STABILITY OF EQUILIBRIA FOR A CLASS OF 
TIME-REVERSIBLE, $D_n \times O(2)$-SYMMETRIC 
HOMOGENEOUS VECTOR FIELDS

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ABSTRACT

First-order, time-reversible n-body problems in three-space whose velocity fields consist of sums of identical two-body interactions are studied under a set of natural symmetry assumptions. Up to linearization about maximally symmetric equilibria, the entire class is shown to be represented by a two-parameter normal form. The symmetries of the class are used to find formulas for the eigenvalues of the linearized problems. The class of problems is divided into two families, one in which vector field components in the spatial directions act in concert, and one in which they act in opposition. When the components act in concert, the equilibria are: (i) unstable when interaction strength grows with distance, (ii) stable when interaction strength decays and \( n = 3, 4 \), and (iii) stable or unstable when interactions strength decays and \( n > 4 \), depending as the singularity of the vector field varies across a critical value. When components act in opposition, stability and instability are interchanged. A nonlinear application of this analysis is the establishment of symmetric near-equilibrium periodic solutions.

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$D_n \times O(2)$-Symmetric Homogeneous Vector Fields

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0. Introduction.

We consider the first-order three-dimensional motion of $n$ bodies under a time-reversible vector field given by the sum of identical two-body interactions, where the time reversal reverses one spatial direction while fixing the other two. We impose a set of natural symmetry assumptions, and ask what can be said about the linearizations of these vector fields about maximally symmetric equilibria entirely on the basis of the symmetries.

First-order $n$-body problems have $3n$-dimensional phase space (whereas second-order problems such as Newtonian mechanical $n$-body problems have $6n$-dimensional phase space). In the notation we will use, we have vector fields

$$\frac{dr}{dt} = f(r) \quad (0.1)$$

where $r = (r_1, \ldots, r_n)$ with $r_j = (x_j, y_j, z_j) \in \mathbb{R}^3$ and $f = (f_1, \ldots, f_n)$ with $f_j = (f_j^x, f_j^y, f_j^z)$. Since the vector field consists of the sum of identical two-body interactions (and under translational symmetry, of course), $f(r)$ is of the form

$$f_j(r) = \sum_{k \neq j} \frac{F(r_j - r_k)}{|r_j - r_k|^\alpha} + \text{const} \quad (0.2)$$

where $F = (F^x, F^y, F^z)$ and the constant is independent of $j$.

Time reversibility is a symmetry. A vector field problem is said to be time-reversible when there exists a phase space reflection $R$ which anticommutes with the vector field. For a time-reversible vector field, whenever $r(t)$ is a solution, so is $Rr(-t)$. A time-symmetric solution is one for which $r(t) = Rr(-t)$, or equivalently, any solution which passes through the time-symmetry plane $\text{Fix} R$. More general reversible systems are discussed in detail in Sevryuk [S]. Here we assume that $f(r)$ is time-reversible with respect to reflection across the $(x,y)$-plane, i.e.,

$$Rf(r) = -f(Rr) \quad (H1)$$

where

$$Rr = (\bar{r}_1, \ldots, \bar{r}_n), \quad \bar{r}_j = (x_j, y_j, -z_j). \quad (0.3)$$
It is natural to assume that the time-reversal direction (here the z-direction) is the only distinguished direction. Hence we assume $f(r)$ is O(2)-symmetric with respect to the $\theta$-axis. Letting $R_\theta$ denote rotation by $\theta$ about the origin applied to each of the $n$ pairs $(x_j, y_j)$, and $K_\theta$ denote reflection in the $(x, y)$-plane across the line obtained when the $x$-axis is rotated by $\theta$ about the origin, we have

$$R_\theta f(r) = f(R_\theta r) \quad \forall \theta,$$

$$K_\theta f(r) = f(K_\theta r) \quad (H2)$$

Finally, we assume that $f(r)$ is homogeneous. To cast this assumption as a symmetry of the problem, we write: there exists $\beta$ such that

$$f(\delta r) = \delta^{-\beta} f(r) \quad (H3)$$

for all positive real $\delta$ (where equality holds up to a constant corresponding to pure translation). We will refer to $\beta$ as the "singularity" of the vector field. Homogeneity is a natural assumption if we consider the approximation of a problem by its least singular term.

We show that vector field problems of the form (0.1–0.2) are $D_n$-symmetric. Under $D_n$ symmetry and time reversibility (H1), regular horizontal $n$-gons are the only maximally symmetric equilibria. In particular, we consider the equilibria given by $r^0$ where

$$r^0_j = (\cos \frac{2\pi j}{n}, \sin \frac{2\pi j}{n}, 0). \quad (0.4)$$

Note that these equilibria are actually relative equilibria in the sense that

$$f_j(r^0) = \text{const. (independent of } j) \quad \forall j. \quad (0.5)$$

Any translation, rotation, or scaling of these equilibria are, of course, also equilibria (corresponding to translational, O(2), and homogeneity symmetries of the problem).

When vertical and horizontal components of the vector field act in concert, we obtain the following stability results:

(i) when interaction strength grows over distance, regular horizontal $n$-gons are unstable;

(ii) for $n = 3, 4$, when interaction strength decays over distance, regular horizontal $n$-gons

are stable;
(iib) for \( n > 4 \), when interaction strength decays over distance, regular horizontal \( n \)-gons are stable or unstable, depending as the singularity \( \beta \) passes through a "stability threshold" \( \beta^*(n) < \infty \);

where stability refers to ellipticity. (Hyperbolic stability is disallowed by time reversibility, so ellipticity is the strongest stability obtainable by linear analysis.) Numerical studies indicate that for problems of type (iib), \( \beta^*(n) \) decreases with increasing \( n \), i.e., that less singular cases are more stable for decaying interaction. When the components of the vector field act in opposition, stability and instability are of course interchanged.

To obtain these results, we use the symmetry properties of the class of problems to exhibit a normal form for linear analysis of the entire class. We show that up to linearization about regular horizontal \( n \)-gons, our class of vector field problems reduces to two 2-parameter families, parametrized by the relative magnitudes of the vertical and horizontal components, and by the singularity strength \( \beta \). One physical example which may be treated under this setting is the sedimentation under gravity of \( n \) small clustered spheres in a highly viscous fluid, studied experimentally by [JMS], and analytically under the Stokeslet model (infinite viscosity and point particle approximation) in [H], [CLLS], [GKL], and [TK]. The Stokeslet interaction is given by

\[
F(x, y, z) = \begin{pmatrix}
xz \\
yz \\
\frac{x^2 + y^2 + 2z^2}{(x^2 + y^2 + z^2)^{3/2}}
\end{pmatrix}, \quad \frac{y^2 + 2z^2}{(x^2 + y^2 + z^2)^{3/2}}.
\]

We obtain formulas for the eigenvalues, essentially as in [H], but for the whole class, and in terms of the assumed symmetries. The (spectral) stability of the equilibria is studied analytically and numerically. The Stokeslet problem falls under case (ii) above, with \( \beta = 1 \). We have \( \beta^*(7) < 1 < \beta^*(6) \) so that regular horizontal \( n \)-gons are stable for \( n = 3, 4, 5, 6 \) and unstable for \( n \geq 7 \), in agreement with [JMS] and [H].

Another example is the dipole-type interaction given by

\[
F(x, y, z) = \begin{pmatrix}
3xz \\
3yz \\
\frac{3x^2 + y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{pmatrix}, \quad \frac{3y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}}.
\]

Here vertical and horizontal components act in opposition and \( \beta = 3 \).

The linear analysis of this class of problems, besides providing stability information, serves as a basis for nonlinear results and investigations. For example, for \( 0 < \beta < \beta^*(n) \), a time-reversible Liapunov center type theorem for spatially symmetric systems ([GKL]) applies, and we are able to establish the existence of families of symmetric, near-equilibrium periodic solutions when a nonresonance condition holds.
1. $D_n$ Symmetry:

We have stated as assumptions the time reversibility and $O(2)$ symmetry of our class. We now show that $D_n$ symmetry is a consequence of the identical two-body interaction form of $f(r)$ given by (0.2). Let

$$C(r_1, r_2, \ldots, r_n) = (r_2, \ldots, r_n, r_1)$$

$$T(r_1, r_2, \ldots, r_n) = (r_n, \ldots, r_2, r_1)$$

and

$$\hat{C} = R_{-2\pi/n} C$$

$$\hat{T} = K_{\pi/n} T$$

with $R_\theta$ and $K_\theta$ as defined in (H2). From the form of $f(r)$ given by (0.2), we see that

$$Cf(r) = f(Cr) \text{ and } Tf(r) = f(Tr).$$

(1.3)

By $O(2)$ symmetry (H2), we have

$$\hat{C}f(r) = f(\hat{C}r) \text{ and } \hat{T}f(r) = f(\hat{T}r).$$

(1.4)

Moreover,

$$\hat{C}r^0 = \hat{T}r^0 = r^0.$$  

(1.5)

Hence we find

$$\begin{cases} 
\gamma f(r) = f(\gamma r) \\
\gamma r^0 = r^0 
\end{cases} \quad \forall \gamma \in \Gamma = (\hat{C}, \hat{T}) \cong D_n.$$  

(1.6)

2. Isotypic Decomposition:

In the eigenvalue analysis to follow, we will require the $\Gamma$-isotypic decomposition for our space $\mathbb{R}^{3n}$ (cf. [GSS]).

Recall that each $\Gamma$-isotypic component of a space is the direct sum of all $\Gamma$-isomorphic copies occurring in a decomposition of the space into $\Gamma$-irreducible subspaces (i.e., into $\Gamma$-invariant subspaces with no proper, nontrivial $\Gamma$-invariant subspaces). Every space has a unique $\Gamma$-isotypic decomposition. We begin by listing
the one- and two-dimensional irreducible representations of $D_n$ (up to isomorphism). When $n$ is odd, the one-dimensional irreducible representations are:

\begin{align}
W_{++} & \quad \hat{C} = 1 \quad \hat{T} = 1 \\
W_{+-} & \quad \hat{C} = 1 \quad \hat{T} = -1
\end{align}

(2.1a)

(2.1b)

When $n$ is even, there are in addition:

\begin{align}
W_{-+} & \quad \hat{C} = -1 \quad \hat{T} = 1 \\
W_{--} & \quad \hat{C} = -1 \quad \hat{T} = -1
\end{align}

(2.1c)

(2.1d)

There are $\text{int}(\frac{n-1}{2})$ distinct two-dimensional irreducible representations:

\begin{align}
W_k & \cong \mathbb{C} \\
\hat{C}_z & = e^{i2\pi k/n} z \\
\hat{T}_z & = \bar{z}
\end{align}

(2.2)

where $z \in \mathbb{C}$ and $\bar{z}$ denotes complex conjugation.

We proceed with the isotypic decomposition of $\mathbb{R}^{3n}$. We write

\begin{align}
\mathbb{R}^{3n} = H \oplus V
\end{align}

(2.3)

where

\begin{align}
H = \{ r \in \mathbb{R}^{3n} \mid z_j = 0 \quad \forall j \}
\end{align}

(2.4a)

and

\begin{align}
V = \{ r \in \mathbb{R}^{3n} \mid z_j = y_j = 0 \quad \forall j \}
\end{align}

(2.4b)

We note that $H$ and $V$ are $\Gamma$-invariant, i.e. $\Gamma H \subseteq H$ and $\Gamma V \subseteq V$. We decompose $H$ as

\begin{align}
H = H_0 \oplus H_1 \oplus \cdots \oplus H_{n-1}
\end{align}

(2.5)

where

\begin{align}
H_1 = \{ r \in H \mid r_{j+1} = R_{2\pi i/n} r_j \}
\end{align}

(2.6)

Identifying $r \in H_1$ such that $r_1 = \rho$ with $\rho = (\xi, \eta, 0) \in \mathbb{R}^3$, we have

\begin{align}
\hat{C}_\rho = R_{2\pi (i-1)/n} \rho \\
\hat{T}_\rho = K_{\pi/n} R_{-2\pi i/n} \rho
\end{align}

(2.7)
We see that

\[
H_0 \cong_{\Gamma} W_1
\]

\[
H_1 \cong_{\Gamma} W_{++} \oplus W_{+-}
\]

\[
H_i \cong_{\Gamma} W_{i-1} \quad \text{for} \quad i = 2, \ldots, \text{int}\left(\frac{n-1}{2}\right) + 1
\]

\[
H_i \cong_{\Gamma} W_{n-(l-1)} \quad \text{for} \quad l = \text{int}\left(\frac{n+2}{2}\right) + 1, \ldots, n - 1
\]

\[
H_{\frac{n}{2}+1} \cong_{\Gamma} W_{++} \oplus W_{--} \quad \text{when } n \text{ even}
\]

where \(\cong_{\Gamma}\) denotes \(\Gamma\)-isomorphism. Next we decompose \(V\) as

\[
V = V_0 \oplus V_1 \oplus \cdots \oplus V_{\text{int}(n/2)}
\]

(2.9)

where \(V_k\) is the real space

\[
V_k = \mathbb{R}(v^*_k, v^i_k)
\]

(2.10α)

with

\[
v^*_k = \left(1, \cos \frac{2\pi k}{n}, \ldots, \cos \frac{2\pi (n-1)}{n}\right)
\]

\[
v^i_k = \left(0, \sin \frac{2\pi k}{n}, \ldots, \sin \frac{2\pi (n-1)}{n}\right)
\]

(2.10β)

In the basis \(\{v^*_k, v^i_k\}\) for \(V_k\), we have for any \(v_k \in V_k\),

\[
\hat{C}v_k = \begin{pmatrix}
\cos \frac{2\pi k}{n} & \sin \frac{2\pi k}{n} \\
-\sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n}
\end{pmatrix} v_k
\]

\[
\hat{T}v_k = \begin{pmatrix}
\cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\
-\sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n}
\end{pmatrix} v_k
\]

(2.11).

We see that

\[
V_0 \cong_{\Gamma} W_{++}
\]

\[
V_k \cong_{\Gamma} W_k \quad \text{for} \quad k = 1, \ldots, \text{int}\left(\frac{n-1}{2}\right)
\]

\[
V_{n/2} \cong_{\Gamma} W_{--} \quad \text{when } n \text{ even}
\]

(2.12)

We arrive at the isotypic decomposition of \(\mathbb{R}^{3n}\) that we want:

\[
\mathbb{R}^{3n} \cong_{\Gamma} W^2_{++} \oplus W_{+-} \oplus W^3_1 \oplus \cdots \oplus W^3_{\frac{n}{2}+1} \quad (n \text{ odd})
\]

\[
\mathbb{R}^{3n} \cong_{\Gamma} W^2_{++} \oplus W_{+-} \oplus W^2_{--} \oplus W^2_{1} \oplus \cdots \oplus W^3_{\frac{n}{2}} \quad (n \text{ even})
\]

(2.13)
3. Eigenvalue Structure:

On the basis of the isotypic decomposition (2.13) and time reversibility (H1), we can make qualitative statements about the eigenvalues of the linearization \( L \) of \( f \) at \( r^0 \). We have

\[
L \equiv df \bigg|_{r^0} = \begin{pmatrix}
\frac{\partial f_1}{\partial r_1} & \frac{\partial f_1}{\partial r_2} & \cdots & \frac{\partial f_1}{\partial r_n} \\
\frac{\partial f_2}{\partial r_1} & \frac{\partial f_2}{\partial r_2} & \cdots & \frac{\partial f_2}{\partial r_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial r_1} & \frac{\partial f_n}{\partial r_2} & \cdots & \frac{\partial f_n}{\partial r_n}
\end{pmatrix}
\bigg|_{r^0} = \left( \frac{\partial f_i}{\partial r_i} \bigg|_{r^0} \right) \quad (3.1)
\]

where the notation on the right means that the \((i,j)\)th \(3 \times 3\) block of \( L \) is given by

\[
\frac{\partial f_i}{\partial r_j} \bigg|_{r^0} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_j} & \frac{\partial f_1}{\partial x_j} & \frac{\partial f_1}{\partial x_j} \\
\frac{\partial f_2}{\partial x_j} & \frac{\partial f_2}{\partial x_j} & \frac{\partial f_2}{\partial x_j} \\
\frac{\partial f_n}{\partial x_j} & \frac{\partial f_n}{\partial x_j} & \frac{\partial f_n}{\partial x_j}
\end{pmatrix}
\bigg|_{r^0} \quad (3.2)
\]

Consider \( W \) any \( L \)-invariant subspace of \( \mathbb{R}^{3n} \). On \( W \), we have \( LR = -RL \) by time reversibility (H1) so when \( Lu = \lambda u \),

\[
L(Ru) = -R(Lu) = -\lambda(Ru),
\]

i.e., \( \lambda \) is an eigenvalue of \( L \big|_W \) if and only if \( -\lambda \) is. Now since \( f \) depends only on the relative positions \( r_j - r_k \) \((j \neq k)\), \( H_0 \) and \( V_0 \) are clearly \( L \)-invariant, and moreover, \( L \big|_{H_0} = 0 \) and \( L \big|_{V_0} = 0 \) (corresponding respectively to horizontal and vertical translational motion). Next, we observe that isotypic components are \( L \)-invariant. To see this, let \( W^* \) be any isotypic component. \( \text{Ker} L \) is \( \Gamma \)-invariant since if \( u \in \text{Ker} L \), we have \( L(\gamma u) = \gamma(Lu) = 0 \). Then \( W \cap \text{Ker} L \) is a \( \Gamma \)-invariant subspace of \( W \) and since \( W \) is irreducible, either \( W \cap \text{Ker} L = W \) or \( W \cap \text{Ker} L = 0 \). We see that either \( LW = 0 \) or \( LW \) is isomorphic to \( W \). Hence \( LW^* \subseteq W^* \). Using the isotypic decomposition (2.13), we are now in a position to perform our eigenvalue structure analysis.

We see immediately that \( L = 0 \) when restricted to \( W^2_{++} \) \((L = 0 \) when restricted to \( V_0 \cong \Gamma \ W_{++} \)), \( W_{+-} \), and \( W_{-+} \) \((n \) even\) since \( \lambda \) an eigenvalue of any \( L \)-invariant subspace implies \( -\lambda \) also an eigenvalue. Together with the two zero eigenvalues corresponding to horizontal translation, we have so far counted five zero eigenvalues when \( n \) is odd, and an additional zero eigenvalue when \( n \) is even.
Next we consider the components $W^2_-$ and $W^3_k$. $W_-$ is one-dimensional so it is trivially absolutely irreducible. The action of $\Gamma$ on $W_k \cong \mathbb{C}$ (given by (2.2)) is also absolutely irreducible (in the sense that when $W_k$ is viewed as a real space, the only linear maps that commute with $\Gamma$ are multiples of the identity). Hence in an appropriate basis,

$$L|_W = \begin{pmatrix} a_{11}I & \cdots & a_{1s}I \\ \vdots & \ddots & \vdots \\ a_{s1}I & \cdots & a_{ss}I \end{pmatrix}$$

(3.4)

where $I = 1$, $s = 2$ for $W^1 = W^2_-$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $s = 3$ for $W^2 = W^3_k$. We see that the eigenvalues of $L|_W$, are exactly the eigenvalues of

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}$$

(3.5)

each with multiplicity $\text{dim}W$. Since $\lambda$ is an eigenvalue of $A$ if and only if $-\lambda$ is, we see that $W^2_-$ has eigenvalues $\pm \lambda$, while $W^3_k$ has eigenvalues $0$, $\pm \lambda$ each of multiplicity two. Since there are $\text{int}(\frac{n-1}{2})$ components $W^3_k$, we have counted $2 \cdot \text{int}(\frac{n-1}{2})$ zero eigenvalues and $\text{int}(\frac{n-1}{2})$ pairs $\pm \lambda$ of multiplicity two. When $n$ is even, $W^2_-$ contributes an additional pair $\pm \lambda$. However, the two zero eigenvalues from $W^3_k$ correspond to $H_0$ so they have already been counted. We actually have $2 \cdot \text{int}(\frac{n-3}{2})$ additional zero eigenvalues.

In fact, we can say more. Noting that $W^2_-$ arises (in the $n$ even case) from $V_{n/2}$ and part of $H_{n/2+1}$, we see that in some basis, $R|_{W^2_-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then if in this basis $A = (a_{ij})$, time reversibility (H1) implies

$$\begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} = \begin{pmatrix} -a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix},$$

(3.6)

i.e., $a_{11} = a_{22} = 0$. We see that the eigenvalues of $A$ are real or pure imaginary. In the case of $W^3_k$, we have (in some basis),

$$R|_{W^3_k} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix}$$

(3.7)

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If in this basis, $A = (a_{ij})$, $RL = -LR$ (H1) implies

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{12} & a_{13} \\ -a_{21} & -a_{22} & a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix},$$

(3.8)

i.e., $a_{11} = a_{12} = a_{21} = a_{22} = a_{33} = 0$. Hence the eigenvalues of $A$ satisfy $\lambda(\lambda^2 - a_{31}a_{13} - a_{32}a_{23}) = 0$. Again, all eigenvalues are real or pure imaginary.

In summary, we have:
Theorem I (Eigenvalue Structure). Consider \( \frac{df}{dt} = f(r) \) where \( f(r) \) has the form (0.2) and satisfies the symmetry relations (H1–H3). Then the linearization of \( f \) at maximally symmetric equilibria has:

(i) 5 zero eigenvalues corresponding to three dimensions of translational motion, one dimension of scale contraction or expansion, and one dimension of rotation about center (average position);

(ii) 1 additional zero eigenvalue when \( n \) is even, corresponding to a type of horizontal displacement which produces vertical translation only—the relevant subspace is that spanned by \( r \in H_{n/2+1} \) where \( r_1 = (-\sin \frac{2\pi}{n}, \cos \frac{2\pi}{n}, 0) \);

(iii) \( \text{int}(\frac{n-1}{2}) \) pairs of real or pure imaginary eigenvalues \( \pm \lambda \), each of multiplicity two;

(iv) \( 2 \cdot \text{int}(\frac{n-3}{2}) \) more zero eigenvalues (forced by time reversibility);

(v) 1 additional pair of real or pure imaginary eigenvalues \( \pm \lambda \) when \( n \) is even.

4. Relation Between Eigenvalues of \( L \) and \( L^2 \):

It turns out that it is convenient to obtain the eigenvalues of \( L \) from the eigenvalues of \( L^2 \mid_V \) so we exhibit the relation between the two sets of eigenvalues. By time reversibility (H1) we have \( fR = -Rf \) so that \( L^2R = RL^2 \). We see that \( L^2H \subset H \) and \( L^2V \subset V \).

Consider \( (\lambda, u) \) an eigenvalue/vector pair for \( L \). From (3.3), \( Lu = \lambda u \) and \( L(Ru) = -\lambda(Ru) \), so for \( \lambda \neq 0 \), \( u \) must have nonzero vertical part. Writing \( u = h + v \) where \( h \in H, v \in V \), we have \( L^2h + L^2v = \lambda^2h + \lambda^2v \). Since \( H \) and \( V \) are \( L^2 \)-invariant, we have in particular,

\[
L^2v = \lambda^2v. \tag{4.1}
\]

Hence every nonzero eigenvalue of \( L \) is the square root of an eigenvalue of \( L^2 \mid_V \).

Conversely, let \( (\mu, v) \) be an eigenvalue/vector pair for \( L^2 \mid_V \). Since \( LW^s \subset W^s \) for any isotypic component \( W^s \), each \( W^s \) is also \( L^2 \)-invariant. For some \( W^s \), there exists \( u \in W^s \) such that \( L^2u = \mu u \). For \( L \mid_W \neq 0 \), we know \( L \mid_W \) has exactly one pair of eigenvalues \( \pm \lambda \) (of some multiplicity but semisimple) and the rest zero. Letting \( E_{\pm} \) denote the eigenspaces corresponding to \( \pm \lambda \), we can write \( u = u_+ + u_- + n \) such that \( u_+ \in E_+ \), \( u_- \in E_- \), and \( n \in \text{Ker}L \). Then

\[
L^2u = L^2(u_+ + u_- + n) = L(\lambda u_+ - \lambda u_-) = \lambda^2(u_+ + u_-) = \mu u = \mu(u_+ + u_- + n). \tag{4.2}
\]
From (4.2), we have

\[(\lambda^2 - \mu)(u_+ + u_-) = \mu n\]  
\tag{4.3}

so when \(\mu \neq 0\), we have \(n = 0\) and \(\mu = \lambda^2\). We see that every nonzero eigenvalue of \(L^2|_V\) is the square of an eigenvalue of \(L\). (In fact, the same would be true even in the nonsemisimple case.)

Thus we can find the (nonzero) eigenvalues of \(L\) by finding the (nonzero) eigenvalues of \(L^2|_V\) and taking their square roots. We define

\[\mathcal{L} \equiv L^2|_V\]  
\tag{4.4}

and note that the eigenvalues of \(\mathcal{L}\) must be real (since the eigenvalues of \(L\) are real or pure imaginary).

5. Eigenvalues of \(\mathcal{L}\):

Since \(\gamma L^2 = L\gamma L = L^2\gamma\) for all \(\gamma \in \Gamma\), we have

\[\hat{\mathcal{L}} = \mathcal{L} \hat{\mathcal{C}} \quad \text{and} \quad \hat{\mathcal{T}} \mathcal{L} = \mathcal{L} \hat{\mathcal{T}}.\]  
\tag{5.1}

This means that (in the basis \((z_1, z_2, \ldots, z_n)\)), advancing the rows of \(\mathcal{L}\) by 1 is equivalent to advancing the columns by \(-1\), and that exchanging the \(j\)th and \((n-j)\)th rows of \(\mathcal{L}\) for all \(j\) is equivalent to exchanging the \(j\)th and \((n-j)\)th columns for all \(j\). Hence

\[\mathcal{L} = \begin{pmatrix}
L_n & L_1 & L_2 & \cdots & L_{n-1} \\
L_{n-1} & L_n & L_1 & \cdots & L_{n-2} \\
L_{n-2} & L_{n-1} & L_n & \cdots & L_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
L_1 & L_2 & L_3 & \cdots & L_n
\end{pmatrix} \quad \text{with} \quad L_j = L_{n-j}  
\tag{5.2}

Direct calculation in this basis gives

\[L_j = \sum_{k=1}^n \left[ \frac{\partial f^*_n}{\partial x_k} \left|_{r_n} \right. \cdot \frac{\partial f^*_k}{\partial z_j} \left|_{r_n} \right. + \frac{\partial f^*_n}{\partial y_k} \left|_{r_n} \right. \cdot \frac{\partial f^*_k}{\partial z_j} \left|_{r_n} \right. \right].\]  
\tag{5.3}

We see by (5.2) that \(\mathcal{L}\) is circulant. Hence \(\mathcal{L}\) can be diagonalized by

\[U = \frac{1}{\sqrt{n}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2}
\end{pmatrix}, \quad \omega = e^{2\pi i/n}  
\tag{5.4}

as

\[U \cdot L U = \text{diag}\left(p(1), p(\omega), \ldots, p(\omega^{n-1})\right)\]  
\tag{5.5a}

where

\[ p(\mu) = L_n + L_1 \mu + \cdots + L_{n-1} \mu^{n-1} \]  

(5.5b)

and \( U^* \) is the conjugate transpose of \( U \) (and equal to \( U^{-1} \)). (For diagonalization of circulant matrices, cf. [O].) Hence the eigenvalues of \( L \) are

\[ p(1), \ p(\omega), \ \ldots, \ p(\omega^{n-1}) \]  

(5.6)

with \( p \) given by (5.5b).

6. **Normal Form for \( F = (F^x, F^y, F^z) \):**

The calculations so far have depended only on the time reversibility and \( D_n \times O(2) \) symmetry of the vector field \( f(\tau) \) and thus apply to general problems with those symmetries. (We may replace the two-body interaction form (0.2) for \( f \) with the assumption that \( f \) commutes with an action of \( D_n \) fixing some horizontal equilibrium and then calculate eigenvalues exactly as described.) Here we obtain the general form of vector fields in our class up to linearization about \( r^0 \) using the full information we have on \( f(\tau) \). We show that the entire class of problems may be divided into two families, parametrized by the degree of singularity \( \beta \) associated with the homogeneity of \( f(\tau) \) and by the relative magnitudes of the vertical versus horizontal components of the vector field.

First, \( f \) is time-reversible (H1) if and only if

\[ F^x(z, y, -z) = -F^x(z, y, z) \]

\[ F^y(z, y, -z) = -F^y(z, y, z) \]  

(6.1)

\[ F^z(z, y, -z) = F^z(z, y, z) \]

Since we are concerned with the linearization of \( f \) about \( r^0 \), we may take as representative,

\[ F^x(z, y, z) = a(z, y)z \]

\[ F^y(z, y, z) = b(z, y)z \]

\[ F^z(z, y, z) = c(z, y) \]  

(6.2)

We must also incorporate the implications of \( O(2) \) symmetry (H2) (cf. [GSS]). \( SO(2) \) symmetry (commutation with rotations) implies \( (F^x, F^y) \) has length dependent only on \( z \) and \(|(z, y)| = \sqrt{z^2 + y^2}\), and direction
given by a rotation \( \phi \) of \( \arg((x, y)) \) where \( \phi \) depends only on \( z \). Moreover, \( F^z \) must depend only on \( z \) and \( |(x, y)| \). Hence we have representative forms
\[
a(x, y) = g(x^2 + y^2)(x \cos \phi - y \sin \phi) \\
b(x, y) = g(x^2 + y^2)(x \sin \phi + y \cos \phi) \tag{6.3}
\]
c(x, y) = c(x^2 + y^2)
where \( \phi \) is a constant. In addition, we require flip symmetry in the \((x, y)\)-plane. We should therefore take \( \sin \phi = 0 \). Finally, homogeneity (H3) forces the functions \( g \) and \( c \) to take particular monomial (possibly fractional powered) forms depending on \( \beta \).

We have
\[
f_j(r) = \sum_{k \neq j} \frac{F(r_j - r_k)}{|r_j - r_k|^\alpha} \tag{0.2}
\]
with \( F = (F^x, F^y, F^z) \) given by
\[
\begin{align*}
F^x(x, y, z) &= g \cdot (x^2 + y^2)^{\alpha/2 - \beta/2 - 1} \cdot zz \\
F^y(x, y, z) &= g \cdot (x^2 + y^2)^{\alpha/2 - \beta/2 - 1} \cdot yz \\
F^z(x, y, z) &= c \cdot (x^2 + y^2)^{\alpha/2 - \beta/2}
\end{align*} \tag{0.4}
\]
where \( g \) and \( c \) are now constants. Although \( \alpha \) appears in this form for \( f(r) \), it is not a parameter since in linearization about \( r^0 \), the effects of \( \alpha \) factor out of the numerator and denominator. Thus we arrive at our normal form for \( f \):
\[
f_j(r) = \sum_{k \neq j} \frac{F(r_j - r_k)}{|r_j - r_k|^\beta+2} \tag{0.2'}
\]
with \( F = (F^x, F^y, F^z) \) given by
\[
\begin{align*}
F^x(x, y, z) &= g \cdot zz \\
F^y(x, y, z) &= g \cdot yz \\
F^z(x, y, z) &= c \cdot (x^2 + y^2)
\end{align*} \tag{0.5}
\]
We see that up to linearization, our class of vector field problems is parametrized by \( g/c \) and \( \beta \). The class may be divided into two families according to the sign of \( g/c \). When \( g/c > 0 \), we say that the vertical and horizontal components of the vector field act "in concert"; when \( g/c < 0 \), we say they act "in opposition". The Stokeslet problem corresponds to \( g/c = \beta = 1 \) and the dipole example to \( g/c = -3, \beta = 3 \).
7. Eigenvalue Calculation:

We are now in a position to obtain formulas for the nonzero eigenvalues of the linearization of $f(r)$ at $r^0$ as functions of the singularity $\beta$ and the constants $g$ and $c$. Combining (5.3), (5.5b), and (6.5), and after much manipulation (see Appendix), we arrive at

$$
\lambda_l^2 = -gc\beta \sum_{j \neq n} \sum_{k \neq n} \frac{(1 + \cos \frac{2\pi kj}{n})(1 - \cos \frac{2\pi k}{n})(1 - \cos \frac{2\pi l}{n}) - \sin \frac{2\pi kj}{n} \sin \frac{2\pi k}{n} \sin \frac{2\pi l}{n}}{(2 \left(1 - \cos \frac{2\pi k}{n}\right))^{\beta/2 + 1} \left(2 \left(1 - \cos \frac{2\pi l}{n}\right)\right)^{\beta/2 + 1}}
$$

(7.1)

or

$$
\lambda_l^2 = -\frac{gc\beta}{4} \sum_{j \neq n} \sum_{k \neq n} \frac{(1 + \cos \frac{2\pi kj}{n})(1 - \cos \frac{2\pi l}{n}) - (1 + \cos \frac{2\pi k}{n})p_l(1 + \cos \frac{2\pi l}{n})p_l(1 + \cos \frac{2\pi k}{n})}{(2 \left(1 - \cos \frac{2\pi k}{n}\right))^{\beta/2} \left(2 \left(1 - \cos \frac{2\pi l}{n}\right)\right)^{\beta/2}}
$$

(7.1')

where $p_l$ is a notational device referring to the polynomial given by

$$
\sin (lt) = \sin t \cdot p_l(\cos t).
$$

(7.2)

(It can be shown easily by induction on $l$ that $\sin (lt)$ always has the form (7.2).) Hence up to sign and the scaling factor $|gc|$, the linearizations about $r^0$ in our class depend only on $\beta$. Throughout the rest of our discussion, we will take $gc = 1$. The implications for other values of $gc$ are obvious.

Noting that $\lambda_l^2 = \lambda_{n-l}^2$ and $\lambda_0^2 = 0$, we see that it is enough to compute $\lambda_l^2$ for $l = 1, \ldots, \text{int}(n/2)$. The double nature of these eigenvalues is as expected from our eigenvalue structure analysis. Figure 1 shows the dependence of $\lambda_l^2$ on $\beta$ for a typical $n$.

(Figure 1)

8. Stability Thresholds:

Recall that by stability of a case in our vector field class we mean that the linearization of $f(r)$ at $r^0$ has only zero and pure imaginary eigenvalues, i.e., that $\lambda_l^2$ is nonpositive for all $l$. When $\lambda_l^2$ is positive for some $l$, then $r^0$ is clearly unstable under $f(r)$. We investigate the cases $\beta < 0$ and $\beta > 0$ separately.

We begin with the case $\beta < 0$. From (7.1') we have for $l = 1$ (with $gc = 1$)

$$
\lambda_1^2 = -\frac{\beta}{4} \sum_{j \neq n} \sum_{k \neq n} \frac{(1 + \cos \frac{2\pi kj}{n})(1 - \cos \frac{2\pi k}{n}) - (1 + \cos \frac{2\pi l}{n})(1 + \cos \frac{2\pi k}{n})}{(2 \left(1 - \cos \frac{2\pi k}{n}\right))^{\beta/2} \left(2 \left(1 - \cos \frac{2\pi l}{n}\right)\right)^{\beta/2}}
$$

$$
= -\frac{\beta}{2} \sum_{k \neq n} \frac{1 + \cos \frac{2\pi k}{n}}{(2 \left(1 - \cos \frac{2\pi k}{n}\right))^{\beta/2}} \sum_{j \neq n} \frac{\cos \frac{2\pi j}{n}}{(2 \left(1 - \cos \frac{2\pi j}{n}\right))^{\beta/2}}.
$$

(8.1)
Since the sum over \( k \) in (8.1) is always positive and \( \beta \) is negative, the sign of \( \lambda_{1}^{2} \) depends only on the sign of the sum over \( j \) in (8.1)

\[
s(\beta, n) = \sum_{j \neq n} \frac{\cos \frac{2\pi j}{n}}{\left(2 \left(1 - \cos \frac{2\pi j}{n}\right)\right)^{\beta/2}}.
\]  

(8.2)

Keeping \( n \) fixed, we first observe that as \( \beta \to 0^- \), \( s(\beta, n) \to -1 \). Next, we note that all terms of \( s(\beta, n) \) are nondecreasing functions of \( \beta \) except when \( j \) is such that \( \pi/3 < \frac{2\pi j}{n} < \pi/2 \) or \( 2\pi - \pi/2 < \frac{2\pi j}{n} < 2\pi - \pi/3 \). On the other hand, when \( \frac{2\pi j}{n} \) falls in one of these regions, we can consider the sum of the \( j \)th and \( 2j \)th (modulo \( n \)) terms. If we define

\[
s_{j}(\beta, n) \equiv \frac{\cos \frac{2\pi j}{n}}{\left(2 \left(1 - \cos \frac{2\pi j}{n}\right)\right)^{\beta/2}} + \frac{\cos \frac{2\pi 2j}{n}}{\left(2 \left(1 - \cos \frac{2\pi 2j}{n}\right)\right)^{\beta/2}}
\]

then differentiating with respect to \( \beta \) and rearranging, we obtain

\[
s'_{j}(\beta, n) > 0
\]

(8.4)

if and only if

\[
\beta < 2 \cdot \ln \left( \frac{\cos \frac{2\pi}{n} \ln (2(1 - \cos \frac{2\pi}{n}))}{-\cos \frac{2\pi}{n} \ln (2(1 - \cos \frac{2\pi}{n}))} \right) \ln (2(1 + \cos \frac{2\pi}{n})).
\]

(8.5)

But the right hand side of (8.5) is always positive so when \( \beta < 0 \), (8.5) and hence (8.4) holds. We see that \( s(\beta, n) \) is nondecreasing and approaches \(-1\) as \( \beta \to 0^- \) so \( s(\beta, n) \) is negative. Then we have \( \lambda_{1}^{2} > 0 \) for all \( \beta < 0 \) (any \( n \)) so that the \( \beta < 0 \) case is always unstable. (Note that by "any \( n \)" we mean \( n \geq 3 \) since the cases \( n = 1, 2 \) are static in relative coordinates.)

Next we consider the \( \beta > 0 \) case. We will require the identities

\[
\sum_{k \neq n} 1 = n - 1
\]

\[
\sum_{k \neq n} \cos \frac{2\pi k}{n} = -1
\]

(8.6)

\[
\sum_{k \neq n} (1 + \cos \frac{2\pi k}{n})p_{l}(\cos \frac{2\pi k}{n}) = n - 2l
\]

The first is obvious, the second is clear when we consider that \( \cos \frac{2\pi k}{n} = \text{Re}(\exp i \frac{2\pi k}{n}) \), and the third is easily proved by induction on \( n \). From the formulas for \( \lambda_{1}^{2} \) (7.1') and the identities (8.6), we have as \( \beta \to 0^+ \),

\[
\lambda_{1}^{2} \to \frac{-\beta}{4} \left[(n - 1 - 1)(n - 1 + 1) - (n - 2l)^{2}\right] = \beta \left[n(\frac{1}{2} - l) + l^{2}\right]
\]

(8.7)
so that \( \lambda_1^2 \to 0^- \) as \( \beta \to 0^+ \). On the other hand, as \( \beta \to +\infty \), the dominant terms in the sum for \( \lambda_1^2 \) are those corresponding to \( j, k = 1, n - 1 \) so that

\[
\lambda_1^2 \to 4\beta \frac{(1 - \cos^2 \frac{2\pi l}{n}) \cos \frac{2\pi}{n}}{(2(1 - \cos \frac{2\pi}{n}))^{\beta+1}} \tag{8.8}
\]
as long as the right hand side of (8.8) is nonzero (otherwise the next term must be examined). We see that as \( \beta \to +\infty \), \( \lambda_1^2 \) is positive as long as \( n > 4 \) (and \( l \neq n/2 \) for \( n \) even). Hence it is clear that for \( n > 4 \), the stability threshold \( \beta^*(n) \) exists. Studying the \( \lambda_1^2 \) numerically for \( n \leq 20 \), the first eigenvalue squared in each case to become positive as \( \beta \) increases is \( \lambda_1^2 \). It can be shown that \( \lambda_1^2 \) is zero for exactly one value of positive \( \beta \) by studying the behavior of \( s(\beta, n) \) defined in (8.2). Table 1 shows approximate values of \( \beta^*(n) \) for \( 4 < n \leq 20 \). For \( n = 5 \), we have in fact \( \beta^*(5) = 2 \) exactly: writing \( s(2, 5) \) in terms of the golden number \( \tau = 2 \cos(\pi/5) = (1 + \sqrt{5})/2 \) (cf. [C]), it is easily seen that \( s(2, 5) = 0 \).

<table>
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<th>( n )</th>
<th>( \beta^*(n) )</th>
<th>( n )</th>
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</tr>
<tr>
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<td>20</td>
<td>0.142</td>
</tr>
</tbody>
</table>

Table 1: Stability Thresholds

The case for \( n \leq 4 \) is somewhat different. The property which distinguishes the cases \( n \leq 4 \) from the cases \( n > 4 \) is that in the former, none of the \( \frac{2\pi k}{n} \) have positive cosine while all in the latter have at least \( \cos \frac{2\pi}{n} > 0 \). In fact, the cases \( n \leq 4 \) are stable for all \( \beta > 0 \). To see this, we simply calculate the eigenvalues explicitly for \( n = 3, 4 \) (the cases \( n = 1, 2 \) are static in relative coordinates). For \( n = 3 \), \( \lambda_1^2 = -\frac{\beta}{2\tau^2} \) and for \( n = 4 \), \( \lambda_1^2 = -\frac{\beta}{2\tau^2}, \lambda_2^2 = -\frac{3\beta}{2\tau^2} \). Hence we have stability for all \( \beta > 0 \).

In summary, we have:

**Theorem II (Stability Threshold).** Consider \( \frac{dT}{dt} = f(r) \) where \( f(r) \) has the form (0.2) and satisfies the symmetry relations (H1–H3). Then for \( gc > 0 \) (with \( g \) and \( c \) as appearing in (6.5)): 
(i) for all \( n \geq 3 \), \( r^0 \) is unstable when \( \beta < 0 \);

(ii) for \( n = 3, 4 \), \( r^0 \) is stable when \( \beta > 0 \);

(iib) for \( n > 4 \), there exists \( \beta^*(n) < \infty \) such that \( r^0 \) is stable for all \( 0 < \beta < \beta^*(n) \) (take \( \beta^*(n) \) to be the largest such \( \beta \) for which this is true).

9. Nonresonant Periodic Solutions:

We begin by stating the time-reversible, spatially symmetric version of the nonresonant Liapunov center theorem essentially as it appears in [GKL]. Consider a system of ODEs

\[
\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n
\]  

(9.1)

with an equilibrium at \( x_0 \). Assume there exists \( R : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
f(Rx) = -Rf(x), \quad Rx_0 = x_0 \quad \text{(time reversibility)};
\]

and a compact Lie group \( \Gamma \) which acts on \( \mathbb{R}^n \) and such that

\[
f(\gamma x) = \gamma f(x), \quad \gamma x_0 = x_0 \quad \forall \gamma \in \Gamma \quad \text{(spatial symmetry)}.
\]

Next, consider \( \Sigma \) a subgroup of \( \Gamma \times S^1 \), where \( \Gamma \times S^1 \) acts on the Banach space of \( 2\pi \)-periodic mappings \( \mathbb{R} \to \mathbb{R}^n \) as

\[
(\gamma, \theta) \cdot x(t) = \gamma x(t + \theta) \quad \forall (\gamma, \theta) \in \Gamma \times S^1.
\]

A periodic solution \( x(t) \) of (9.1) is said to have symmetry \( \Sigma \) when \( (\gamma, \theta) \cdot x(t) = x(t) \) for all \( (\gamma, \theta) \in \Sigma \).

From [GKL] we have:

**Theorem.** Assume that the \( \Gamma \)-equivariant system (9.1) has a \( \Gamma \)-invariant equilibrium \( x_0 \). Assume

\[
\pm \omega_0 i \quad \text{are nonzero eigenvalues of } (df)_{x_0}, \text{ and}
\]

\[
k \omega_0 i \quad \text{is not an eigenvalue of } (df)_{x_0} \text{ for } k = 2, 3, \ldots
\]

Assume that the generalized eigenspace \( V_i \) corresponding to the eigenvalues \( \pm \omega_0 i \) has the form

\[
V_i = W \oplus W
\]
where $\Gamma$ acts absolutely irreducibly on $W$. Assume finally that (9.1) has time-reversal symmetry $R$ fixing $z_0$ and that the subgroup $\Sigma \subseteq \Gamma \times S^1$ satisfies

$$\dim \text{Fix}(\Sigma) \cap V_i = 2$$

$$R(\text{Fix}(\Sigma) \cap V_i) = \text{Fix}(\Sigma) \cap V_i$$

$$R|_{\text{Fix}(\Sigma) \cap \text{Ker}(df)_{z_0}} = I$$

Then there exists an $m+1$-parameter family of periodic solutions to (9.1), with period near $2\pi/\omega_0$ and symmetry $\Sigma$, where

$$m = \dim \text{Fix}(\Sigma) \cap \text{Ker}(df)_{z_0}.$$ 

To apply this theorem to our class of problems, we need to check several conditions. First, for $0 < \beta < \beta^*(n)$, we know all eigenvalues are pure imaginary so we may take $\omega_0 = |\tilde{\lambda}t|$, with any $t$ from $1, \ldots, \text{int}(n/2)$. The nonresonance condition ($k\omega_0 t$ not an eigenvalue for $k = 2, 3, \ldots$) must be checked but seems to hold generically in this class. That $V_i \subseteq W \oplus W$ holds can be seen by taking the isotypic decomposition (2.13) and arguing (by uniqueness of isotypic decompositions) that since the null part of any component $W_k^3$ is $\Gamma$-isomorphic to $W_k$, $V_i$ must be of the required form. (For $l = n/2$, the relevant component is $W_2^{2-}$, which is also of the required form.) We note finally that, apart from the null direction corresponding to vertical translation which may clearly be ignored, all zero eigenvalues of $(df)_r^o$ correspond to horizontal null directions so that $R|_{\text{Fix}(\Sigma) \cap \text{Ker}(df)_{z_0}} = I$ holds for any $\Sigma$ we choose. Hence to apply the theorem, it is sufficient to check nonresonance and to look for two-dimensional fixed point subspaces which are $R$-invariant.

We obtain:

**Theorem III (Nonresonant Periodic Solutions).** Consider $\frac{dr}{dt} = f(r)$ where $f(r)$ has the form (0.2) and satisfies the symmetry relations (H1–H3). Let $g_c > 0$ (with $g$ and $c$ as appearing in (6.5)) and let $0 < \beta < \beta^*(n)$. Let $\lambda$ be any nonzero eigenvalue of $(df)_r^o$ and $E_{\pm\lambda}$ the eigenspace corresponding to $\pm\lambda$. Assume the nonresonance condition holds that

$$k\lambda \quad \text{is not an eigenvalue of } (df)_r^o \text{ for } k = 2, 3, \ldots$$

If $\Sigma$ is any subgroup of $\Gamma \times S^1 \cong D_n \times S^1$ such that
\[(i) \dim \text{Fix}(\Sigma) \cap E_{\pm\lambda} = 2; \]

\[(ii) \text{Fix}(\Sigma) \cap E_{\pm\lambda} \text{ is left invariant by } R; \]

\[(iii) \dim \text{Fix}(\Sigma) \cap \text{Ker}(df),v = m; \]

then there exists an \(m + 1\)-parameter family of periodic solutions to \(\frac{df}{dt} = f(r)\) with period near \(2\pi/|\lambda|\) and symmetry \(\Sigma\).

The resulting symmetric families of periodic solutions are listed and described in [GKL] for \(3 \leq n \leq 6\) (there for the Stokeslet problem, but the results apply equally here). The relevant subgroups of \(D_n \times S^1\) are listed in full in [GSS]. Independent calculations for the period of small orbits in the Stokeslet problem when \(n = 4\) for synchronous rhombi (corresponding here to the eigenvalues \(\pm\lambda_2\)) are performed in [TK].

10. Remarks:

The focus here has been on exploiting symmetry properties to extract information about our class of problems. Several related vector field classes (e.g. inhomogeneous problems satisfying the other symmetry relations of our class, or \(Z_n \times SO(2)\) rather than \(D_n \times O(2)\)) can be studied in essentially the same way. The information obtained here about the linearized problem provides the basis for nonlinear investigations such as the determination of families of near-equilibrium periodic solutions performed above.

Since physical problems tend to have integer values of the singularity strength \(\beta\), and since \(\beta^*(n) < 1\) for \(n > 6\), we do not expect to find clusters of more than six bodies persisting in nature for problems of the type considered when vertical and horizontal components act in concert (e.g. in the sedimentation problem). In fact, when \(\beta \neq 1\), under decay-type interaction, we expect only to find clusters of 3, 4, or 5 bodies. When the components act in opposition (as in the dipole example), we do not expect to find clusters of 3 or 4 bodies under decay-type interaction.

The parameter \(\beta\) may also be viewed as a bifurcation parameter for the family of decay-type problems in the class considered above. It can be shown that the family undergoes a time-reversible, equivariant pitchfork bifurcation as \(\beta\) passes through the critical value \(\beta^*(n)\) [M].

References:


Appendix: Derivation of Eigenvalue Formulas

(Note: indices should be taken modulo \( n \) where appropriate.)
The nonzero eigenvalues of the linearization of \( f(\mathbf{r}) \) at \( \mathbf{r}^0 \) are given by

\[
\lambda_i^2 = p(\omega^i)
\]  
(5.6)

where \( \omega = \exp \, i 2\pi / n \) and

\[
p(\mu) = L_n + L_1 \mu + \cdots + L_{n-1} \mu^{n-1}
\]  
(5.5b)

with

\[
L_j = \sum_{k=1}^{n} \left[ \frac{\partial f_x}{\partial x_k} \right]_{\mathbf{r}^0} \cdot \left( \frac{\partial f_y}{\partial z_j} \right)_{\mathbf{r}^0} + \left( \frac{\partial f_x}{\partial y_k} \right)_{\mathbf{r}^0} \cdot \left( \frac{\partial f_y}{\partial z_j} \right)_{\mathbf{r}^0} \right].
\]  
(5.3)

If we define

\[
A_{13}(i,j) = \left. \frac{\partial f_x}{\partial z_j} \right|_{r^0}, \quad A_{23}(i,j) = \left. \frac{\partial f_y}{\partial z_j} \right|_{r^0}, \quad A_{31}(i,j) = \left. \frac{\partial f_x}{\partial y_k} \right|_{r^0}, \quad A_{32}(i,j) = \left. \frac{\partial f_y}{\partial y_k} \right|_{r^0}
\]  
(A.1)

we have

\[
L_j = \sum_{k=1}^{n} \left[ A_{31}(n,k)A_{13}(k,j) + A_{32}(n,k)A_{23}(k,j) \right].
\]  
(5.3')

Taking the appropriate derivatives and substituting in \( \mathbf{r}^0 \), we have:

for \( k \neq n \)

\[
A_{31}(n,k) = -\frac{\beta c \left( 1 - \cos \frac{2\pi k}{n} \right)}{(2 \left( 1 - \cos \frac{2\pi k}{n} \right))^{1/2} + 1}
\]

\[
A_{31}(n,n) = -\sum_{k \neq n} A_{31}(n,k)
\]  
(A.2)

\[
A_{32}(n,k) = \frac{\beta c \left( -\sin \frac{2\pi k}{n} \right)}{(2 \left( 1 - \cos \frac{2\pi k}{n} \right))^{1/2} + 1}
\]

\[
A_{32}(n,n) = -\sum_{k \neq n} A_{32}(n,k)
\]

for \( k \neq j \)

\[
A_{13}(k,j) = \frac{g \left( \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right)}{(2 \left( 1 - \cos \frac{2\pi (j-k)}{n} \right))^{1/2} + 1}
\]

\[
A_{13}(j,j) = -\sum_{k \neq j} A_{13}(j,k) = \sum_{k \neq j} A_{13}(k,j)
\]  
(A.3)

\[
A_{23}(k,j) = \frac{g \left( \sin \frac{2\pi j}{n} - \sin \frac{2\pi k}{n} \right)}{(2 \left( 1 - \cos \frac{2\pi (j-k)}{n} \right))^{1/2} + 1}
\]

\[
A_{23}(j,j) = -\sum_{k \neq j} A_{23}(j,k) = \sum_{k \neq j} A_{23}(k,j)
\]

We see that for \( j \neq n \), we have

\[
L_j = \sum_{k=1}^{n} \left[ A_{31}(n,k)A_{13}(k,j) + A_{32}(n,k)A_{23}(k,j) \right]
\]

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\[ = \sum_{k \neq j,n} [A_{31}(n,k)A_{13}(k,j) + A_{32}(n,k)A_{23}(k,j)] + \sum_{k \neq j} [A_{31}(n,j)A_{13}(k,j) + A_{32}(n,j)A_{23}(k,j)] \\
- \sum_{k \neq j,n} [A_{31}(n,k)A_{13}(n,j) + A_{32}(n,k)A_{23}(n,j)] \\
= \sum_{k \neq j,n} [A_{31}(n,k)A_{13}(k,j) + A_{32}(n,k)A_{23}(k,j)] + \sum_{k \neq j,n} [A_{31}(n,j)A_{13}(k,j) + A_{32}(n,j)A_{23}(k,j)] \\
+ [A_{31}(n,j)A_{13}(n,j) + A_{32}(n,j)A_{23}(n,j)] - \sum_{k \neq j,n} [A_{31}(n,k)A_{13}(n,j) + A_{32}(n,k)A_{23}(n,j)] \\
- [A_{31}(n,j)A_{13}(n,j) + A_{32}(n,j)A_{23}(n,j)] \\
= \sum_{k \neq j,n} [A_{31}(n,k)A_{13}(k,j) + A_{32}(n,k)A_{23}(k,j)] + \sum_{k \neq j,n} [A_{31}(n,j)A_{13}(k,j) + A_{32}(n,j)A_{23}(k,j)] -\sum_{k \neq j,n} [A_{31}(n,k)A_{13}(n,j) + A_{32}(n,k)A_{23}(n,j)] \quad (A.4) \]

We also have

\[ L_n = \sum_{k=1}^{n} [A_{31}(n,k)A_{13}(k,n) + A_{32}(n,k)A_{23}(k,n)] \]
\[ = \sum_{k \neq n} [A_{31}(n,k)A_{13}(k,n) + A_{32}(n,k)A_{23}(k,n)] + \sum_{k \neq n} A_{31}(n,k) \sum_{j \neq n} A_{13}(n,j) + \sum_{k \neq n} A_{32}(n,k) \sum_{j \neq n} A_{23}(n,j) \]
\[ = \sum_{k \neq n} [A_{31}(n,k)(A_{13}(k,n) + \sum_{j \neq n} A_{13}(n,j)) + A_{32}(n,k)(A_{23}(k,n) + \sum_{j \neq n} A_{23}(n,j)) - \sum_{k \neq n} A_{31}(n,k)A_{13}(n,j) + A_{32}(n,k)A_{23}(n,j)] \]
\[ = \sum_{k \neq n} \sum_{j \neq k,n} [A_{31}(n,k)A_{13}(n,j) + A_{32}(n,k)A_{23}(n,j)] \quad (A.5) \]

Noting that

\[ \sum_{j \neq n} L_j = \sum_{j \neq n} \sum_{k \neq j,n} [A_{31}(n,k)A_{13}(k,j) + A_{32}(n,k)A_{23}(k,j)] + \sum_{j \neq n} \sum_{k \neq j,n} [A_{31}(n,j)A_{13}(k,j) + A_{32}(n,j)A_{23}(k,j)] - L_n \]

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\[
\begin{align*}
&= \sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(k, j) + A_{32}(n, k)A_{23}(k, j) \right] \\
&\quad - \sum_{k \neq n} \sum_{j \neq k, n} \left[ A_{31}(n, j)A_{13}(j, k) + A_{32}(n, j)A_{23}(j, k) \right] - L_n \\
&= -L_n
\end{align*}
\]
we see that the eigenvalues we seek are

\[
\lambda_i^2 \equiv p(\omega^i) = -\sum_{j=1}^n L_j \left( 1 - \cos \frac{2\pi j}{n} \right) + i \sum_{j=1}^n L_j \left( \sin \frac{2\pi j}{n} \right).
\]

But \( L_j = L_{n-j} \) so \( \text{Im}(\lambda_i^2) = 0 \) (as expected). Hence we have

\[
\lambda_i^2 = -\sum_{j=1}^n L_j \left( 1 - \cos \frac{2\pi j}{n} \right)
\]

where \( L_j \) is given by (A.4). Note that \( \lambda_0^2 = \lambda_{n-1}^2 \) and \( \lambda_n^2 = 0 \).

Substituting (A.4) into (A.8) we obtain

\[
\begin{align*}
\lambda_i^2 &= -\sum_{j \neq n} L_j \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&= -\sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(k, j) + A_{32}(n, k)A_{23}(k, j) \right] \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&\quad - \sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, j)A_{13}(j, k) + A_{32}(n, j)A_{23}(j, k) \right] \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&\quad + \sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(n, j) + A_{32}(n, k)A_{23}(n, j) \right] \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&= -\sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(k, j) + A_{32}(n, k)A_{23}(k, j) \right] \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&\quad + \sum_{k \neq j, n} \sum_{j \neq k, n} \left[ A_{31}(n, k)A_{13}(k, j) + A_{32}(n, k)A_{23}(k, j) \right] \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&\quad + \sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(n, j) + A_{32}(n, k)A_{23}(n, j) \right] \left( 1 - \cos \frac{2\pi j}{n} \right) \\
&= \sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(k, j) + A_{32}(n, k)A_{23}(k, j) \right] \left( \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right) \\
&\quad + \sum_{j \neq n} \sum_{k \neq j, n} \left[ A_{31}(n, k)A_{13}(n, j) + A_{32}(n, k)A_{23}(n, j) \right] \left( 1 - \cos \frac{2\pi j}{n} \right)
\end{align*}
\]

Now substituting according to (A.2–A.3),

\[
\begin{align*}
\lambda_i^2 &= \underbrace{g_c \beta \sum_{j \neq n} \sum_{k \neq j, n}}_{(A.10)} \left\{ \frac{(1 - \cos \frac{2\pi k}{n})(\cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n}) - \sin \frac{2\pi k}{n}(\sin \frac{2\pi j}{n} - \sin \frac{2\pi k}{n})}{(2 \left( 1 - \cos \frac{2\pi j}{n} \right))^\beta/2 + 1} \right\} \\
&\quad \cdot \left( \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right) \\
&= \underbrace{g_c \beta \sum_{j \neq n} \sum_{k \neq j, n}}_{(A.10)} \left\{ \frac{(1 - \cos \frac{2\pi k}{n})(1 - \cos \frac{2\pi j}{n}) + \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n}}{(2 \left( 1 - \cos \frac{2\pi j}{n} \right))^\beta/2 + 1} \right\} \\
&\quad \cdot \left( \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right)
\end{align*}
\]
Simplifying (A.10),

\[
\lambda^2 = g c_\beta \sum_{j \neq n} \sum_{k \neq j, n} \left[ \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right] + \left( 1 - \cos \frac{2\pi j}{n} \right) \left( \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right) (2 \left( 1 - \cos \frac{2\pi j}{n} \right))^{\beta/2+1} \left( \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \right)
\]

\[
- g c_\beta \sum_{j \neq n} \sum_{k \neq j, n} \left[ \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right] + \left( 1 - \cos \frac{2\pi j}{n} \right) \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right) (2 \left( 1 - \cos \frac{2\pi j}{n} \right))^{\beta/2+1} \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right)
\]

\[
= g c_\beta \sum_{k \neq j, n} \sum_{j \neq n} \left[ \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right] + \left( 1 - \cos \frac{2\pi j}{n} \right) \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right) (2 \left( 1 - \cos \frac{2\pi j}{n} \right))^{\beta/2+1} \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right)
\]

\[
= g c_\beta \sum_{k \neq j, n} \sum_{j \neq n} \left[ \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right] + \left( 1 - \cos \frac{2\pi j}{n} \right) \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right) (2 \left( 1 - \cos \frac{2\pi j}{n} \right))^{\beta/2+1} \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right)
\]

Defining

\[
C(k) = \frac{\beta C}{(2 \left( 1 - \cos \frac{2\pi k}{n} \right))^{\beta/2+1}}
\]

\[
G(j) = \frac{g}{(2 \left( 1 - \cos \frac{2\pi j}{n} \right))^{\beta/2+1}}
\]

we have

\[
\lambda^2 = \sum_{j \neq n} \sum_{k \neq n} C(k) G(j) \left[ \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right) + \left( 1 - \cos \frac{2\pi j}{n} \right) \left( \cos \frac{2\pi j(k-i)}{n} - \cos \frac{2\pi k}{n} \right) \right]
\]

\[
= \sum_{j \neq n} \sum_{k \neq n} C(k) G(j) \left[ \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} + \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} \right]
\]

\[
= \sum_{j \neq n} \sum_{k \neq n} C(k) G(j) \left\{ \begin{array}{l}
\cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} + \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} + \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} (1 - \cos \frac{2\pi j}{n}) \\
- \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} + \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} (1 - \cos \frac{2\pi j}{n}) \\
+ \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \sin \frac{2\pi j}{n} + \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} (1 - \cos \frac{2\pi j}{n}) \\
- \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \sin \frac{2\pi j}{n} + \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} (1 - \cos \frac{2\pi j}{n}) \\
- \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} (1 - \cos \frac{2\pi j}{n}) \\
+ \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} - \cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} (1 - \cos \frac{2\pi j}{n}) \\
(1 - \cos \frac{2\pi j}{n}) \cos \frac{2\pi k}{n} - (1 - \cos \frac{2\pi j}{n}) \sin \frac{2\pi j}{n} \sin \frac{2\pi k}{n} \\
+ (1 - \cos \frac{2\pi j}{n}) \cos \frac{2\pi k}{n} - (1 - \cos \frac{2\pi j}{n}) (1 - \cos \frac{2\pi j}{n})
\end{array} \right\}
\]
Eliminating terms which are odd in $k$ (since they sum to zero over $k \neq n$) and rearranging,

$$
\lambda^2_i = \sum_{j \neq n} \sum_{k \neq n} C(k)G(j) \cdot \begin{cases}
\cos \frac{2\pi k}{n} \cos \frac{2\pi j}{n} \\ + \cos \frac{2\pi k}{n} \left( \cos \frac{2\pi j}{n} - \cos \frac{2\pi j}{n} \right) \\ + \cos \frac{2\pi k}{n} \left( \cos \frac{2\pi j}{n} + \left( 1 - \cos \frac{2\pi j}{n} \right) \right) \\
- \left( 1 - \cos \frac{2\pi j}{n} \right) \left( 1 - \cos \frac{2\pi j}{n} \right) + \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} \sin \frac{2\pi j}{n} \left( \frac{2\pi j}{n} \right)
\end{cases}
$$

$$
= -\sum_{j \neq n} \sum_{k \neq n} C(k)G(j) \cdot \begin{cases}
(1 + \cos \frac{2\pi k}{n} - \cos \frac{2\pi k}{n} - \cos \frac{2\pi k}{n} \cos \frac{2\pi k}{n}) \left( 1 - \cos \frac{2\pi j}{n} \right) \left( 1 - \cos \frac{2\pi j}{n} \right) \\
- \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} \sin \frac{2\pi j}{n} \sin \frac{2\pi j}{n}
\end{cases}
$$

$$
= -\sum_{j \neq n} \sum_{k \neq n} C(k)G(j) \cdot \begin{cases}
(1 + \cos \frac{2\pi k}{n}) \left( 1 - \cos \frac{2\pi k}{n} \right) \left( 1 - \cos \frac{2\pi j}{n} \right) \left( 1 - \cos \frac{2\pi j}{n} \right) \\
- \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} \sin \frac{2\pi j}{n} \sin \frac{2\pi j}{n}
\end{cases}
$$

(A.14)

Hence the nonzero eigenvalues of the linearization of $f$ at $r^0$ are $\lambda_i$ such that

$$
\lambda^2_i = -g_{ij} \sum \left( 1 + \cos \frac{2\pi k}{n} \right) \left( 1 - \cos \frac{2\pi k}{n} \right) \left( 1 - \cos \frac{2\pi j}{n} \right) \left( 1 - \cos \frac{2\pi j}{n} \right) - \sin \frac{2\pi k}{n} \sin \frac{2\pi k}{n} \sin \frac{2\pi j}{n} \sin \frac{2\pi j}{n}
$$

(7.1)
Figure 1: $\lambda_2^2$ as $\beta$ varies

The case $n = 12$ is shown, with curves for $l = 1, \ldots, 6$ appearing from top to bottom ($g = c = 1$).
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