H_0, D(A^{1/2}) \times D(A^{1/2})^* \right). $$

(21)

If we take the initial data $[z_0, z_1, v_0, v_1]$ to be in $H_1 \times H_0$, then with $[z, z_t, v, v_t] \equiv [\overline{z}, \overline{v}]$, we can use the definitions above to rewrite (1)–(2) (formally) as the coupled pair of operator equations

$$ z_{tt} = (-A + I)z + ANh(v_t) + f_1(z) + g_1 $$(22)

$$ v_{tt} = -\overline{A}v - \overline{A}v_t + f_2(v) + f_3(v_t) - N^*Az_t + g_2 $$

$$ \begin{bmatrix} z(0), v(0) \end{bmatrix} = [\overline{z_0}, \overline{v_0}, \overline{v_1}], $$

and it is this system with which we will work to achieve our ends.

2 Statement of the Main Result

**Theorem 1** (i) Let the assumptions on the geometry $\{\Omega, \Gamma, \Gamma_0\}$ remain in place, as well as those on the functions $\{h, f_1, f_2, f_3, g_1, g_2\}$ prescribed in (H0)–(H5); then for initial data $\overline{z_0} \in H^1(\Omega) \times L^2(\Omega)$ and $[\overline{v^0}, \overline{v^1}] \in H^2_0(\Gamma_0) \times H^\eta(\Gamma_0)$ (again, $\eta \in (0, \frac{1}{2})$), there exists a unique (local) solution $[\overline{z}, \overline{v}]$ to (22)–(23), and hence to (1)–(2), for some finite time $T > 0$, with

$$ \overline{z} \in C([0,T]; H^1(\Omega) \times L^2(\Omega)); $$

$$ \overline{v} \in C([0,T]; H^2_0(\Gamma_0) \times H^\eta(\Gamma_0)) \cap L^2(0,T; H^2_0(\Gamma_0) \times H^2_0(\Gamma_0)). $$

(24)

(25)

(ii) If in particular $v^1 \in H^{1/2}(\Gamma_0)$, then the restrictions imposed on growth of the functions $h$ and $f_3$ (reflected in the magnitudes of $p$ of (H0) and $s$ of (H3) respectively), can be lifted so that for $h$ obeying the assumption in (H0) with $p \in [2, \infty)$ and $f_3$ obeying that of (H3) with $s \in [1, \infty)$ (and all else remaining the same), there exists a unique local solution $[\overline{z}, \overline{v}]$ to (1)–(2), for $T$ small enough, with

$$ \overline{z} \in C([0,T]; H^1(\Omega) \times L^2(\Omega)) $$

$$ \overline{v} \in C([0,T]; H^2_0(\Gamma_0) \times H^{1/2-\epsilon}(\Gamma_0)) \cap L^2(0,T; H^2_0(\Gamma_0) \times H^2_0(\Gamma_0)), $$

(26)

(27)

where $\epsilon > 0$ is arbitrarily small.

**Remark 1** Theorem 1 and the parameter $\rho_\eta$ of (H0) implies that the degree of nonlinearity which can be allowed in the system (1)–(2) so as to obtain finite energy solutions depends on the smoothness of the component $v^1$ and the geometry $\Omega$. 

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2.1 Outline of the Proof of Theorem 1

For the sake of clarity, we sketch briefly our motivation and strategy for the proof of Theorem 1. The strategy to be adopted for the proof of Theorem 1 is motivated by relatively simple heuristics: to wit, using the sets of dynamics \( \{ e^{A_1 t} \}_{t \geq 0} \) \( \{ e^{A_0 t} \}_{t \geq 0} \) generated by \( A_1 \) and \( A_0 \) respectively (see (A1), (A0) above), and the operator \( C \) (defined by (21)), one might be tempted to straightaway construct the solution \( \bar{\varphi}, \bar{\psi} \) to (22)--(23) through the following implicit representation, again with initial data taken in \( H_1 \times D\left( \tilde{A}^{\frac{1}{2}} \right) \times D\left( \tilde{A}^{\frac{3}{2}} \right) \) (after using the characterization in (13)):

\[
\bar{\varphi}(t) = e^{A_1 t} \bar{\varphi}_0 + \int_0^t e^{A_1 (t-\tau)} C \begin{bmatrix} 0 \\ h(v_i(\tau)) \end{bmatrix} d\tau + \int_0^t e^{A_1 (t-\tau)} \begin{bmatrix} f_1(\bar{z}(\tau)) + g_1(\tau) \\ 0 \end{bmatrix} d\tau; \tag{28}
\]

\[
\bar{\psi}(t) = e^{A_0 t} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} - \int_0^t e^{A_0 (t-\tau)} C^* \bar{\varphi}(\tau) d\tau + \int_0^t e^{A_0 (t-\tau)} \begin{bmatrix} f_2(v(\tau)) + f_3(v_1(\tau)) + g_2(\tau) \\ 0 \end{bmatrix} d\tau. \tag{29}
\]

(where \( C^* \) above denotes the adjoint of \( C \)) which would lead, by a subsequent substitution, to the determination that the solution \( \bar{\varphi} \) of (22)--(23) is a fixed point of the integral operator

\[
K_\eta \bar{\psi}(t) = e^{A_0 t} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} - \int_0^t e^{A_0 (t-\tau)} C^* e^{A_1 \tau} \bar{\varphi}_0 d\tau - \int_0^t e^{A_0 (t-\tau)} C^* \bar{\varphi}(\tau)(v_i(\tau)) d\tau + \int_0^t e^{A_1 (t-\tau)} \begin{bmatrix} f_2(v(\tau)) + f_3(v_1(\tau)) + g_2(\tau) \\ 0 \end{bmatrix} d\tau; \tag{30}
\]

here, we are operating in a purely formal manner, with many technical issues ignored, including justification of the presumption that \( \bar{\varphi} \) can be written as some continuous function of \( v_i \). We will work towards finding a unique fixed point of \( K_\eta \), in some suitable topology, resolving functional analytical technicalities as we go along. Regarding this choice of topology, the regularity manifested for the linear version of the structural acoustics model (4)--(5) in [1] suggests that we should target some subspace \( C([0, T]; H_1) \times C([0, T]; H_0) \cap L^2 \left( 0, T; D(\tilde{A}^{\frac{1}{2}}) \right) \) as our space of (local) well-posedness for the solution \( \bar{\varphi}, \bar{\psi} \). Accordingly for \( T \) small enough, we will search for a fixed point \( \bar{\psi} \) of \( K_\eta \) in a subspace of \( C([0, T]; H_0) \cap L^2 \left( 0, T; D(\tilde{A}^{\frac{1}{2}}) \right) \) which depends upon the parameter \( \eta \); having found this component \( \bar{\psi} \) of the solution, \( \bar{\varphi} \in H_1 \) will come quickly through the use of the formula (28): subsequently the existence and uniqueness of the pair \( \bar{\varphi}, \bar{\psi} \) as a solution to the coupled pair of integral equations (28)--(29), and hence the coupled system (1)--(2) will be ascertained.

However, \( K_\eta \) as defined in (30), has no clear mathematical meaning \textit{a priori}, constructed as it is from the rather glib representation (28)--(29), and consequently a preponderant portion of the proof of Theorem 1 (spread out over three propositions) is devoted to providing a well-definition to \( K_\eta \); we briefly sketch our steps in this direction:

(S1) To begin with, the regularity of the “solution” \( \bar{\varphi} \) given in (28) is immediately in question, owing to the nonlinearity presented by the function \( h \) (the linear case has only been treated recently in [2] with \( v_i \) of sufficiently smoothness); and even if it should be found that \( \bar{\varphi} \) lives in the desired space \( C([0, T]; H_1) \), we must insist that \( \bar{\varphi} \) be still “smoother”. More precisely, because of the appearance of the term \( C^* \bar{\varphi} \) in the integrand of the right hand side of (29),
we must give some meaning to boundary values on $\Gamma_0$ of $z_t$; note that such an assignment of a “trace” for $z_t$ will not come from classical Sobolev theory, given that we plan on showing that $z_t(t)$ takes values pointwise no better than in $L^2(\Omega)$. Thus, we will prove first that $\mathcal{Z}$ as the solution of (28) is an element of $C([0, T]; H_1)$, and one can make some sense of $z_t|_{\Gamma_0}$. This is to be done by tweaking, in an appropriate way, the regularity theory already in place for linear second-order hyperbolic equations (see Theorem A below), while taking into account the assumptions posted in (H0) and (H1) above. Specifically, we use a classical interpolation result (see Theorem B below) to determine an magnitude of the exponent $p$ (of the growth bound posted in (H0)) which will allow for the application of Theorem A, and further exploit the Lipschitz continuity of $f_1$ to solve the semilinear equation (1) (for zero initial data) via the Contraction Mapping Principle. Looking at the range for the parameter $\rho_n$ posted in (3), we see that the size of the nonlinearity $p$ depends ultimately on the smoothness of the initial data component $v^1$ and on the geometry $\Omega$.

(SII) With a valid expression in hand for $C^* \mathcal{Z}(v_t)$, we now deal directly with the term $K_\eta \mathcal{V}$, and proceed to make sense of this quantity: this is done in part by exploiting the deep result that $A_0$ generates an analytic semigroup (see Theorem C(i) below), which allows for the extensive use of recently developed techniques applicable to such dynamics (see ([14]) and references therein). Also of indispensable use here is the form of $A_0$, a so-called elastic operator, which not only generates a semigroup with even greater smoothing properties than those usually seen in analytic generators, but which further allows for a crucial characterization of the spaces of well-posedness (of $\mathcal{V}$) by those of the fractional powers of $A$ (see Theorem C(ii) and (iii)). Ultimately, we validate, as was shown in the linear analog of $K_\eta$ in [1], that $K_\eta : L^\infty(0, T; D(\tilde{A}^\frac{1}{2})) \times L^\infty(0, T; D(\tilde{A}^\frac{1}{2})) \cap L^2(0, T; D(\tilde{A}^\frac{1}{2})) \rightarrow C([0, T]; D(\tilde{A}^\frac{3}{2})) \times C([0, T]; D(\tilde{A}^\frac{3}{2})) \cap L^2(0, T; D(\tilde{A}^\frac{1}{2}))$, with this mapping being locally Lipschitz continuous. As was the case for the exponent $p$, a tolerable size of $s$ (of (H3)) depends on the smoothness of the component $v^1$.

Having provided meaning to the operator $K_\eta$ as a continuous mapping, the assertion of a fixed point of $K_\eta$, and subsequently a local solution to the system (1)--(2) will quickly follow, as we have said, from the Contraction Mapping Principle.

3 Proof of Theorem 1

In proving our main theorem, we move by a procession of propositions suggested by (SI) and (SII).

As hinted in (SI) above, to handle the entire coupled structure (1)--(2) (or (22)--(23)), we will need to first ascertain the regularity of semilinear wave equations with input data of a predetermined smoothness in the Neumann boundary condition. In view of this necessity, we will prove the following preliminaries.

Proposition 1 (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a globally Lipshitz function, viz. $\exists C_f$ such that $|f(\xi_1) - f(\xi_2)| \leq C_f |\xi_1 - \xi_2|$ for all $\xi_1, \xi_2 \in \mathbb{R}$, and $q \in L^2(0, T; L^2(\Omega))$; furthermore, let boundary data $u \in L^2(0, T; H^\frac{3}{2}(\Gamma))$, where $\beta \equiv 3$ if $\Omega$ is a bounded open set with smooth boundary $\Gamma$, and $\beta \equiv 4$ in the special case that $\Omega$ is a rectangle. Then for arbitrary initial data $z_0 \in H_1$ (again, $H_1 \equiv H^1(\Omega) \times L^2(\Omega)$) and all time $T < \infty$, there exists a $\mathcal{Z}(u) \equiv [z(u), z(u)_t] \in C([0, T]; H_1)$ which
uniquely solves the following semilinear wave equation

\[ \frac{\partial z}{\partial t} = \Delta z + f(z) + g \quad \text{on} \quad (0, T) \times \Omega \]

\[ \frac{\partial z}{\partial \nu} = u \quad \text{on} \quad (0, T) \times \Gamma \]

(31)

\[ \mathcal{F}(u) \text{ has the implicit representation} \]

\[ \mathcal{F}(u)(t) = e^{A_1 t} \mathcal{F}_0 + \int_0^t e^{A_1 (t-\tau)} C \begin{bmatrix} 0 \\ u(\tau) \end{bmatrix} d\tau + \int_0^t e^{A_1 (t-\tau)} \begin{bmatrix} 0 \\ f(z(u)(\tau)) + g(\tau) \end{bmatrix} d\tau. \]

(32)

(ii) \( \mathcal{F}(u) \) varies continuously as a function of \( u \) with the following norm estimates being satisfied for all \( u_1, u_2 \in L^2(0, T; H^{3/2}(\Gamma)) \):

\[ \| \mathcal{F}(u_1) - \mathcal{F}(u_2) \|_{C([0, T]; H_1)} \leq C(T, C_f) \| u_1 - u_2 \|_{L^2(0, T; H^{3/2}(\Gamma))}. \]

(33)

(iii) In addition, \( z(u)(t) \in L^2(0, T; H^{-3/2}(\Gamma)) \), with continuous dependence on the boundary data manifested by the following estimate holding for all \( u_1, u_2 \in L^2(0, T; H^{3/2}(\Gamma)) \) (the parameter \( \beta \) being as in (i)):

\[ \| \mathcal{F}(u_1) - \mathcal{F}(u_2) \|_{L^2(0, T; H^{-\beta/2}(\Gamma))} \leq C(T, C_f) \| u_1 - u_2 \|_{L^2(0, T; H^{3/2}(\Gamma))}. \]

(34)

\[ \| \mathcal{F}(u_1) - \mathcal{F}(u_2) \|_{L^2(0, T; H^{-\beta/2}(\Gamma))} \leq C(T, C_f) \| u_1 - u_2 \|_{L^2(0, T; H^{3/2}(\Gamma))}. \]

(35)

Proof of (i): We first note that the solution \( \mathcal{F}(u) \equiv [z, z_t] \) which solves (31) appears tantamount to finding a fixed point for the map \( \mathcal{F}(\mathcal{F}) \) defined as

\[ \mathcal{F}(u)(t) = e^{A_1 t} \mathcal{F}_0 + \int_0^t e^{A_1 (t-\tau)} C \begin{bmatrix} 0 \\ u(\tau) \end{bmatrix} d\tau + \int_0^t e^{A_1 (t-\tau)} \begin{bmatrix} 0 \\ f(z(u)(\tau)) + g(\tau) \end{bmatrix} d\tau. \]

(37)

if the right hand side of (37) is verified to make sense in \( C([0, T]; H_1) \) for arbitrary \( T < \infty \), then a unique fixed point \( \mathcal{F}(u) \in C([0, T]; H_1) \) can be found in a very straightforward fashion, given the Lipschitz continuity imposed upon \( f \) via the Contraction Mapping Principle and repeated iteration. Indeed, we find, using the regularity theory derived by S. Miyatake for mixed second order hyperbolic problems (see [17, Theorem 1]), that for fixed \( z \in C([0, T]; H_1) \), the quantity

\[ \mathcal{F}(u)(t) \equiv e^{A_1 t} \mathcal{F}_0 + \int_0^t e^{A_1 (t-\tau)} \begin{bmatrix} 0 \\ f(z(u)(\tau)) + g(\tau) \end{bmatrix} d\tau. \]

(38)

satisfies \( \mathcal{F}(\mathcal{F}(u)) \in C([0, T]; H^{3/2}(\Omega)) \times C([0, T]; L^2(\Omega)) \times L^2(0, T; H^{-3/2}(\Gamma)) \) with the estimate (valid for all \( t \in [0, T) \))

\[ \| \omega(t) \|_{H^3(\Omega)} + \| \omega(t) \|_{L^2(\Omega)} + \int_0^T \| \omega(s) \|_{H^{-3/2}(\Gamma)}^2 ds \leq C(T) \left[ \| f(z) + g \|_{L^2(0, T; L^2(\Omega))} + \| \mathcal{F}_0 \|_{H_1}^2 \right]. \]

(39)

---

\(^1\)Throughout, \( C(T, \cdot) \) and \( C(\cdot, \cdot) \) will denote constants that diminish as \( T \) does.
\( \omega \) being as it is a solution of the following wave equation:

\[
\begin{align*}
\omega_{tt} &= \Delta \omega + f(z(t)) + g(t) \quad \text{on } \Omega \times (0, T) \\
\frac{\partial \omega}{\partial \nu} &= 0 \quad \text{on } \Gamma \times (0, T) \\
[\omega(0), \omega_t(0)] &= \mathbf{z}_0.
\end{align*}
\]  
(40)

To handle the other term on the right hand side of (37), we must validate the regularity of the map

\[
u(\cdot) \rightarrow \int_0^t e^{A_t(-\tau)}C \left[ \begin{array}{c} 0 \\ u(\tau) \end{array} \right] d\tau \equiv \overline{\psi}(\cdot).
\]
(41)

Differentiation will give that \( \overline{\psi}(\cdot) \) as defined solves (weakly) the wave equation

\[
\begin{align*}
\psi_{tt} &= \Delta \psi \quad \text{on } \Omega \times (0, T) \\
\frac{\partial \psi}{\partial \nu} &= u \quad \text{on } \Gamma \times (0, T) \\
\psi(0) &= \psi_t(0) = 0.
\end{align*}
\]
(42)

and we have the following recent result concerning the well-definition of the so-called Neumann–Dirichlet map which gives \( \overline{\psi} \) to be in \( C([0, T]; H_1) \):

**Theorem A.** Let \( \psi \) be a weak solution of the wave equation (42), with \( u \in L^2 \left( 0, T; H^{\frac{3}{2}}(\Gamma) \right) \), where \( \beta \equiv 4 \) if \( \Omega \) is rectangular and \( \beta \equiv 3 \) in the case that \( \Omega \) is a bounded open set with smooth boundary \( \Gamma \). Then we have that

\[
[\psi, \psi_t, \psi_{tt}] \in C \left( [0, T]; H^1(\Omega) \right) \times C \left( [0, T]; L^2(\Omega) \right) \times L^2 \left( 0, T; H^{-\frac{1}{2}}(\Gamma) \right),
\]

with continuous dependence on the datum \( u \): viz. \( \forall t \in [0, T] \exists C \) (independent of \( t \)), such that

\[
\|\psi(t)\|_{H^1(\Omega)}^2 + \|\psi_t(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\psi(s)\|_{H^{-\frac{1}{2}}(\Gamma)}^2 ds \leq C \|u\|_{L^2(0, T, H^{\frac{3}{2}}(\Gamma))}^2.
\]
(43)

**Remark 2** The result above was proved in [2] for the case that \( \Omega \) is a rectangle, and in ([12]) when \( \Omega \) is a bounded open set with smooth boundary \( \Gamma \) (see also [13] for related regularity results). Notice that the result posted in **Theorem A** states “sharp” regularity estimates for the Neumann–Dirichlet map which do not follow from standard PDE theory. In fact, the standard estimates (see [16]) require that \( u \in H^{\frac{3}{2}}((0, T) \times \Gamma) \) in order to obtain only that \( \psi \in C([0, T]; H^1(\Omega)) \). Thus, our result improves the classical regularity by a \( \frac{1}{2} - \frac{1}{\beta} \) derivative in space and a \( \frac{1}{2} \) derivative in time. More recent and refined PDE estimates such as Miyatake’s in [17] with initial datum \( u \in L^2 \left( 0, T; H^{\frac{3}{2}}(\Gamma) \right) \)

produce \( [\psi, \psi_t, \psi_{tt}] \in C([0, T]; H^1(\Omega)) \times C([0, T]; L^2(\Omega)) \times L^2(0, T; H^{-\frac{1}{2}}(\Gamma)). \) Thus, **Theorem A** still better that of Miyatake’s by \( \frac{1}{2} - \frac{1}{\beta} \) space derivative to obtain the same state regularity and further allows for the “trace” \( \gamma_{\Gamma, t} \) to be defined in an Sobolev space narrower than \( H^{-\frac{1}{2}}(\Gamma) \). The “anisotropic” quality of our result (only square integrability is imposed on the time variable of the input \( u \)) and the sharp regularity obtained for the regularity of the boundary traces are absolutely crucial for the analysis of these structural acoustic models.

With the input \( u \) yielding \( \overline{\psi} \) as defined in (41) to be in \( C([0, T]; H_1) \), then as mentioned above, we can hence proceed, for arbitrary \( u \in L^2 \left( 0, T; H^{\frac{3}{2}}(\Gamma) \right) \), to find a unique \( \mathcal{F}(u) \in C([0, T]; H_1) \) which solves the integral equation (37), and *a fortiori*, this \( \mathcal{F}(u) \) will be a solution of the semilinear equation (31).
Proof of (ii)–(iii): The norm estimates (33)–(36) are a standard consequence of the explicit representation of $\mathcal{F}(u)$ in (37), the estimates (39) and (43), the Lipschitz continuity of $f$ with its accompanying Lipschitz constant $C_f$, and Gronwall’s inequality. 

**Proposition 2** Let $\Gamma_0$ be a segment of the real line, $0 < T < \infty$ and fixed $\eta \in [0, \frac{1}{2})$; for $q = 3$ or 4, set $\rho^q_\eta$ to be

$$\rho^q_\eta = \begin{cases} \frac{\eta^{1-\frac{10}{q}}}{3(1-2\eta)} & \text{if } q = 3 \\ \frac{3-\rho}{3-2\eta} & \text{if } q = 4. \end{cases} \quad (44)$$

then:

(i) With $h$ a $C^2(\mathbb{R})$ function satisfying the growth condition

$$|h''(\xi)| \leq M(1 + |\xi|^{p-2})$$

for some $p \in [2, \rho^q_\eta]$ and $|\xi| > M > 1$, we have the mapping

$$u \rightarrow h(u) \quad (46)$$

to be bounded and locally Lipschitz continuous from $L^2(0, T; H^2(\Gamma_0)) \cap L^\infty(0, T; H^\eta(\Gamma_0))$ into $L^2(0, T; H^2(\Gamma_0)) \cap L^1(0, T; H^1(\Gamma_0))$, with the following norm estimate\(^2\) being valid for all $u, v \in L^2(0, T; H^2(\Gamma_0)) \cap L^\infty(0, T; H^\eta(\Gamma_0))$:

$$||h(u)||_{L^2(0, T; H^2(\Gamma_0)) \cap L^1(0, T; H^1(\Gamma_0))} \leq C_0(T, \text{meas}(\Gamma_0)) + C_1 ||u||^p_{L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))},$$

$$||h(u) - h(v)||_{L^2(0, T; H^2(\Gamma_0)) \cap L^1(0, T; H^1(\Gamma_0))} \leq C (||u||, ||v||) ||u - v||_{L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))}. \quad (47)$$

(ii) In addition, for all $p \in [1, \rho^q_\eta]$, the mapping (46) is bounded and locally Lipschitz continuous from $L^2(0, T; H^2(\Gamma_0)) \cap L^\infty(0, T; H^\eta(\Gamma_0))$ into $L^2(0, T; H^{\beta(q)}(\Gamma_0))$, where $\beta(q)$ is defined as

$$\beta(q) = \begin{cases} 4 & \text{if } q = 3 \\ 3 & \text{if } q = 4. \end{cases} \quad (49)$$

with the following norm estimate being valid for all $u, v \in L^2(0, T; H^2(\Gamma_0)) \cap L^\infty(0, T; H^\eta(\Gamma_0))$:

$$||h(u)||_{L^2(0, T; H^{\beta(q)}(\Gamma_0))} \leq C_0(T) ||u||^p_{L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))} + C_1(T),$$

$$||h(u) - h(v)||_{L^2(0, T; H^{\beta(q)}(\Gamma_0))} \leq C (||u||, ||v||) ||u - v||_{L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))}. \quad (50)$$

**Proof of (i):** At this point, we invoke here a classical interpolation statement to be used throughout:

**Theorem B** (see [19]). Let $X$ and $Y$ be Hilbert spaces satisfying $X \subset Y$ with this inclusion being dense and continuous; then we have the following characterization for all $p_0, p_1 \geq 1$ and $0 \leq \theta \leq 1$:

$$[L^p_0(X), L^p_1(Y)]_\theta = L^{p_2}(\theta X + (1-\theta) Y). \quad (52)$$

\(^2\)In the arguments of the constant $C$ in (48), \(||\cdot||\) denotes the $L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))$-norm throughout, in expressing constants which depend upon this norm, we will, for space considerations, neglect to insert the usual subscripts.
where \( \frac{1}{p_2} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \). Moreover, for \( 0 \leq \theta \leq 1 \), there exists a positive constant \( C_\theta \) such that for all \( u \in L_{p_0}(X) \cap L_{p_1}(Y) \),

\[
\|u\|_{L_{p_0}(X) \cap L_{p_1}(Y)} \leq C_\theta \|u\|_{L_{p_0}(X)}^{1-\theta} \|u\|_{L_{p_1}(Y)}^\theta .
\]  

(53)

**Step 1:** Show the mapping (46) is bounded and locally Lipshitz continuous from \( L^2(0,T;H^2(\Gamma)) \cap L^\infty(0,T;H^q(\Gamma)) \) into \( L^p(0,T;L^2(\Gamma)) \) for all \( p \in [1,\rho_0^q] \).

As (45) provides that

\[
|h(\xi)| \leq C_0 + C_1 |\xi| + C_2 |\xi|^p
\]

for all \( \xi \in \mathbb{R} \), in proving the foregoing assertion, it will then suffice to show that for \( p \in (1,\rho_0^q) \) the mapping

\[
u \to v^p
\]

is bounded into \( L^q(0,T;L^2(\Gamma_0)) \), with the accompanying norm estimate

\[
\|v^p\|_{L^q(0,T;L^2(\Gamma_0))} \leq C_0(T) \|v\|^{p}_{L^\infty(0,T;H^q(\Gamma)) \cap L^2(0,T;H^2(\Gamma_0))}.
\]

(56)

To prove the boundedness of (55), we first note that

\[
\theta \equiv \frac{3q + 1}{2p(2-q)} \in (0,1) \text{ for } p \in \left[ \frac{1}{1-2q}, \frac{2q}{1-2q} \right]
\]

(57)

(where again, \( q = 3 \) or \( 4 \) and \( \eta \in \left( \frac{1}{2}, \frac{3}{2} \right) \)); consequently, for \( u \in L^2(0,T;H^2(\Gamma_0)) \cap L^\infty(0,T;H^q(\Gamma_0)) \) we have, upon using the interpolation result (52) with \( \theta \) as defined in (57), that

\[
u \in L^{\frac{6q(2-q)}{2q+1}}(0,T;H^{\frac{2q}{2q+1}}(\Gamma_0)).
\]

(58)

Moreover, for \( q = 3 \) or \( 4 \) and \( p \in \left( \frac{1}{1-2q}, \frac{2q}{1-2q} \right) \), we get \( q \leq \frac{4q-1}{p-2q-1} \), which coupled with (58) gives

\[
\int_0^T \left( \int_{\Gamma_0} |u(t)|^p \, d\Gamma_0 \right)^\frac{q}{p} \, dt = \int_0^T \|u(t)\|_{L^{\frac{pq}{2q-p+1}}(\Gamma_0)}^q \, dt \leq C \int_0^T \|u(t)\|_{H^{\frac{2q-1}{2q-p+1}}(\Gamma_0)}^q \, dt.
\]

(by the Sobolev Embedding Theorem)

\[
= \|u\|_{L^{\frac{pq}{2q-p+1}}(0,T;H^{\frac{2q-1}{2q-p+1}}(\Gamma_0))}^q
\]

\[
\leq C \left( \|u\|_{L^\infty(0,T;H^q(\Gamma_0))}^q \|u\|_{L^2(0,T;H^2(\Gamma_0))}^{2q-p-1} \right)^\frac{q}{p} \leq C \|u\|_{L^\infty(0,T;H^q(\Gamma_0))}^q \|u\|_{L^2(0,T;H^2(\Gamma_0))}^{2q-p-1} \|u\|_{L^2(0,T;H^2(\Gamma_0))} \frac{q}{p}.
\]

(59)

after using the interpolation inequality (53), and where \( C \) above is independent of \( p \) or \( \eta \).

To establish the boundedness of (55) for arbitrary \( p \in (1,\frac{1}{1-2q}) \), we simply take \( \eta \equiv \frac{p}{2p-1} \); then \( p \in \left( \frac{1}{1-2q}, \frac{2q}{1-2q} \right) \) for \( \epsilon > 0 \) small enough, and a fortiori \( 0 < \eta \leq \frac{1}{2} \), so estimate (59) applies for these parameters \( p \) and \( \eta \) as well, giving us the boundedness of (55) with the inequality

\[
\int_0^T \left( \int_{\Gamma_0} |u(t)|^p \, d\Gamma_0 \right)^\frac{q}{p} \, dt \leq C \|u\|_{L^\infty(0,T;H^q(\Gamma_0))}^q \|u\|_{L^2(0,T;H^2(\Gamma_0))}^{2q-p-1} \|u\|_{L^2(0,T;H^2(\Gamma_0))} \frac{q}{p}.
\]

(60)
(after using the fact that $H^\eta(\Gamma_0) \subseteq H^\xi(\Gamma_0)$). As mentioned above, establishing the boundedness of (55) into $L^q(0,T;L^2(\Gamma_0))$ for $p \in \left[1, \rho^q_\theta + \frac{2\eta}{1-2\eta}\right]$ implies that of (46) into $L^q(0,T;L^2(\Gamma_0))$ likewise, with the estimate

$$\|u\|_{L^q(0,T;L^2(\Gamma_0))} \leq C_0(T,\text{meas}(\Gamma_0)) + C_1 \|u\|_{L^\infty(0,T;H^\eta(\Gamma_0)) \cap L^2(0,T;H^2(\Gamma_0))}^q$$

(61)

following from (54), (59), and (60).

Inasmuch as

$$|h'(\xi)| \leq C_0 + C_1 |\xi|^{p-1} \quad \text{for all } \xi \in \mathbb{R},$$

(62)

we next observe, by the Mean Value theorem, that for all $(t, x) \in (0, T) \times \Gamma_0$ and $u, v \in L^\infty(0,T;H^\eta(\Gamma)) \cap L^2(0,T;H^2(\Gamma_0))$

$$|h(u(t, x)) - h(v(t, x))| = |h'(\xi(t, x))(u(t, x) - v(t, x))|$$

(where $\xi(t, x) \in (u(t, x), v(t, x))$)

$$\leq \left(C_0 + C_1 |\xi(t, x)|^{p-1}\right)|u(t, x) - v(t, x)|$$

$$\leq \left(C_0 + C_1 \sup \left\{|u(t, x)|^{p-1} : |v(t, x)|^{p-1}\right\}\right)|u(t, x) - v(t, x)|;$$

(63)

and so to show the continuity of (46) into $L^q(0,T;L^2(\Gamma_0))$, we must consider the quantity

$$\int_0^T \left(\int_{\Gamma_0} \left(\sup \left\{|u(t)|^{p-1} : |v(t)|^{p-1}\right\}\right)^2 |u(t) - v(t)|^2 d\Gamma_0\right)^{\frac{q}{2}} dt.$$

(64)

To deal with (64) (again with $p \in (1, \rho^q_\theta)$), the interpolation result (52) with $\theta \equiv \frac{3 - 2\epsilon}{2(2 - \eta)}$ (where $\epsilon > 0$ is small enough) yields that for $u \in L^\infty(0,T;H^\eta(\Gamma_0)) \cap L^2(0,T;H^2(\Gamma_0))$

$$u \in L^{\frac{4q^2 - 1}{4q^2 - 2q + 2\epsilon}}(0,T;H^{\frac{3}{2} + \epsilon}(\Gamma_0));$$

(65)

hence for $q = 3$ or 4.

$$\int_0^T \left(\int_{\Gamma_0} \left(\sup \left\{|u(t)|^{p-1} : |v(t)|^{p-1}\right\}\right)^2 \|u(t) - v(t)\|^q_{L^2(\Gamma_0)} dt\right)$$

(66)

$$\leq C \int_0^T \left(\|u(t)\|_{H^{\frac{3}{2} + \epsilon}(\Gamma_0)}^{p-1} + \|v(t)\|_{H^{\frac{3}{2} + \epsilon}(\Gamma_0)}^{p-1}\right) \left(\|u(t) - v(t)\|_{L^2(\Gamma_0)}^q\right) dt$$

(by the Sobolev Embedding Theorem)

$$\leq C \int_0^T \left(\|u(t)\|_{L^{\frac{3q}{2 - 2\epsilon}}(\Gamma_0)}^{p - q} + \|v(t)\|_{L^{\frac{3q}{2 - 2\epsilon}}(\Gamma_0)}^{p - q}\right) dt \cdot \|u - v\|_{L^\infty(0,T;H^\eta(\Gamma_0))}^q$$

(after Hölder’s inequality)

$$= C \left(\|u\|_{L^{\frac{3q}{2 - 2\epsilon}}(0,T;H^{\frac{3}{2} + \epsilon}(\Gamma_0))}^{p - q} + \|v\|_{L^{\frac{3q}{2 - 2\epsilon}}(0,T;H^{\frac{3}{2} + \epsilon}(\Gamma_0))}^{p - q}\right) \|u - v\|_{L^\infty(0,T;H^\eta(\Gamma_0))}^q$$

$$\leq C \left(\|u\|_{L^{\frac{3q}{2 - 2\epsilon}}(0,T;H^{\frac{3}{2} + \epsilon}(\Gamma_0))}^{p - q} + \|v\|_{L^{\frac{3q}{2 - 2\epsilon}}(0,T;H^{\frac{3}{2} + \epsilon}(\Gamma_0))}^{p - q}\right) \|u - v\|_{L^\infty(0,T;H^\eta(\Gamma_0))}^q$$

$$\leq C \left(\|u\|_{L^\infty(0,T;H^\eta(\Gamma_0)) \cap L^2(0,T;H^2(\Gamma_0))}^{p - q} + \|v\|_{L^\infty(0,T;H^\eta(\Gamma_0)) \cap L^2(0,T;H^2(\Gamma_0))}^{p - q}\right) \|u - v\|_{L^\infty(0,T;H^\eta(\Gamma_0))}^q.$$
after using the fact that $\frac{4(2-\eta)}{1-2\eta+2\xi} \leq 1$ for $p \in \left[1, \rho_0^g \right]$, followed by the interpolation result (53) with $\theta \equiv \frac{3-2\xi}{2(2-\eta)}$. This foregoing estimate, coupled with (63), gives the continuity of the mapping (46) into $L^q(0, T; L^2(\Gamma_0))$, with the locally Lipschitz estimate

$$||h(u) - h(v)||_{L^q(0, T; L^2(\Gamma_0))} \leq C \left(||u|| \cdot ||v||\right) ||u - v||_{L^\infty(0, T; H^\gamma(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))}.$$  

(67)

This concludes Step 1.

Step 2: Show the mapping (46) is bounded and locally Lipschitz continuous from $L^2(0, T; H^2(\Gamma_0)) \cap L^\infty(0, T; H^\eta(\Gamma_0))$ into $L^1(0, T; H^1(\Gamma_0))$ for all $p \in \left[2, \rho_0^g \right]$: Because of Step 1 and the growth estimate (62), to show the boundedness of the mapping (46) into $L^1(0, T; H^1(\Gamma_0))$, we need only prove that the mapping

$$u \rightarrow u^{p-1} u_x$$

(68)

is likewise bounded from $L^2(0, T; H^2(\Gamma_0)) \cap L^\infty(0, T; H^\eta(\Gamma_0))$ into $L^1(0, T; L^2(\Gamma_0))$ (here, $u_x$ denotes the first spatial derivative of $u$).

In establishing the boundedness of (68) into the given space, we have by the interpolation result (52) with $\theta \equiv \frac{1}{2}$, that for $u \in L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))$ \Rightarrow $u \in L^4(0, T; H^{1+\frac{2}{3}}(\Gamma_0))$ and hence

$$u_x \in L^2(0, T; H^1(\Gamma_0)) \cap L^4(0, T; H^{\frac{5}{3}}(\Gamma_0))$$

(69)

with

$$||u_x||_{L^2(0, T; H^{\frac{5}{3}}(\Gamma_0))} \leq ||u||_{L^\infty(0, T; H^\eta(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))}.$$  

(70)

after using the inequality (53); hence combining (69) and (65), we have for $p \in \left[1, \rho_0^g \right]$

$$\int_0^T \left( \int_{\Gamma_0} |u(t)|^{2p-2} u_x(t)^2 d\Gamma_0 \right)^{\frac{1}{2}} dt \leq \int_0^T ||u(t)||^{p-1}_{L^\infty(\Gamma_0)} \left( \int_{\Gamma_0} |u_x(t)|^2 d\Gamma_0 \right)^{\frac{1}{2}} dt \leq C \int_0^T ||u(t)||^{p-1}_{H^{\frac{1}{2}+\epsilon}(\Gamma_0)} ||u_x(t)||_{H^{\frac{5}{3}}(\Gamma_0)} dt$$

(by the Sobolev Embedding Theorem)

$$\leq C \left( \int_0^T ||u(t)||^{\frac{3}{2}(p-1)}_{H^{\frac{3}{4}+\epsilon}(\Gamma_0)} dt \right)^{\frac{3}{4}} \cdot \left( \int_0^T ||u_x(t)||^{4}_{H^{\frac{5}{3}}(\Gamma_0)} dt \right)^{\frac{1}{4}}$$

(by Hölder's Inequality)

$$\leq C \left( \int_0^T ||u(t)||^{\frac{3}{10}(2-\eta)}_{L^1(0, T; H^{\frac{5}{3}}(\Gamma_0))} \cdot ||u_x||_{L^2(0, T; H^{\frac{5}{3}}(\Gamma_0))} \right)^{\frac{3}{4}} 

(71)

after using the fact that $\frac{4(2-\eta)}{1-2\eta+2\xi} \leq \frac{4p-4}{3}$ for $p \in \left[1, \rho_0^g \right]$, followed by the interpolation inequalities (53) and (70). Consequently, with (62) and (71) we obtain $h(u) \in L^1(0, T; H^1(\Gamma_0))$ with the estimate

$$||h(u)||_{L^1(0, T; H^1(\Gamma_0))} \leq C_0(T, \text{meas}(\Gamma_0)) + C_1 \left( ||u||^p_{L^\infty(0, T; L^2(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))} \right).$$  

(72)
To show the continuity of (46): Using the Mean Value theorem, the chain rule and the estimate (45), we can proceed in a fashion analogous to that in Step 1 to have for \( u, v \in L^\infty(0, T; H^q(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0)) \),

\[
\left| \frac{d}{dx} h(u(t, x)) - \frac{d}{dx} h(v(t, x)) \right| = \left| h'(u(t, x)) u_x(t, x) - h'(v(t, x)) v_x(t, x) \right|
\]

\[
\leq \left| h'(u(t, x)) \right| \left| u_x(t, x) - v_x(t, x) \right| + \left| h''(\xi(t, x)) \right| \left| u(t, x) - v(t, x) \right| \left| v_x(t, x) \right|
\]

(\text{where } \xi(t, x) \in (u(t, x), v(t, x)) )

\[
\leq \left( C_0 + C_1 \left| u(t, x) \right|^{p-1} \right) \left| u_x(t, x) - v_x(t, x) \right|
\]

\[+ \left( C_2 + C_3 \sup \left\{ \left| u(t, x) \right|^{p-2}, \left| v(t, x) \right|^{p-2} \right\} \right) \left| u(t, x) - v(t, x) \right| \left| v_x(t, x) \right|. \quad (73)\]

and consequently, we must estimate

\[
\int_0^T \left( \int_{\Gamma_0} \left| u(t) \right|^{2p-2} \left| u_x(t) - v_x(t) \right|^2 \, d\Gamma_0 \right)^{\frac{1}{2}} \, dt; \quad (74)
\]

\[
\int_0^T \left( \int_{\Gamma_0} \left( \sup \left\{ \left| u(t) \right|^{p-2}, \left| v(t) \right|^{p-2} \right\} \right)^2 \left| u(t) - v(t) \right|^2 \left| v_x(t) \right|^2 \, d\Gamma_0 \right)^{\frac{1}{2}} \, dt. \quad (75)
\]

To handle the integral (74), the use of (65) and (69) give for \( p \in [2, p^*_H] \)

\[
(74) \quad \leq \int_0^T \left| u(t) \right|^{p-1}_{L^\infty(\Gamma_0)} \left( \int_{\Gamma_0} \left| u_x(t) - v_x(t) \right|^2 \, d\Gamma_0 \right)^{\frac{1}{2}} \, dt
\]

\[
\leq C \int_0^T \left| u(t) \right|^{p-1}_{H^{\frac{1}{2}+r}(\Gamma_0)} \left\| u_x(t) - v_x(t) \right\|_{H^{\frac{3}{2}}(\Gamma_0)} \, dt
\]

(by the Sobolev Embedding Theorem)

\[
\leq C \left| u \right|^{p-1}_{L^{\frac{3}{2}(1-r)}(0, T; H^{\frac{1}{2}+r}(\Gamma_0))} \left\| u - v \right\|_{L^{4}(0, T; H^{\frac{3}{2}}(\Gamma_0))}
\]

\[
\leq C \left| u \right|^{p-1}_{L^{\infty}(0, T; L^2(\Gamma_0))} \left\| u - v \right\|_{L^\infty(0, T; L^2(\Gamma_0))} \left\| u - v \right\|_{L^2(0, T; H^2(\Gamma_0))} \quad (76)
\]

after again considering that \( \frac{4(2-n)}{1-2n+\epsilon} \geq \frac{4n-4}{3} \) for \( p \in [2, p^*_H] \), followed by the interpolation ineqaul-
ities (53) and (70). To majorize the integral (75), we have

\[
\begin{aligned}
&\leq \int_0^T \left(\|u(t)\|_{L^\infty(\Gamma_0)}^{p-2} + \|v(t)\|_{L^\infty(\Gamma_0)}^{p-2}\right)\|v_x(t)\|_{L^\infty(\Gamma_0)} \left(\int_{\Gamma_0} |u(t) - v(t)|^2 \, d\Gamma_0\right)^{\frac{1}{2}}
\leq C \int_0^T \left(\|u(t)\|_{H^{\frac{1}{2}+\gamma}(\Gamma_0)}^{p-2} + \|v(t)\|_{H^{\frac{1}{2}+\gamma}(\Gamma_0)}^{p-2}\right)\|v_x(t)\|_{H^{\frac{1}{2}+\gamma}(\Gamma_0)} \|u(t) - v(t)\|_{H^{\gamma}(\Gamma_0)} \, dt
\end{aligned}
\]

(after the use of the Sobolev Embedding Theorem)

\[
\leq C \left(\|u\|_{L^{2p-1}(0,T;H^{\frac{1}{2}+\gamma}(\Gamma_0))}^{p-2} + \|v\|_{L^{2p-1}(0,T;H^{\frac{1}{2}+\gamma}(\Gamma_0))}^{p-2}\right)\|v_x\|_{L^2(0,T;H^{\frac{1}{2}+\gamma}(\Gamma_0))} \|u - v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}
\]

(after Cauchy–Schwarz)

\[
\leq C \left[\|u\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}^{p-2} \|v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))} + \|v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}^{p-2}\right] \|v_x\|_{L^2(0,T;H^2(\Gamma_0))} \|u - v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}
\]  

(after using (65))

\[
\leq C \left[\|u\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}^{p-2} \|v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))} + \|v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}^{p-2}\right] \|v\|_{L^\infty(0,T;H^2(\Gamma_0))} \|u - v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))} \|v_x\|_{L^2(0,T;H^2(\Gamma_0))}
\]

(77)

after using the fact \(\frac{4(2-\eta)}{1-2\eta+2\theta} \geq 2p - 4\) for \(p \in [2, \rho_0]\), followed by the interpolation inequality (53).

Coupling (73), (76) and (77), the locally Lipschitz continuity of (46) is deduced, with the estimate

\[
\|h(u)|_{L^1(0,T;H^1(\Gamma_0))} \leq C (\|u\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))} + \|v\|_{L^\infty(0,T;H^{\gamma}(\Gamma_0))}) \|v_x\|_{L^2(0,T;H^2(\Gamma_0))}
\]

being established, thereby concluding Step 2. The proof of (i) is now complete, with the estimate (47) following from the coupling of (61) and (72), and that of (48) upon the collection of (67) and (78). (ii) and its accompanying estimates (50) and (51) now follow from the interpolation result (52) (with \(X \equiv H^1(\Gamma_0), Y \equiv L^2(\Gamma_0), p_0 \equiv 1, p_1 \equiv q, \) and \(\theta \equiv \frac{\eta}{2(\eta - 1)}\)), the interpolation inequality (53), (47) and (48).

\[\blacksquare\]

**Splicing together Propositions 1 and 2(ii), we easily have:**

**Corollary 1** Let \(\Omega \subset \mathbb{R}^2\) be an bounded open set with Lipschitz boundary \(\Gamma\), \(\Gamma_0\) a smooth, simply connected segment of \(\Gamma\) and \(0 < T < \infty\). Furthermore, let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a globally Lipschitz function with Lipshitz constant \(C_f\), arbitrary initial data \(\varphi_0 \in H_1\), and boundary data \(u \in L^2(0,T;H^1_0(\Gamma_0)) \cap L^\infty(0,T;H^\eta(\Gamma_0))\) for some \(\eta \in (0, \frac{1}{2})\). Then:

(i) If \(h\) is a \(C^2(\mathbb{R})\) function which satisfies the growth condition \(\|h''(\xi)\| \leq M (1 + |\xi|^{p-2})\) for \(p \in [2, \rho_0]\), \(|\xi| > M > 1\) and arbitrary \(\xi \in \mathbb{R}\), where \(\rho_0\) is defined as

\[
\rho_0 \equiv \begin{cases} 
3 - \frac{3\eta}{1-2\eta} & \text{if } \Omega \text{ has a smooth boundary } \Gamma \\
\frac{11-10\eta}{3(1-2\eta)} & \text{if } \Omega \text{ is a rectangular domain;}
\end{cases}
\]

(79)

we then have that \(h(u) \in L^2(0,T;H^\beta(\Gamma_0))\), where \(\beta\) is defined as

\[
\beta \equiv \begin{cases} 
3 & \text{if } \Omega \text{ has a smooth boundary } \Gamma \\
4 & \text{if } \Omega \text{ is a rectangular domain;}
\end{cases}
\]

(80)
consequently, there exists a $z^* \equiv [z, z_1] \in C([0, T]; H_1)$ which solves uniquely the following semilinear wave equation

$$ z_{tt} = \Delta z + f(z) + g \text{ on } (0, T) \times \Omega $$

$$ \frac{\partial z}{\partial \nu} = \begin{cases} h(u) & \text{on } (0, T) \times \Gamma_0 \\ 0 & \text{on } (0, T) \times \Gamma \setminus \Gamma_0. \end{cases} \quad (81) $$

$z^*$ has the implicit representation

$$ z^*(t) = e^{A_1^*} z_0 + \int_0^t e^{A_1(t-\tau)} C \begin{bmatrix} 0 \\ h(u(\tau)) \end{bmatrix} \, d\tau + \int_0^t e^{A_1^*(t-\tau)} \begin{bmatrix} 0 \\ f(z(\tau)) + g(\tau) \end{bmatrix} \, d\tau. \quad (82) $$

(ii) $z^*(\cdot)$ is bounded and locally Lipschitz continuous as a function of $u$ with the following estimates being satisfied for all $u, v \in L^\infty(0, T; H^0(\Omega)) \cap L^2(0, T; H^2(\Gamma_0))$: 

$$ \| z^*(u) \|_{C([0, T]; H_1)} \leq C_0(T, C_f) \left[ \| z_0 \|_{H_1} + \| u \|_{L^\infty(0, T; H^0(\Omega))} \| \nabla \|_{L^2(0, T; H^2(\Gamma_0))} + \| g \|_{L^2(0, T; L^2(\Omega))} + \| f(0) \| \right] + C_1(T, C_f); \quad (83) $$

$$ \| z^*(u) - z^*(v) \|_{C([0, T]; H_1)} \leq C(T, C_f, \| u \|_1, \| v \|_1) \| u - v \|_{L^\infty(0, T; H^0(\Omega))} \| \nabla \|_{L^2(0, T; H^2(\Gamma_0))}. \quad (84) $$

(iii) In addition, $z_1|_{\Gamma_0} \in L^2(0, T; H^{-\beta}(\Gamma))$, where $\beta$ is as defined above in (80), with the map $u \mapsto z(u)|_{\Gamma_0}$ being bounded and locally Lipschitz continuous, with the following inequality being satisfied for all $u, v \in L^\infty(0, T; L^2(\Gamma_0)) \cap L^2(0, T; H^2(\Gamma_0))$: 

$$ \| z^*(u)|_{\Gamma_0} \|_{L^2(0, T; H^{-\beta}(\Gamma_0))} \leq C_0(T, C_f) \left[ \| z_0 \|_{H_1} + \| u \|_{L^\infty(0, T; H^0(\Gamma_0))} \| \nabla \|_{L^2(0, T; H^2(\Gamma_0))} + \| g \|_{L^2(0, T; L^2(\Omega))} + \| f(0) \| \right] + C_1(T, C_f); \quad (85) $$

$$ \| z^*(u)|_{\Gamma_0} - z^*(v)|_{\Gamma_0} \|_{L^2(0, T; H^{-\beta}(\Gamma_0))} \leq C(T, C_f, \| u \|_1, \| v \|_1) \| u - v \|_{L^\infty(0, T; H^0(\Gamma_0))} \| \nabla \|_{L^2(0, T; H^2(\Gamma_0))}. \quad (86) $$

**Remark 3** Note how the specification of the geometry of $\Omega$ and the presence of the parameter $\eta$ in (79) dictates in part the degree of nonlinearity (allowed with respect to the exponent $p$) for the provision of finite energy solutions to the wave equation (81).

As outlined in (SII) above, we will now prove the main result which will validate the use of the integral representation (30), and consequently the well-posedness of (22)–(23). In what follows, we will use extensively the following properties of the semigroup $\{e^{A_0 t}\}_{t \geq 0}$ associated to the “elastic” operator $A_0$, defined by (19) and (20) of **Section 1.2**, this operator being used to abstractly describe the parabolic component of the coupled dynamics (22)–(23).

**Theorem C** (see [7], [8]). Let $A_0$ be defined as in (19) and (20); then

(i) $A$ generates a $C_0$-semigroup of contractions $\{e^{A_0 t}\}$ which is also analytic on $H_0$ and $\rho(A_0) \subseteq \{\lambda \ni \Re \lambda \geq 0\}$ (hence fractional powers of $(-A_0)$ are well-defined).
(ii) (a) For $0 \leq \zeta \leq \frac{1}{2}$ we have the following characterization of the domain of $(-A_0)^\zeta$:

$$D((-A_0)^\zeta) = D\left(\tilde{A}^{\frac{1}{2}}\right) \times D\left(\tilde{A}^\zeta\right);$$

\hspace{1cm} (87)

(b) For $\frac{1}{2} \leq \zeta \leq 1$, the characterization of the domain of $(-A_0)^\zeta$ is as follows:

$$D((-A_0)^\zeta) = \left\{ [v_1, v_2] \in H_0 : [v_1, v_2] \in D\left(\tilde{A}^{\frac{1}{2}}\right) \times D\left(\tilde{A}^\zeta\right) \text{ and } v_1 + v_2 \in D\left(\tilde{A}^\zeta\right) \right\}.$$

(iii) For all real $\zeta$, the map

$$e^{A_0(-\cdot)} \in \mathcal{L}\left(\left[D((-A_0)^\zeta), C([0, T]; D((-A_0)^\zeta)) \cap L^2\left(0, T; D\left((-A_0)^{\zeta+\frac{1}{2}}\right)\right)\right]\right),$$

\hspace{1cm} (88)

(iv) For $\overline{f} \in L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right) \times L^2(\Omega)\right)$, we have that the mapping

$$\overline{f}(\cdot) \rightarrow \int_0^\tau e^{A_0(-\tau)} \overline{f}(\tau) d\tau$$

\hspace{1cm} (89)

is continuous from $L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right) \times L^2(\Omega)\right)$ to $C\left([0, T]; [D\left(\tilde{A}^{\frac{1}{2}}\right)]^2\right)$.

Proposition 3 (i) Let the assumptions on $\Omega$, $\Gamma$, $\Gamma_0$, as given for the system (1)–(2) remain in place, as well as those posted in (H0)–(H5); furthermore, for $\eta \in \left[0, \frac{1}{2}\right)$, suppose that the initial data $z_0^\eta \in H_1$ and $[v^0, v^1] \in D\left(\tilde{A}^{\frac{1}{2}}\right) \times D\left(\tilde{A}^\eta\right)$. Define the operator $\mathcal{K}_\eta$ by having for all $\overline{v} \equiv [v_1, v_2] \in L^\infty\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right) \times L^\infty\left(0, T; D\left(\tilde{A}^\frac{1}{2}\right)\right) \cap L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right)$

$$\mathcal{K}_\eta \overline{v}(\cdot) \equiv e^{A_0(-\cdot)} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} - \int_0^\tau e^{A_0(-\tau)} C^* e^{A_1^\eta} z_0^\eta d\tau + \int_0^\tau e^{A_0(-\tau)} C^* \overline{z}(v_2)(\tau) d\tau$$

$$+ \int_0^\tau e^{A_0(-\tau)} \begin{bmatrix} f_2(v_1(\tau)) + f_3(v_2(\tau)) + g_2(\tau) \\ 0 \end{bmatrix} d\tau,$$

\hspace{1cm} (90)

where $\overline{z}(v_2)$ denotes the solution to the semilinear wave equation

$$z_{tt} = \Delta z + f_1(z) \quad \text{on } (0, T) \times \Omega$$

$$\frac{\partial z}{\partial \nu} = \begin{cases} h(v_2) & \text{on } (0, T) \times \Gamma_0 \\ 0 & \text{on } (0, T) \times \Gamma \setminus \Gamma_0 \end{cases}$$

\hspace{1cm} (91)

then for all $\eta \in \left[0, \frac{1}{2}\right)$, the operator $\mathcal{K}_\eta$ is bounded and locally Lipshitz continuous as a mapping of $L^\infty\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right) \times L^\infty\left(0, T; D\left(\tilde{A}^\frac{1}{2}\right)\right) \cap L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right)$ into $C\left([0, T]; D\left(\tilde{A}^{\frac{1}{2}}\right)\right) \times C\left([0, T]; D\left(\tilde{A}^{\frac{1}{2}}\right)\right) \cap L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right)$ with the following estimates holding for all
\( \overline{v} \in L^\infty(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)) \times L^\infty(0, T; D\left(\tilde{A}^{\frac{3}{2}}\right)) \cap L^2(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)) \):

\[
\|\mathcal{K}_n \overline{v}\|_{C([0,T];D(\tilde{A}^{\frac{1}{2}})) \times C([0,T];D(\tilde{A}^{\frac{3}{2}})) \cap L^2(0,T;D(\tilde{A}^{\frac{1}{2}}))} \leq C_0(T, C_f)
\]

\[
+ C_1(T, C_f) \left[ \|\overline{v}_0\|_{H^1} + \left\|\begin{array}{c} \overline{v}^0 \\ \overline{v}^1 \end{array}\right\|_{D(\tilde{A}^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{3}{2}})} + \|\overline{v}\|^p + \|\overline{v}\|^r + \|\overline{v}\|^s \right]
\]

\[
+ C_2(T, C_f) \left[ \|g_1\|_{L^2(0,T;L^2(\Omega))} + \|g_2\|_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma_0))} + |f_1(0)| \right]
\]

(92)

(where the exponents \(p, r, s\) are from the assumptions (H0), (H2) and (H3));

\[
\left\|\mathcal{K}_n \overline{v} - \mathcal{K}_n \overline{v}\right\|_{C([0,T];D(\tilde{A}^{\frac{1}{2}})) \times C([0,T];D(\tilde{A}^{\frac{3}{2}})) \cap L^2(0,T;D(\tilde{A}^{\frac{1}{2}}))} \leq C \left( T, \|\overline{v}\|, \left\|\overline{v}\right\| \right) \left\|\overline{v} - \overline{v}\right\|_{L^\infty(0,T;D(\tilde{A}^{\frac{1}{2}})) \times L^\infty(0,T;D(\tilde{A}^{\frac{3}{2}})) \cap L^2(0,T;D(\tilde{A}^{\frac{1}{2}}))}.
\]

(93)

\[\text{(ii) For } T > 0 \text{ small enough, } \mathcal{K}_n \text{ has a unique fixed point } \overline{v} \in C([0,T]; D\left(\tilde{A}^{\frac{1}{2}}\right)) \times C([0,T]; D\left(\tilde{A}^{\frac{3}{2}}\right)) \cap L^2(0,T; D\left(\tilde{A}^{\frac{1}{2}}\right)).\]

Proof of (i): As noted in (S1), (SII) above, it is incumbent upon us to show the well-definition along with the continuity of \(\mathcal{K}_n\), and to this end we will scrutinize each term on the right hand side of (90) for arbitrary \(\overline{v} \in L^\infty\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right) \times L^\infty\left(0, T; D\left(\tilde{A}^{\frac{3}{2}}\right)\right) \cap L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right).

First off, we handle the third term on the right hand side of (90): By Corollary 1(i)–(ii), \(\overline{z}^*(v_2)\) as a solution of (91) is well-defined in \(C([0,T]; H_1)\) and is locally Lipschitz continuous function of \(v_2\); moreover, Corollary 1(iii) and the definition of \(C\) in (21) give that

\[
v_2 \rightarrow C^* \overline{z}^*(v_2) = \left(\begin{array}{c} 0 \\ z_1(v_2)|_{\Gamma_0} \end{array}\right)
\]

is a locally Lipschitz continuous mapping of \(L^\infty\left(0, T; \tilde{A}^{\frac{3}{2}}\right) \cap L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right)\right)\) into \(L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right) \times \left[D\left(\tilde{A}^{\frac{1}{2}}\right)\right]^\top\right)\) (hence we are using freely the characterizations in (13)). Now, making use of the explicit representation of \((-A_0)^{\frac{1}{2}}\) given in [11] (p.62), we then have upon application of this operator to an arbitrary element \(\overline{v} \equiv [u_1, u_2]\) of \(L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right) \times \left[D\left(\tilde{A}^{\frac{1}{2}}\right)\right]^\top\right)\) the pointwise equality

\[
(-A_0)^{-\frac{1}{2}} \overline{u}(t) = \begin{bmatrix} \tilde{A}^{-\frac{3}{2}} \left(2I + \tilde{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} u_2(t) \\ \tilde{A}^{-\frac{3}{2}} \left(2I + \tilde{A}^{\frac{1}{2}}\right)^{-\frac{1}{2}} u_2(t) \end{bmatrix}:
\]

(95)

a fortiori then, we have, after employing the functional calculus for self-adjoint operators, that

\[
(-A_0)^{-\frac{1}{2}} \overline{u}(\cdot) \in L^2\left(0, T; D\left(\tilde{A}^{\frac{1}{2}}\right) \times D\left(\tilde{A}^{\frac{3}{2}}\right)\right),
\]

(96)

and in particular from Theorem C(ii)(a).

\[
(-A_0)^{-\frac{1}{2}} \overline{u}(\cdot) \in L^2\left(0, T; D\left((-A_0)^{\frac{3}{2}}\right)\right).
\]

(97)
Thus, as
\[( -A_0 )^{\frac{1}{2}} \int_0^t e^{A_0(t-\tau)} \overline{u}(\tau) \, d\tau = \int_0^t e^{A_0(t-\tau)} ( -A_0 )^{\frac{1}{2}} ( -A_0 )^{\frac{1}{2}} \overline{u}(\tau) \, d\tau, \]
(98)

(97) and **Theorem C(iv)** provides that the mapping
\[
\overline{u}(\cdot) \to \int_0^\cdot e^{A_0(\cdot-\tau)} \overline{u}(\tau) \, d\tau
\]
\[
\in \mathcal{L} \left( L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \times \left[ D \left( \tilde{A}^\frac{1}{2} \right) \right] ', C \left( \left[ 0, T \right]; D \left( ( -A_0 )^\frac{1}{2} \right) \right) \right) \right) . \]
(99)
Moreover, using the standard result for generators of analytic semigroups (see [10], Appendix A) that
\[
\overline{u}(\cdot) \to -A_0 \int_0^\cdot e^{A_0(\cdot-\tau)} \overline{u} \, d\tau
\]
is a continuous mapping from \( L^2(0, T; H_0) \) into itself, we further attain from (97) and (98) that
\[
\overline{u}(\cdot) \to \int_0^\cdot e^{A_0(\cdot-\tau)} \overline{u}(\tau) \, d\tau
\]
\[
\in \mathcal{L} \left( L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \times \left[ D \left( \tilde{A}^\frac{1}{2} \right) \right] ', L^2 \left( 0, T; D \left( ( -A_0 )^\frac{1}{2} \right) \right) \right) \right) ; \]
(100)
and so (99), (100), (94), (85) and (86) in combination give the operator
\[
\overline{u}(\cdot) \to \int_0^\cdot e^{A_0(\cdot-\tau)} C^{\ast} \mathcal{F}(v_2)(\tau) \, d\tau
\]
to be bounded and locally Lipschitz continuous from \( L^\infty \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \times L^\infty \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \cap L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \) into \( C \left( \left[ 0, T \right]; D \left( \tilde{A}^\frac{1}{2} \right) \right) \times C \left( \left[ 0, T \right]; D \left( \tilde{A}^\frac{1}{2} \right) \right) \cap L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \) (after the using the characterizations provided in **Theorem C(ii)(b)**), with the following norm estimates being satisfied for all \( \overline{u} \). \( \overline{v} \in L^\infty \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \times L^\infty \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \cap L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) ; \)
\[
\left\| \int_0^\cdot e^{A_0(\cdot-\tau)} C^{\ast} \mathcal{F}(v_2)(\tau) \, d\tau \right\|_{C \left( \left[ 0, T \right]; D \left( \tilde{A}^\frac{1}{2} \right) \right) \times C \left( \left[ 0, T \right]; D \left( \tilde{A}^\frac{1}{2} \right) \right) \cap L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) } \leq C \left( T, C_{f_1} \right) \left[ \left\| g_1 \right\|_{L^2(0, T; \Omega)} + \left\| \mathcal{F}_0 \right\|_{H_1} + \left\| \mathcal{F} \right\| + \left\| f_1 \right\| \right] + C_1 \left( T, C_{f_1} \right) ; \]
(102)
\[
\left\| \int_0^\cdot e^{A_0(\cdot-\tau)} C^{\ast} \left( \mathcal{F}(v_2)(\tau) - \mathcal{F}(\overline{v}_2)(\tau) \right) \, d\tau \right\|_{C \left( \left[ 0, T \right]; D \left( \tilde{A}^\frac{1}{2} \right) \right) \times C \left( \left[ 0, T \right]; D \left( \tilde{A}^\frac{1}{2} \right) \right) \cap L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) } \leq C \left( T, C_{f_1}, \left\| \overline{v} \right\|, \left\| \mathcal{F} \right\| \right) \left\| \mathcal{F} - \mathcal{F} \right\|_{L^\infty \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \times L^\infty \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) \cap L^2 \left( 0, T; D \left( \tilde{A}^\frac{1}{2} \right) \right) } , \]
(103)

*To deal with the last term on the right hand side of (90):*

(i) We first note that the mapping
\[
u \to f_2 (u)
\]
(104)
is bounded and locally Lipschitz continuous from $L^\infty(0, T; D(\hat{A}^\frac{1}{2}))$ into $L^2(0, T; L^2(\Gamma_0))$. Indeed, given the growth assumption (H2), it suffices, by a train of logic analogous to that employed in the proof of Proposition 2, to majorize the quantities

$$\int_0^T \left( \int_{\Gamma_0} |u(t)|^{2r} d\Gamma_0 \right) dt;$$  
(105) 

$$\int_0^T \left( \int_{\Gamma_0} \left( \sup \left( |u(t)|^{r-1}, |v(t)|^{r-1} \right) \right)^2 |u(t) - v(t)|^2 dt. \right.$$  
(106) 

for arbitrary $u, v \in L^\infty(0, T; D(\hat{A}^\frac{1}{2}))$. To estimate (105), we have easily by the Sobolev Embedding Theorem that for $r \geq 1$

$$\int_0^T \left( \int_{\Gamma_0} |u(t)|^{2r} d\Gamma_0 \right) dt;$$  
(105) 

$$\leq T \text{ meas} (\Gamma_0) \|u\|_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))}^{2r}; \right.$$  
(107) 

moreover,

$$\int_0^T \left( \int_{\Gamma_0} \left( |u(t)|^{r-1}_{L^\infty(\Gamma_0)} + |v(t)|^{r-1}_{L^\infty(\Gamma_0)} \right)^2 |u(t) - v(t)|^2_{L^\infty(\Gamma_0)} dt \right.$$  
(106) 

$$\leq T \text{ meas} (\Gamma_0) \left( |u|^{r-1}_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))} + |v|^{r-1}_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))} \right)^2 \|u - v\|^{2r}_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))} . \right.$$  
(108) 

Thus, the assumption (H2), (105), (106), (107) and (108) allow the deduction of the asserted boundedness and Lipschitz continuity for (111), with the following estimates valid for all $u, v \in L^\infty(0, T; D(\hat{A}^\frac{1}{2}))$:

$$\|f_2(u)\|_{L^2(0, T; L^2(\Gamma_0))} \leq C_0(T) \|u\|_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))}^r + C_1(T);$$  
(109) 

$$\|f_2(u) - f_2(v)\|_{L^2(0, T; L^2(\Gamma_0))} \leq C(T, \|u\|, \|v\|) \|u - v\|_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))}.$$  
(110) 

Subsequently, (109), (110) and Theorem C(iv) give that the mapping

$$\bar{v}\left(\cdot\right) \rightarrow \int_0^\tau e^{As(-\tau)} \begin{bmatrix} 0 \\ f_2(v_1(\tau)) \end{bmatrix} d\tau$$  
(111) 

is a bounded and locally Lipschitz continuous from $L^\infty(0, T; D(\hat{A}^\frac{1}{2})) \times L^\infty(0, T; D(\hat{A}^\frac{1}{2})) \cap L^2(0, T; D(\hat{A}^\frac{1}{2}))$ into $C\left(\left[0, T\right]; D\left(\hat{A}^\frac{1}{2}\right)\right) \times C\left(\left[0, T\right]; D\left(\hat{A}^\frac{1}{2}\right)\right)$ with the following estimates valid for all $\bar{v}, \bar{v} \in L^\infty(0, T; D(\hat{A}^\frac{1}{2})) \times L^\infty(0, T; D(\hat{A}^\frac{1}{2})) \cap L^2(0, T; D(\hat{A}^\frac{1}{2}))$:

$$\left\| \int_0^\tau e^{As(-\tau)} \begin{bmatrix} 0 \\ f_2(v_1(\tau)) \end{bmatrix} d\tau \right\|_{C\left(\left[0, T\right]; D\left(\hat{A}^\frac{1}{2}\right)\right) \times C\left(\left[0, T\right]; D\left(\hat{A}^\frac{1}{2}\right)\right)} \leq C_0(T) \left\| \bar{v}\right\|_{L^\infty(0, T; D(\hat{A}^\frac{1}{2}))}^r \|f_2(v_1(\tau))\|_{C\left(\left[0, T\right]; D\left(\hat{A}^\frac{1}{2}\right)\right) \times C\left(\left[0, T\right]; D(\hat{A}^\frac{1}{2})\right)} + C_1(T);$$  
(112) 

$$\left\| \int_0^\tau e^{As(-\tau)} \begin{bmatrix} 0 \\ f_2(v_1(\tau)) - f_2(\bar{v}(\tau)) \end{bmatrix} d\tau \right\|_{C\left(\left[0, T\right]; D\left(\hat{A}^\frac{1}{2}\right)\right) \times C\left(\left[0, T\right]; D(\hat{A}^\frac{1}{2})\right)} \leq C(T, \|v\|, \|\bar{v}\|) \left\| \bar{v} - \bar{v}\right\|_{L^\infty(0, T; D(\hat{A}^\frac{1}{2})) \times L^\infty(0, T; D(\hat{A}^\frac{1}{2})) \cap L^2(0, T; D(\hat{A}^\frac{1}{2}))}.$$  
(113)
(ii) In just the same way that the estimates (109) and (110) of (i) above and (61) and (67) of Proposition 2 were established, we can show, through the use of the growth condition (H3) and again with exponent \( s \in \left[ 1, \frac{5 + \alpha_2}{1 - \frac{\alpha_2}{N}} \right] \), that for \( u \in L^\infty(0, T; D(\mathbf{A}^{1/2})) \cap L^2(0, T; D(\mathbf{A}^{1/2})) \),

\[
u \rightarrow f_3(\nu) \]

is bounded and locally Lipschitz continuous as a mapping of \( L^\infty(0, T; D(\mathbf{A}^{1/2})) \cap L^2(0, T; D(\mathbf{A}^{1/2})) \) into \( L^2(0, T; L^2(\Gamma_0)) \), with the following estimates valid for all \( u, v \in L^\infty(0, T; D(\mathbf{A}^{1/2})) \cap L^2(0, T; D(\mathbf{A}^{1/2})) \):

\[
\|f_3(u)\|_{L^2(0,T;L^2(\Gamma_0))} \leq C_0(T) \|u\|_{L^\infty(0,T;D(\mathbf{A}^{1/2})) \cap L^2(0,T;D(\mathbf{A}^{1/2}))} + C_1(T), \tag{114}
\]

\[
\|f_3(u) - f_3(v)\|_{L^2(0,T;L^2(\Gamma_0))} \leq C(T) \|u - v\|_{L^\infty(0,T;D(\mathbf{A}^{1/2})) \cap L^2(0,T;D(\mathbf{A}^{1/2}))}. \tag{115}
\]

Subsequently, along with Theorem C(iv), (114) and (115) (like the analogous estimates in (ii)) provide that for \( s \) in the given range,

\[
\nu(\cdot) \rightarrow \int_0^t e^{A_0(-\tau)} \begin{bmatrix} 0 \\ f_3(v_2(\tau)) \end{bmatrix} d\tau
\]

is bounded and locally Lipschitz continuous as a mapping of \( L^\infty([0,T]; D(\mathbf{A}^{1/2})) \times L^\infty([0,T]; D(\mathbf{A}^{1/2})) \cap L^2([0,T]; D(\mathbf{A}^{1/2})) \) into \( C([0,T]; D(\mathbf{A}^{1/2})) \times C([0,T]; D(\mathbf{A}^{1/2})) \) with the following norm estimates being satisfied for all \( \nu, \nu' \in L^\infty([0,T]; D(\mathbf{A}^{1/2})) \times L^\infty([0,T]; D(\mathbf{A}^{1/2})) \):

\[
\left\| \int_0^t e^{A_0(-\tau)} \begin{bmatrix} 0 \\ f_3(v_2(\tau)) \end{bmatrix} d\tau \right\|_{C([0,T]; D(\mathbf{A}^{1/2})) \times C([0,T]; D(\mathbf{A}^{1/2}))}
\leq C_0(T) \|
u'\|_{L^\infty([0,T]; D(\mathbf{A}^{1/2})) \times L^\infty([0,T]; D(\mathbf{A}^{1/2})) \cap L^2([0,T]; D(\mathbf{A}^{1/2}))} + C_1(T); \tag{116}
\]

\[
\left\| \int_0^t e^{A_0(-\tau)} \begin{bmatrix} 0 \\ f_3(v_2(\tau)) - f_3(\nu_2(\tau)) \end{bmatrix} d\tau \right\|_{C([0,T]; D(\mathbf{A}^{1/2})) \times C([0,T]; D(\mathbf{A}^{1/2}))}
\leq C(T) \|
u' - \nu\|_{L^\infty([0,T]; D(\mathbf{A}^{1/2})) \times L^\infty([0,T]; D(\mathbf{A}^{1/2})) \cap L^2([0,T]; D(\mathbf{A}^{1/2}))}. \tag{117}
\]

(iii) Also, proceeding as we did in obtaining the regularity for the map (101), we have for \( g_2 \in L^2(0, T; H^{-\frac{\alpha_2}{2}} - \epsilon(\Gamma_0)) \),

\[
\int_0^t e^{A_0(-\tau)} \begin{bmatrix} 0 \\ g_2(\tau) \end{bmatrix} d\tau \in C([0,T]; D(\mathbf{A}^{1/2})) \times C([0,T]; D(\mathbf{A}^{1/2} - \frac{\alpha_2}{2})) \cap L^2([0,T]; D(\mathbf{A}^{1/2}))
\subset C([0,T]; D(\mathbf{A}^{1/2})),
\]

\[
\times C([0,T]; D(\mathbf{A}^{1/2})),
\]

and \( \cap L^2([0,T]; D(\mathbf{A}^{1/2})) \) (this containment above valid for \( \eta \) in the given range) with the estimate

\[
\left\| \int_0^t e^{A_0(-\tau)} \begin{bmatrix} 0 \\ g_2(\tau) \end{bmatrix} d\tau \right\|_{C([0,T]; D(\mathbf{A}^{1/2})) \times C([0,T]; D(\mathbf{A}^{1/2}) \cap L^2([0,T]; D(\mathbf{A}^{1/2}))}
\leq C(T) \|g_2\|_{L^2(0, T; H^{-\frac{\alpha_2}{2}} - \epsilon(\Gamma_0))}. \tag{119}
\]
Finally, to deal with the first two (constant) terms on the right hand side of (90): We observe that $e^{A_1(t)}z_0d\tau$ is a weak solution of
\begin{align}
\begin{cases}
    w_{tt} = \Delta w & \text{on } (0, T) \times \Omega \\
    \partial w / \partial \nu = 0 & \text{on } (0, T) \times \Gamma_0 \\
    \bar{w}(0) = \bar{z}_0,
\end{cases}
\end{align}

thus by [17] (Theorem 1), we have that $C \cdot e^{A_1(t)}\bar{z}_0 \in L^2 \left( 0, T; D \left( \mathbf{A}^{1/2} \right) \right) \times H^{-1/2} \left( \Gamma_0 \right)$; we can then invoke the same analysis used to establish the well-definition of the map (101) to show that the term
\begin{align}
\int_0^t e^{A_0(-\tau)}C \cdot e^{A_1\tau}z_0 d\tau \in C \left( [0, T]; D \left( \mathbf{A}^{1/2} \right) \right) \times C \left( [0, T]; D \left( \mathbf{A}^{3/2} \right) \right) \cap L^2 \left( 0, T; D \left( \mathbf{A}^{1/2} \right) \right),
\end{align}

with the following estimate for all $\bar{z}_0 \in H_0^1$:
\begin{align}
\left\| \int_0^t e^{A_0(-\tau)}C \cdot e^{A_1\tau}z_0 d\tau \right\|_{L^2(0,T;D(\mathbf{A}^{1/2})) \times L^2(0,T;D(\mathbf{A}^{3/2})) \cap L^2(0,T;D(\mathbf{A}^{1/2}))} \leq C(T) \| \bar{z}_0 \|_{H_0^1}.
\end{align}

Moreover, because of the regularity of the component $v^1$, we have by Theorem C(iii),
\begin{align}
e^{A_0(t)} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \in C([0,T];D((-A_0)^{3/2})) \cap L^2 \left( 0, T; D \left( (-A_0)^{3/2} + \mathbf{A}^{1/2} \right) \right);
\end{align}

in addition, using the the characterization in Theorem C(ii) and the boundedness of the mapping described in Theorem C(iii),
\begin{align}
\left\| e^{A_0(t)} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \right\|_{C([0,T];D(\mathbf{A}^{1/2})) \times C([0,T];D(\mathbf{A}^{3/2})) \cap L^2(0,T;D(\mathbf{A}^{1/2}))} & \leq \left\| e^{A_0(t)} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \right\|_{C([0,T];D((-A_0)^{3/2}) \cap L^2(0,T;D((-A_0)^{3/2} + \mathbf{A}^{1/2}))} \\
& \leq C(T) \left\| \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \right\|_{D((-A_0)^{3/2}) \times D(\mathbf{A}^{1/2})} \leq C(T) \left\| \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \right\|_{D(\mathbf{A}^{1/2}) \times D(\mathbf{A}^{3/2})}.
\end{align}

The proof of (i) is completed, after the collection of the estimates (102), (112), (116), (119), (122) and (124) so as to attain (92), and (103), (113) and (117) to acquire the locally Lipschitz condition (93).

**Proof of (ii):** The existence of a fixed point $\bar{v}$ to $K_\eta$ (for small time $T$) with the improved regularity in time, will now follow easily from the Contraction Mapping Principle, given the norm estimates (92) and (93).

**Proof of Theorem 1(i):** Having in hand the unique (local in time) fixed point $\bar{v} \equiv [v, v_1] \in C([0, T]; H_0^2(\Gamma_0) \times H^\eta(\Gamma_0)) \cap L^2 \left( 0, T; \left[ H_0^2(\Gamma_0) \right]^2 \right)$ of the operator $K_\eta$, assured by Proposition 3(ii), we set $\bar{z}$ to be the unique weak solution of
\begin{align}
z_{tt} = \Delta z + f_1(z) + g_1 & \text{ on } (0, T) \times \Omega \\
\frac{\partial z}{\partial \nu} = \begin{cases} 
    h(v_1) & \text{on } (0, T) \times \Gamma_0 \\
    0 & \text{on } (0, T) \times \Gamma \setminus \Gamma_0
\end{cases}
\end{align}

$\bar{z}(0) = \bar{z}_0$. 

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and by Corollary 1(i), $z^r \equiv [z, z_t] \in C([0, T]; H^1(\Omega) \times L^2(\Omega))$. Moreover, by Corollary 1(iii), $z_t|_{t_0} \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma_0))$, and hence a simple uniqueness argument allows us to deduce that we have obtained a pair $[z^r, \bar{v}]$ which uniquely solves the coupled system of fixed point equations

$$\bar{z}(t) = e^{A_1 t} \bar{z}_0 + \int_0^t e^{A_1(t-\tau)} C \begin{bmatrix} 0 \\ h(v_t(\tau)) \end{bmatrix} d\tau + \int_0^t e^{A_1(t-\tau)} \begin{bmatrix} 0 \\ \bar{f}_1(z(\tau)) + g_1(\tau) \end{bmatrix} d\tau; \quad (126)$$

$$\bar{v}(t) = e^{A_0 t} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} - \int_0^t e^{A_1(t-\tau)} C^* \bar{z} d\tau + \int_0^t e^{A_0(t-\tau)} C \begin{bmatrix} 0 \\ f_2(v(\tau)) + f_3(v_t(\tau)) + g_2(\tau) \end{bmatrix} dt. \quad (127)$$

Upon differentiation of the quantities (126)-(127), one then has that $[\bar{z}^r, \bar{v}]$ uniquely solves the operator equations (22) and (23), which is tantamount to the pair being the unique weak solution of (1)-(2).

Proof of (ii): If $v^1$ is additionally in $H^\frac{1}{2}(\Gamma_0)$, then we simply apply the result of Theorem 1(i) with $\eta \equiv \frac{1}{2} - \epsilon$, and hence any given $p$ and $s$ will be in the range prescribed in (H0) and (H1) for $\epsilon > 0$ small enough.

The proof of Theorem 1 is now complete.

References


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