$\mathcal{H}_2$ AND $\mathcal{H}_\infty$ DESIGNS OF MULTIRATE SAMPLED-DATA SYSTEMS

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Abstract

Treating causality constraints, this paper studies the optimal syntheses of multirate sampled-data systems with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criteria. Explicit solutions to both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems are obtained by input-output space extensions (lifting) and frequency-domain techniques.

Keywords: multirate systems, digital control, sampled-data systems, $\mathcal{H}_2$ optimization, $\mathcal{H}_\infty$ optimization, causality constraint.

1 Introduction

In industry, most control systems are implemented digitally via microprocessors. However, many digital designs are performed by rules of thumb. There are essentially two conventional methods: Design an analog controller and then implement it digitally, sampling “fast enough”; discretize the plant and then design a discrete controller, ignoring intersample behavior.

The recent trend is to perform direct digital design, i.e., design digital controllers directly using continuous-time performance measures. This should be the preferred approach because most sampled-data systems operate in real time and the input and output signals are naturally in continuous time. Many pieces of work treating design issues have been completed in this direction; they include solutions to several $\mathcal{H}_2$ sampled-data control problems [8, 24, 5], several

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solutions to the $\mathcal{H}_\infty$ sampled-data control problem [20, 36, 4, 33, 35, 32, 21], and approximate solutions to the $\mathcal{L}_1$ sampled-data control problem [12, 3]. A general mathematical tool, the lifting technique, has been developed [36, 38, 4, 6] for attacking problems in single-rate sampled-data systems.

All work mentioned above is in the single-rate setting. However, multirate sampled-data systems arise in a more natural way. Consider the system depicted in Figure 1, where $G$

![Figure 1: A sampled-data system](image)

is the generalized analog plant and $K_d$ the digital controller, interfaced via A/D and D/A conversions. One way to improve the performance of such a system is to use A/D and D/A converters at their maximum allowable speeds. Very often the maximum capacities of the two converters do not match and this therefore gives rise to a multirate sampled-data system. In multirate setting, $K_d$ is usually periodic in a certain sense; so it can often be implemented on microprocessors via periodic difference equations. Thus the thrust for our study of multirate sampled-data systems comes from the fact that they provide improved performance and reduced implementation cost but do not violate the finite memory constraint in microprocessors.

The concept of multirate sampling was pioneered by Kranc [26]. Recent interests in multirate systems are reflected in the LQG/LQR designs [7, 1, 27, 9] (but no effective way was proposed to tackle the causality constraint), the parametrization of stabilizing controllers [28, 30], and the work in [2, 19]. While the research on single-rate direct digital design has been active, little work has been done on multirate systems using the direct design approach. The main obstacle is perhaps the so-called causality constraint [28, 30], which presents a unique difficulty for synthesizing the feedthrough term in lifted controllers. In this paper we shall study how to treat the causality constraint in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design frameworks.

Causality constraints also arise in discrete-time periodic control [25], where the feedthrough terms in lifted controllers must be block lower-triangular. Explicit and interesting solutions were obtained for the one-block $\mathcal{H}_\infty$ problem [13, 18] and $\mathcal{H}_2$ problem [37]. Our $\mathcal{H}_2$ and $\mathcal{H}_\infty$ solutions in this paper are more general in that first, we treat multirate designs directly from a sampled-data point of view; and second, we tackle four-block problems directly. We remark that most multirate design problems do not fall into the one-block framework, i.e., the transfer matrices in the associated model-matching problems are in general nonsquare. Finally, we
note that a treatment of causality constraints in the $\ell_1$ framework for discrete-time periodic systems was recently reported in [11].

The organization of this paper is as follows. Section 2 presents the multirate sampled-data configuration for our subsequent study; in particular, desirable properties of multirate controllers are discussed.

Section 3 extends the lifting macros in [6] to the multirate case; these formulas are useful in converting a sampled-data problem to an associated discrete-time one.

Section 4 formulates and solves explicitly the multirate $\mathcal{H}_2$-optimal control problem using the lifting presented in Section 3; the optimal solution is expressed in terms of its optimal feedthrough term, which can be obtained via solving a matrix 2-norm optimization problem.

Section 5 is devoted to the multirate $\mathcal{H}_\infty$ control problem. We show how to reduce the multirate sampled-data problem to a discrete-time $\mathcal{H}_\infty$ problem with causality constraint. The latter problem is then studied in detail using frequency-domain methods and an explicit solution is obtained.

Finally, in Section 6 we offer some concluding remarks.

The main contributions of this paper are contained in Sections 4 and 5. To our best knowledge, this paper is the first in directly and effectively addressing the causality constraint in multirate sampled-data control syntheses.

The notation is quite standard. We use $\ell$ to denote the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. The external direct sum of $n$ copies of $\ell$ is denoted $\ell^n$. The space $\ell_2$ is a subspace of $\ell$ of square-summable sequences. Similarly for the external direct sum $\ell^n_2$. Finally, if $G$ is a linear time-invariant (LTI) system, we shall not distinguish $G$ from its transfer function.

\section{Setup}

The setup of the paper is shown in Figure 2. Here we have used continuous lines for continuous

\begin{figure}[h]
\centering
\includegraphics{system.png}
\caption{A multirate control system}
\end{figure}

signals and dotted lines for discrete signals. In Figure 2, $S_{mh}$ is an ideal sampler with period $mh$, $H_{nh}$ a zero-order hold with period $nh$, and $K_d$ a multirate digital controller which is
synchronized with $S_{mh}$ and $H_{nh}$ by a clock in the sense that $K_d$ takes in a value of the sampled measurement $\psi$ at times $t = k(mh)$, $k \geq 0$, and outputs a value of the control sequence $v$ to the hold device at $t = k(nh)$, $k \geq 0$. We shall assume throughout the paper that $m$ and $n$ are coprime integers. In general, given any sampling period $\tau_1$ and hold period $\tau_2$ with a rational ratio, there exists a unique three-tuple $(m, n, h)$ such that $\tau_1 = mh$, $\tau_2 = nh$, and $m$ and $n$ are coprime integers.

This setup is not the most general one as in [28, 30]; in fact, it has a uniform sampling rate and a uniform hold rate. But since the ratio of the two rates can be any positive rational number, this setup captures all the essential features in multirate systems while maintaining some clarity in the exposition. The extension of our results pertinent to this setup to the more general one is a conceptually simple and routine task.

We shall consider only the analog $G$ which are LTI, causal, and finite-dimensional. What are the corresponding concepts for the multirate controller $K_d$? Throughout $K_d$ is regarded as a linear map from $\ell$ to $\ell$. Since the input and output time scales are not compatible, the single-rate definitions must be modified.

The sampled-data controller $H_{nh}K_dS_{mh}$ as a continuous-time operator is in general time-varying. However, note that both $S_{mh}$ and $H_{nh}$ are periodic elements, their least common period being $T = mnh$; so, by proper choice of $K_d$ it is possible that $H_{nh}K_dS_{mh}$ is $T$-periodic in continuous time. Now let $U$ be the unit time delay on $\ell$ and $U^*$ the unit time advance. We define $K_d$ to be $(m, n)$-periodic if

$$(U^*)^m K_d U^n = K_d.$$  

Then it is not hard to see that $H_{nh}K_dS_{mh}$ is $T$-periodic iff $K_d$ is $(m, n)$-periodic.

This periodicity implies a deeper fact if we use the standard discrete-time lifting procedure, see, e.g., [25], and extend the input and output spaces of $K_d$ so as to be compatible with the period $T$. Define the discrete lifting operator $L_m : \ell \rightarrow \ell^m$ via $v = L_m v$:

$$\{v(0), v(1), \ldots\} \mapsto \left\{ \begin{bmatrix} v(0) \\ \vdots \\ v(m) \\ \vdots \\ v(m-1) \end{bmatrix}, \begin{bmatrix} v(m) \\ \vdots \\ v(2m-1) \end{bmatrix}, \ldots \right\}.$$  

Similarly for $L_n$. Now define the lifted controller

$$K_d := L_m K_d L_n^{-1}. \tag{1}$$

This is now single-rate with the underlying period being $T$. Then $K_d$ is time-invariant iff $K_d$ is $(m, n)$-periodic.

Next is causality. Again we require that $H_{nh}K_dS_{mh}$ be causal in continuous time. This condition translates to an interesting constraint on $K_d$. To see this more clearly, we look at the lifted controller $K_d$. The feedthrough term $D$ in $K_d$ is an $m \times n$ block matrix, namely,

$$D = \begin{bmatrix} D_{00} & \cdots & D_{0,n-1} \\ \vdots & \ddots & \vdots \\ D_{m-1,0} & \cdots & D_{m-1,n-1} \end{bmatrix},$$
where each $D_{ij}$ is a matrix with dimensions compatible to the dimensions of $\psi$ and $v$. Now the causality of $H_{nh}K_dS_{mh}$ translates exactly to the causality of $\overline{K_d}$ and a constraint on $D$, namely,

$$D_{ij} = 0, \quad \text{whenever } jm > in.$$  

This condition on $D$ will be called the $(m, n)$-causality constraint. For example, if $m = 3$ and $n = 2$, then this constraint means that $D$ is of the form

$$D = \begin{bmatrix} X & 0 \\ X & 0 \\ X & X \end{bmatrix},$$

where $X$ denotes any unconstrained block. For ease of reference, the set of all $D$ satisfying the $(m, n)$-causality constraint is denoted by $\Omega(m, n)$.

We say $K_d$ is $(m, n)$-causal if the single-rate $K_d$ is causal and $D$ satisfies the $(m, n)$-causality constraint. It follows then that the sampled-data controller $H_{nh}K_dS_{mh}$ is causal in continuous time iff $K_d$ is $(m, n)$-causal. More general treatment of these concepts can be found in, e.g., [28, 30].

A similar notion is that of strict causality. We say $D$ satisfies the strict $(m, n)$-causality constraint if

$$D_{ij} = 0, \quad \text{whenever } jm \geq in.$$  

The set of all such $D$ is $\Omega_s(m, n)$. It follows that $H_{nh}K_dS_{mh}$ is strictly causal in continuous time iff $\overline{K_d}$ is causal and $D \in \Omega_s(m, n)$. For the same example, $D \in \Omega_s(3, 2)$ iff $D$ is of the form

$$D = \begin{bmatrix} 0 & 0 \\ X & 0 \\ X & X \end{bmatrix}.$$

Finally, we turn to finite dimensionality of the controller $K_d$. This is again best explained in terms of $K_d$. Assume $K_d$ is $(m, n)$-periodic and $(m, n)$-causal. Then from the previous discussion $\overline{K_d}$ is LTI and causal. We furthermore assume $K_d$ is finite-dimensional. Thus $K_d$ has a state model

$$K_d = \begin{bmatrix} A & B_0 & \cdots & B_{n-1} \\ C_0 & D_{00} & \cdots & D_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m-1} & D_{m-1,0} & \cdots & D_{m-1,n-1} \end{bmatrix}.$$  

The corresponding difference equations for $K_d$ ($v = K_d\psi$) are

$$\eta(k+1) = A\eta(k) + \sum_{j=0}^{n-1} B_j\psi(nk + j), \quad (2)$$

$$v(mk + i) = C_i\eta(k) + \sum_{j=0}^{n-1} D_{ij}\psi(nk + j), \quad i = 0, 1, \cdots, m - 1. \quad (3)$$
Here \( \eta \), the state for \( K_d \), is updated every \( T = mnh \) seconds and \( \psi \) every \( nh \) seconds. Such difference equations can be implemented on microprocessors with only finite memory because the vector \( \eta \) is finite-dimensional.

In summary, in this paper we are interested in the class of multirate \( K_d \) which are \((m,n)\)-periodic, \((m,n)\)-causal, and finite-dimensional; this class is called the admissible class of \( K_d \) and can be modeled by the difference equations (2) and (3) with \( D_{ij} = 0 \) when \( jm > in \). The corresponding admissible class of \( K_d \) is characterized by LTI, causal, and finite-dimensional \( K_d \) with the same constraint on \( D \). The causality constraint on the feedthrough term of \( K_d \) is the new feature in multirate systems. Good multirate design methods should tackle this constraint effectively.

### 3 Multirate Lifting

The single-rate lifting technique \([36, 38, 4, 6]\) is very powerful in sampled-data control because it converts a periodic sampled-data system into an LTI discrete system with infinite-dimensional input and output spaces. In this section we shall extend this technique to the multirate case.

In Figure 2, partition \( G \) according to its inputs and outputs and bring in a state model:

\[
G = \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix} = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
C_2 & 0 & D_{22}
\end{bmatrix}.
\] (4)

The zero block in \( D_{21} \) guarantees the proper functioning of the sampler; it also ensures that the sampled-data system is stabilizable in some input-output sense \([8]\). The zero block in \( D_{11} \) is necessary for the finiteness of the \( \mathcal{H}_2 \) measure in the next section and is often used as a simplifying condition in the \( \mathcal{H}_\infty \) case. Now move \( S_{mh} \) and \( H_{nh} \) into the plant to get Figure 3, where

![Figure 3: An equivalent multirate system](image)

\[
G_{sd} = \begin{bmatrix}
G_{11} & G_{12}H_{nh} \\
S_{mh}G_{21} & S_{mh}G_{22}H_{nh}
\end{bmatrix}.
\]
With our assumptions on $K_d$, this multirate system is $T$-periodic in continuous time. So the idea of lifting can be used.

Following [6], let $E$ be any finite-dimensional Euclidean space, $E^n$ be the external direct sum of $n$ copies of $E$, and $K$ be $L_2[0,T)$. The sequence space $\ell_2(K)$ is defined to be

$$\ell_2(K) := \{ \psi : \psi_k \in K, \sum_{k=0}^{\infty} \| \psi_k \|^2 < \infty \}.$$ 

The norm for $\psi_k$ is the one on $K$ and the norm for $\ell_2(K)$ is given by

$$\| \psi \|_2 := \left( \sum_{k=0}^{\infty} \| \psi_k \|^2 \right)^{1/2}.$$ 

To handle unbounded signals, we bring in the two extended spaces $L_{2e}[0,\infty)$ and $\ell_{2e}(K)$ defined in the obvious way. The lifting operator $L_T$, mapping $L_{2e}[0,\infty)$ to $\ell_{2e}(K)$ is defined by

$$\psi = L_T y \iff \psi_k(t) = y(t + kT), \quad 0 \leq t < T.$$ 

As before, we denote the lifted signal $L_T y$ by $y$.

Now we lift the system in Figure 3 with respect to the period $T$. Define

$$G_{sd} := \begin{bmatrix} L_T & 0 \\ 0 & L_n \end{bmatrix} G_{sd} \begin{bmatrix} L_T^{-1} & 0 \\ 0 & L_m^{-1} \end{bmatrix} = \begin{bmatrix} L_T G_{11} L_T^{-1} & L_T G_{12} H_{nh} L_m^{-1} \\ L_n S_{nh} G_{21} L_T^{-1} & L_n S_{nh} G_{22} H_{nh} L_m^{-1} \end{bmatrix}. \quad (5)$$

and $K_d$ again as in (1) to get the lifted system configuration in Figure 4. Here the signals

\begin{center}
\begin{tikzpicture}
  \node [draw, inner sep=0] (G) at (0,0) {$G_{sd}$};
  \node [draw, inner sep=0] (K) at (0,-1) {$K_d$};
  \draw [->, dashed] (G) -- ++(0,1) node [above] {$z$};
  \draw [->, dashed] (G) -- ++(0,-1) node [below] {$w$};
  \draw [->] (K) -- ++(0,1) node [above] {$\psi$};
  \draw [->] (K) -- ++(0,-1) node [below] {$\psi$};
\end{tikzpicture}
\end{center}

Figure 4: The lifted system

are all lifted; e.g., $w = L_T w$ and $\psi = L_n \psi$. We saw in Section 2 that $K_d$ is LTI; it is not hard to see that $G_{sd}$ too is LTI. So Figure 4 represents a discrete LTI system. Let $T_{zw}$ be the closed-loop map $w \mapsto z$ in Figure 2. Then the closed-loop map $T_{zw} : w \mapsto z$ in Figure 4 is the lifted $T_{zw}$, namely, $L_T T_{zw} L_T^{-1}$. The usefulness of this relationship is due to the fact that the operators $L_T$ and $L_T^{-1}$ preserve norms.
Now with a state model for $G$ in (4), we can derive a state-space representation for $G_{sd}$. But first, let us find state models for the four blocks of $G_{sd}$ in (5) as they may be of independent interest.

**Lifting $G_{11}$**

The lifted $G_{11}$, namely, $G_{11} := L_T G_{11} L_T^{-1}$, maps $\ell_{2e}(\mathcal{K})$ to $\ell_{2e}(\mathcal{K})$ and can be represented by a state model with finite-dimensional state space [6]:

$$G_{11} = \begin{bmatrix} A_d & B_1 \\ C_1 & D_{11} \end{bmatrix},$$

where

$$A_d : \mathcal{E} \to \mathcal{E}, \quad A_d = e^{T A},$$

$$B_1 : \mathcal{K} \to \mathcal{E}, \quad B_1 w = \int_0^T e^{(T - \tau)A} B_1 w(\tau) \, d\tau,$$

$$C_1 : \mathcal{E} \to \mathcal{K}, \quad (C_1 \xi)(t) = C_1 e^{tA} \xi,$$

$$D_{11} : \mathcal{K} \to \mathcal{K}, \quad (D_{11} w)(t) = \int_0^t C_1 e^{(t - \tau)A} B_1 w(\tau) \, d\tau.$$  

**Lifting $S_{mh} G_{21}$**

Now we derive a state model for $S_{mh} G_{21}$, namely, $L_n S_{mh} G_{21} L_T^{-1}$, which maps $\ell_{2e}(\mathcal{K})$ to $\ell^n$.

Write $\psi = S_{mh} G_{21} w$ and let the state for the realization in (4) be $x$. Then the state equations for $G_{21}$ ($y = G_{21} w$) are

$$\begin{align*}
\dot{x}(t) &= A x(t) + B_1 w(t), \\
y(t) &= C_2 x(t).
\end{align*}$$

Integrate (6) from $kT$ to $(k + 1)T$ and define the sequence $\xi$ by $\xi(k) = x(kT)$ to get

$$\xi(k + 1) = A_d \xi(k) + B_1 w_k.$$  

By the definition of $L_n$,

$$\psi(k) = \begin{bmatrix} \psi(kn) \\ \vdots \\ \psi(kn + n - 1) \end{bmatrix},$$

where $\psi(k) = y(kmh) = C_2 x(kmh)$, $k \geq 0$. For $j = 0, 1, \ldots, n - 1$,

$$\begin{align*}
\psi(kn + j) &= C_2 x(kT + jmh) \\
&= C_2 e^{jmh} \xi(k) + \int_{kT}^{kT+jmh} C_2 e^{(kT+jmh-\tau)A} B_1 w(\tau) \, d\tau \\
&= C_2 e^{jmh} \xi(k) + \int_0^{jmh} C_2 e^{(jmh-\tau)A} B_1 w_k(\tau) \, d\tau.
\end{align*}$$
For notational convenience, define
\[ \Psi_j(\tau) = C_2 e^{(j m h - \tau) A} B_1 \chi_{(0, j m h)}(\tau) \] 
for \( j = 0, 1, \ldots, n - 1 \), where \( \chi_{[a,b]} \) is the characteristic function on the interval \([a, b]\). Then
\[ \psi(k n + j) = C_2 e^{j m h A} \xi(k) + \int_0^T \Psi_j(\tau) w_k(\tau) \, d\tau. \]

Putting things together, we get the state model
\[ \begin{aligned}
\xi(k + 1) &= A_d \xi(k) + B_1 w_k, \\
\psi(k) &= C_{2d} \xi(k) + D_{21} w_k,
\end{aligned} \]
or equivalently
\[ S_{m h} G_{21} = \begin{bmatrix} A_d & B_1 \\ C_{2d} & D_{21} \end{bmatrix}, \]
where \( A_d \) and \( B_1 \) were given before and
\[ C_{2d} : \mathcal{E} \to \mathcal{E}^n, \quad C_{2d} = \begin{bmatrix} C_2 \\ C_2 e^{m h A} \\ \vdots \\ C_2 e^{(n-1) m h A} \end{bmatrix}, \]
\[ D_{21} : \mathcal{K} \to \mathcal{E}^n, \quad D_{21} w = \int_0^T \begin{bmatrix} \Psi_0(\tau) \\ \vdots \\ \Psi_{n-1}(\tau) \end{bmatrix} w(\tau) \, d\tau. \]

The lifting formulas for the other two blocks in \( G_{sd} \) can be derived similarly; they are summarized below.

**Lifting** \( G_{12} H_{nh} \)

The lifted operator \( G_{12} H_{nh} \), namely, \( L_T G_{12} H_{nh} L_m^{-1} : \mathcal{E}^m \to \ell_2(\mathcal{K}) \), also has a state model. For \( i = 0, 1, \ldots, (m - 1) \), define
\[ \Phi_i(t) = D_{12} \chi_{(inh, (i+1)nh)}(t) + \int_0^t C_1 e^{(t-\tau) A} B_2 \chi_{(inh, (i+1)nh)}(\tau) \, d\tau. \] 
(8)

Next define
\[ B_{2d} : \mathcal{E}^m \to \mathcal{E}, \quad B_{2d} = \int_0^T e^{(T-\tau) A} B_2 \begin{bmatrix} \chi_{[0, nh]}(\tau) & \cdots & \chi_{[(m-1) nh, T]}(\tau) \end{bmatrix} \, d\tau, \]
\[ D_{12} : \mathcal{E}^m \to \mathcal{K}, \quad (D_{12} v)(t) = \begin{bmatrix} \Phi_0(t) & \cdots & \Phi_{m-1}(t) \end{bmatrix} v. \]

Then
\[ G_{12} H_{nh} = \begin{bmatrix} A_d & B_{2d} \\ C_1 & D_{12} \end{bmatrix}. \]

Here \( A_d \) and \( C_1 \) were given before.
Lifting $S_{mh}G_{22}H_{nh}$

The lifted operator $S_{mh}G_{22}H_{nh} := L_m S_{mh} G_{22} H_{nh} L_m^{-1}$ maps $\ell^m$ to $\ell^n$; it is a standard discrete-time system. A state model is

$$S_{mh}G_{22}H_{nh} = \begin{bmatrix} A_d & B_{2d} \\ C_{2d} & D_{22d} \end{bmatrix},$$

where $A_d, B_{2d}, C_{2d}$ were already given and

$$D_{22d} : \mathcal{E}^m \to \mathcal{E}^n, \quad [D_{22d}]_{ij} = D_{22x[ih,(i+1)nh]}(j mh) + \int_{ih}^{(i+1)nh} C_{2e(jmh-\tau)}A B_{2x[0,jmh]}d\tau.$$

It can be verified that $D_{22d}$ satisfies the $(n,m)$-causality constraint. Furthermore, $D_{22d}$ satisfies the strict $(n,m)$-causality constraint if $G_{22}$ is strictly causal ($D_{22} = 0$).

Lifting $G_{sd}$

We remark that all the four lifted blocks in $G_{sd}$ share the same state vector $\xi(k) = \pi(kT)$. Moreover, their state models fit nicely together to form a state model for $G_{sd}$ which maps $\ell_{2e}(\mathcal{K}) \oplus \ell^m$ to $\ell_{2e}(\mathcal{K}) \oplus \ell^n$:

$$G_{sd} = \begin{bmatrix} A_d & B_1 & B_{2d} \\ C_1 & D_{11} & D_{12} \\ C_{2d} & D_{21} & D_{22d} \end{bmatrix}.$$ 

This lifted model will be exploited in the design problem of the next section.

4 $\mathcal{H}_2$-Optimal Control

This section treats the first synthesis problem: Design an admissible $K_d$ to achieve internal stability and minimize some generalized $\mathcal{H}_2$ performance measure.

First of all, let us look at the performance measure. Recall that for an admissible $K_d$, the closed-loop system $T_{2w}$ in Figure 2 is $T$-periodic. Thus we adopt the generalized $\mathcal{H}_2$ measure proposed for periodic systems in [24, 5].

Let $F$ be a continuous-time, $T$-periodic, causal system described by the following integral operator

$$(Fu)(t) = \int_0^t f(t, \tau) u(\tau) d\tau.$$ 

We assume that $f$, the matrix-valued impulse response of $F$, is locally square-integrable, i.e., every element is square-integrable on any compact subset of $\mathcal{R}^2$. The periodicity of $F$ implies $f(t + T, \tau + T) = f(t, \tau)$, and the causality implies that $f(t, \tau) = 0$ if $\tau > t$. If $f$ is square-integrable on $[0, \infty) \times [0, T)$, we can define a norm for $F$ as follows [24, 5]:

$$\|F\|_{\text{per}} = \left\{ \frac{1}{T} \int_0^T \int_0^\infty \text{trace} \left[ f'(t, \tau) f(t, \tau) \right] dt \, d\tau \right\}^{1/2}.$$
This is a generalization of the $\mathcal{H}_2$ norm for transfer functions: If $F$ is furthermore time-invariant, the right-hand side reduces to the $\mathcal{H}_2$ norm of the transfer function $F(s)$. Moreover, it admits a sensible stochastic interpretation [5].

Now we lift $F$ to get $F := L_T F L_T^{-1}$. The lifted system $F : \ell_2(\mathcal{K}) \mapsto \ell_2(\mathcal{K})$ can be described by $(y = F u)$

$$y_k = \sum_{j=0}^{k} f_{k-j} y_j, \quad k \geq 0,$$

where $f_k, \ i \geq 0$, map $\mathcal{K}$ to $\mathcal{K}$ via

$$(f_i u)(t) = \int_0^T f(t + iT, \tau) u(\tau) \, d\tau, \quad 0 \leq t < T.$$

$F$ is LTI in discrete time; its transfer function is defined as

$$F(\lambda) = \sum_{i=0}^{\infty} f_i \lambda^i.$$

The local square-integrability of $f(t, \tau)$ implies that $f_i, \ i \geq 0$, are Hilbert-Schmidt operators [39]. Moreover, the set of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, $\| \cdot \|_{HS}$, is a Hilbert space [18]. Thus the transfer function $F$ is a Hilbert-space vector-valued function on some subset of $\mathcal{C}$. We say the function $F$ belongs to $\mathcal{H}_2$ if

$$\left( \sum_{i=0}^{\infty} \|f_i\|_{HS}^2 \right)^{1/2} < \infty,$$

and the left-hand side is defined to be its $\mathcal{H}_2$ norm, denoted $\|F\|_2$ [34]. A result connecting the time- and frequency-domain definitions is given below.

**Lemma 1** [5] $F$ is in $\mathcal{H}_2$ iff every element of $f$ is square-integrable on $[0, \infty) \times [0, T)$; in this case, $\frac{1}{\sqrt{T}} \|F\|_2 = \|F\|_{per}$.

Now we turn to internal stability of Figure 2. There are many ways to define closed-loop stability. Here we choose the way in terms of the plant state $x$ and the controller state $\eta$ ($K_d$ is admissible). Define the continuous-time vector

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \eta(k) \end{bmatrix}, \quad kT \leq t < (k + 1)T.$$

The (autonomous) multirate sampled-data system is internally stable, or $K_d$ internally stabilizes $G$, if for any initial value $x_{sd}(t_0), \ 0 \leq t_0 < T$, $x_{sd} \rightarrow 0$ as $t \rightarrow \infty$. Note that by finite dimensionality, internal stability implies that $x_{sd} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

We need a few standing assumptions in this section about the plant $G$ in (4):

1. $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable;
2. the period $T$ is non-pathological with respect to $G$ [23, 8];

3. $D_{22} = 0$.

Assumptions 1 and 2 are mild and standard. Assumption 3 is for the well-posedness of the closed-loop system. The system in Figure 2 is well-posed iff the lifted system in Figure 4 is well-posed. The latter property holds if the matrix $I - D_{22}D$ is nonsingular, where $D_{22}$ is in $\Omega_s(n, m)$ as we commented before and $D$, the feedthrough term in $K_d$, is in $\Omega(m, n)$. Invoking the fact that $M_1, M_2 \in \Omega_s(p, q)$ if $M_1 \in \Omega_s(p, l)$ and $M_2 \in \Omega(l, q)$ [28, 30], we get that $D_{22}D \in \Omega_s(n, n)$, or equivalently, $D_{22}D$ is strictly (block) lower-triangular. Hence the matrix $I - D_{22}D$ is nonsingular.

Let us write

$$G_{sd} = \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix} := \begin{bmatrix}
A_d & B_1 & B_{2d} \\
C_1 & D_{11} & D_{12} \\
C_{2d} & D_{21} & D_{22d}
\end{bmatrix}$$

and note that $G_{sd}$ is a standard discrete-time system. Now we relate internal stability of the multirate system to that of the LTI discrete-time system shown in Figure 5.

![Diagram](image)

Figure 5: A discrete-time system for stability

**Lemma 2** Under Assumptions 1-3, $K_d$ internally stabilizes $G$ iff $K_d$ internally stabilizes $G_{22}$ in discrete time.

**Proof** Under Assumptions 1 and 2, we have that $(A_d, B_T)$ is stabilizable, where

$$B_T := \int_0^T e^{(T-\tau)A}B_d \, d\tau,$$

and $(C_2, A_d)$ is detectable [23]. It follows easily that $(C_{2d}, A_d)$ is detectable. The stabilizability of $(A_d, B_{2d})$ can be obtained from the fact that

$$B_T = B_{2d} \begin{bmatrix}
I \\
\vdots \\
I
\end{bmatrix}.$$
Finally, using a similar argument as in [15], we can conclude that \( x_{sd} \to 0 \) as \( t \to \infty \) iff
\[
\begin{bmatrix}
\xi(k) \\
\eta(k)
\end{bmatrix} \to 0, \quad k \to \infty,
\]
where \( \xi \) is the state for \( G_{sd} \) or \( G_{22} \).
\[ \text{QED} \]

We can now state the \( \mathcal{H}_2 \)-optimal control problem precisely: Given \( G, m, n, \) and \( h \), design an admissible \( K_d \) to provide internal stability and minimize \( \| T_{zw} \|_{\text{per}} \) in Figure 2. By Lemmas 1 and 2, we can recast the problem exactly in the lifted spaces: Design an admissible \( K_d \) to internally stabilize \( G_{22} \) and minimize the \( \mathcal{H}_2 \) norm of \( T_{zw} \) in Figure 4.

In what follows we shall solve explicitly this \( \mathcal{H}_2 \) problem using a frequency-domain approach. The problem is much harder than the single-rate one [24, 5] due to the facts that \( D_{21} \) is nonzero and that \( K_d \) must satisfy the causality constraint.

In (9), \( A_d, B_{2d}, C_{2d}, D_{22d} \) are matrices and \( B_1, D_{11}, D_{12}, D_{21} \) are operators. However, all the operators but \( D_{11} \) are of finite rank. This fact can be exploited: Define the real-rational matrices
\[
\bar{G}_{11} = \begin{bmatrix} A_d & I \\ I & 0 \end{bmatrix}, \quad \bar{G}_{12} = \begin{bmatrix} A_d & B_{2d} \\ I & 0 \\ 0 & I \end{bmatrix}, \quad \bar{G}_{21} = \begin{bmatrix} A_d & I \\ C_{2d} & 0 \end{bmatrix}
\]
to get
\[
\begin{align*}
G_{11} &= D_{11} + C_1 \bar{G}_{11} B_1 \\
G_{12} &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \bar{G}_{12} \\
G_{21} &= \begin{bmatrix} B_1 & D_{21} \end{bmatrix} \bar{G}_{21}.
\end{align*}
\]

Now bring in a special doubly-coprime factorization for the real rational transfer matrix \( \bar{G}_{22} \):
\[
\bar{G}_{22} = NM^{-1} = \bar{M}^{-1} \bar{N},
\]
\[
\begin{bmatrix}
\bar{X} & -\bar{Y} \\
-\bar{N} & \bar{M}
\end{bmatrix}
\begin{bmatrix}
M & Y \\
N & X
\end{bmatrix} = I,
\]
with \( M(0) = I \) and \( \bar{M}(0) = I \). The latter conditions on \( M(0) \) and \( \bar{M}(0) \) yield
\[
\begin{align*}
N(0) &= \bar{N}(0) = D_{22d}, \\
X &= I, \quad \bar{X} = I, \\
Y(0) &= \bar{Y}(0) = 0.
\end{align*}
\]
(The standard procedure in [14] generates such a factorization.) Then by Youla’s parametrization, every real-rational, proper stabilizing controller \( K_d \) for \( \bar{G}_{22} \) has the form
\[
K_d = (Y - MQ)(X - NQ)^{-1}
\]  \( \text{(10)} \)
for some \( Q \in \mathcal{RH}_\infty \) with \( X - NQ \) invertible. Now we consider the causality constraint on \( K_d \), namely, the condition that

\[
K_d(0) = Q(0)[I - D_{22d}Q(0)]^{-1}
\]

must lie in \( \Omega(m, n) \). By [28, 30], \( K_d(0) \in \Omega(m, n) \) iff \( Q(0) \in \Omega(m, n) \). Moreover, the same argument used in the well-posedness discussion yields that \( X - NQ \) is invertible if \( Q(0) \in \Omega(m, n) \).

On summarizing, the set of admissible \( K_d \) which internally stabilize \( G \) is parametrized by

\[
K_d = (Y - MQ)(X - NQ)^{-1}, \quad Q \in \mathcal{RH}_\infty, \quad Q(0) \in \Omega(m, n).
\]

With this controller applied, the closed-loop map in Figure 4 is

\[
T_{\text{scw}} = T_1 - T_2QT_3,
\]

where \( T_1, T_2, T_3 \) are given by

\[
T_1 = D_{11} + [C_1 \ D_{12} \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \tilde{G}_{12}MY_2G_{21}] \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix},
\]

\[
T_2 = \begin{bmatrix} C_1 \\ D_{12} \end{bmatrix} \tilde{G}_{12}M,
\]

\[
T_3 = \{MG_{21}\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}.
\]

Therefore, the multirate \( \mathcal{H}_2 \) problem is equivalent to the following constrained \( \mathcal{H}_2 \) model-matching problem

\[
\inf_{Q \in \mathcal{RH}_\infty, Q(0) \in \Omega} \|T_1 - T_2QT_3\|_2.
\]

Here we used \( \Omega \) for \( \Omega(m, n) \) to simplify notation. Note that \( T_1, T_2, T_3 \) are all operator-valued. For an operator-valued transfer function \( T(\lambda) \), denote the transfer function of the adjoint system by \( T^\sim(\lambda) := T^*(1/\lambda) \). To proceed further, we need one additional assumption:

4. For every \( \lambda \) on the unit circle, \( T_2(\lambda) \) and \( T_3^\sim(\lambda) \) are both injective.

Since for a fixed \( \lambda \), the domains of \( T_2(\lambda) \) and \( T_3^\sim(\lambda) \) are finite-dimensional but their codomains are infinite-dimensional, Assumption 4 will be generically satisfied. It is not hard to show that Assumption 4 holds if for every \( \lambda \) on the unit circle, the two operator matrices

\[
\begin{bmatrix} A_d - \lambda I & B_{2d} \\ C_1 & D_{12} \end{bmatrix}, \quad \begin{bmatrix} A_d - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix}
\]

are injective and surjective respectively.
Note that $T_2\hspace{-0.5em}^\sim T_2$ and $T_3 T_3\hspace{-0.5em}^\sim$ are both matrix-valued. Bring in constant matrices $E_{12}$ and $E_{21}$ satisfying
\[
E'_{12} E_{12} = \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} \begin{bmatrix} C_1 & D_{12} \end{bmatrix},
E_{21} E'_{21} = \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1^* & D_{21}^* \end{bmatrix},
\]
(computational issues will be addressed at the end of the section) to get
\[
T_2\hspace{-0.5em}^\sim T_2 = (E_{12} \hat{G}_{12} M)^\sim (E_{12} \hat{G}_{12} M),
T_3 T_3\hspace{-0.5em}^\sim = (\hat{M} \hat{G}_{21} E_{21})(\hat{M} \hat{G}_{21} E_{21})^\sim.
\]

It follows that $T_2\hspace{-0.5em}^\sim T_2$ and $T_3 T_3\hspace{-0.5em}^\sim$ are both para-symmetric real-rational matrices and have full ranks on the unit circle (Assumption 4). So we can perform spectral factorizations $T_2\hspace{-0.5em}^\sim T_2 = T_2\hspace{-0.5em}^\sim T_2$ and $T_3 T_3\hspace{-0.5em}^\sim = T_{3\hspace{-0.5em}co} T_{3\hspace{-0.5em}co}$ with $T_{2\hspace{-0.5em}co}, T_{2\hspace{-0.5em}co}^{-1}, T_{3\hspace{-0.5em}co}, T_{3\hspace{-0.5em}co}^{-1} \in \mathcal{RH}_\infty$. An inner-outer factorization $T_2 = T_{2i} T_{2o}$ and a co-inner-outer factorization $T_3 = T_{3\hspace{-0.5em}co} T_{3ci}$ can be obtained by defining
\[
T_{2i} = T_{2o} T_{2o}^{-1} = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \hat{G}_{12} M T_{2o}^{-1},
T_{3ci} = T_{3\hspace{-0.5em}co}^{-1} T_3 = T_{3\hspace{-0.5em}co}^{-1} \hat{M} \hat{G}_{21} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}.
\]

Note that the operator-valued inner factor $T_{2i}$ satisfies $T_{2i} T_{2i} = I$ and co-inner factor $T_{3ci}$ satisfies $T_{3ci} T_{3ci}^\sim = I$.

Define the constant matrix
\[
E_{11} := \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} D_{11} \begin{bmatrix} B_1^* & D_{21}^* \end{bmatrix}
\]
and the real-rational matrix in $\mathcal{L}_2$
\[
R_{11} = (\hat{G}_{12} M T_{2o}^{-1})^\sim \left[ E_{11} + E'_{12} E_{12} \left[ \begin{bmatrix} \hat{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \hat{G}_{12} \hat{M} \hat{G}_{21} \right] E_{21} E'_{21} \right] (T_{3\hspace{-0.5em}co}^{-1} \hat{M} \hat{G}_{21})^\sim.
\]
Denote the constant term of $R_{11}$ by $R_{110}$. (Since $R_{11}$ is in general noncausal, it follows that in general $R_{110} \neq R_{11}(0)$). Let $\Pi_{\mathcal{H}_2} : \mathcal{L}_2 \rightarrow \mathcal{H}_2$ and $\Pi_{\mathcal{H}_2^\perp} : \mathcal{L}_2 \rightarrow \mathcal{H}_2^\perp$ be the orthogonal projections. We are now set up to state the main result of this section.

**Theorem 1** The optimal $Q$ in (12) is given by
\[
Q_{opt} = Q_0 + \lambda T_{2o}^{-1} \left[ \Pi_{\mathcal{H}_2} \left[ \lambda^{-1} (R_{11} - T_{2o} Q_{0} T_{3\hspace{-0.5em}co}) \right] \right] T_{3\hspace{-0.5em}co}^{-1},
\]
where the constant matrix $Q_0$ is the optimal $Q(0)$ solving
\[
\min_{Q(0) \in \mathcal{U}} \| R_{110} - T_{2o}(0) Q(0) T_{3\hspace{-0.5em}co}(0) \|_2.
\]
Proof Apply unitary transformations to $T_1 - T_2 Q T_3$ and define

$$
\begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix} = \begin{bmatrix}
T_{2i}^{-1} \\
I - T_{2i} T_{2i}^{-1}
\end{bmatrix} T_1 \begin{bmatrix}
T_{3i}^{-1} \\
I - T_{3i} T_{3i}^{-1}
\end{bmatrix}
$$

(simple calculation shows that this $R_{11}$ is exactly the one given in (13)) to get

$$
\|T_1 - T_2 Q T_3\|_2^2 = \| \begin{bmatrix}
T_{2i}^{-1} \\
I - T_{2i} T_{2i}^{-1}
\end{bmatrix} (T_1 - T_2 Q T_3) \begin{bmatrix}
T_{3i}^{-1} \\
I - T_{3i} T_{3i}^{-1}
\end{bmatrix} \|_2^2
$$

$$
\| \begin{bmatrix}
R_{11} - T_{2o} Q T_{3o} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix} \|_2^2
$$

$$
= \|R_{11} - T_{2o} Q T_{3o}\|_2^2 + \|R_{12}\|_2^2 + \|R_{21}\|_2^2 + \|R_{22}\|_2^2.
$$

The last three terms are independent of $Q$; so the problem in (12) reduces to minimizing the first term. On writing

$$
Q = Q(0) + \lambda Q_1, \quad Q_1 \in \mathcal{R} \mathcal{H}_\infty,
$$

this is

$$
\inf_{Q(0) \in \Omega} \inf_{Q_1 \in \mathcal{R} \mathcal{H}_\infty} \|R_{11} - T_{2o} Q T_{3o}\|_2^2
$$

$$
\inf_{Q(0) \in \Omega} \inf_{Q_1 \in \mathcal{R} \mathcal{H}_\infty} \|R_{11} - T_{2o} Q(0) T_{3o} - \lambda T_{2o} Q_1 T_{3o}\|_2^2
$$

$$
\inf_{Q(0) \in \Omega} \inf_{Q_1 \in \mathcal{R} \mathcal{H}_\infty} \|\lambda^{-1}[R_{11} - T_{2o} Q(0) T_{3o}] - T_{2o} Q_1 T_{3o}\|_2^2
$$

$$
\geq \inf_{Q(0) \in \Omega} \|\Pi_{\mathcal{H}_2} \left\{ \lambda^{-1}[R_{11} - T_{2o} Q(0) T_{3o}] \right\} \|_2^2
$$

$$
\|\Pi_{\mathcal{H}_2} R_{11}\|_2^2 + \inf_{Q(0) \in \Omega} \|R_{110} - T_{2o} (0) Q(0) T_{3o}(0)\|_2^2.
$$

The equality is achieved by setting

$$
Q_1 = T_{2o}^{-1} \Pi_{\mathcal{H}_2} \left[ \lambda^{-1}(R_{11} - T_{2o} Q(0) T_{3o}) \right] T_{3o}^{-1}.
$$

Thus the optimal $Q$ can be obtained in two steps: Solve the matrix 2-norm optimization in (15) to get $Q_0$ (this will be discussed in Lemma 3); and then substitute $Q_0$ into (14) to get $Q_{opt}$. QED

Now let us look at how to compute the optimal cost, $\inf_Q \|T_{2o} w\|_2^2$. From the above proof we see that the optimal cost is

$$
\|R_{12}\|_2^2 + \|R_{21}\|_2^2 + \|R_{22}\|_2^2 + \|\Pi_{\mathcal{H}_2} R_{11}\|_2^2 + \|R_{110} - T_{2o} (0) Q_0 T_{3o}(0)\|_2^2.
$$

(16)

The computation of $Q_0$ is our next topic; now we compute the first several terms in (16). Instead of a direct effort, we note for a short cut that the sum of the first four terms equals

$$
\| \begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix} \|_2^2 - \|\Pi_{\mathcal{H}_2} R_{11}\|_2^2 = \|T_1\|_2^2 - \|\Pi_{\mathcal{H}_2} R_{11}\|_2^2.
$$
and the norm of $T_1$ can be computed with relative ease:

$$
\|T_1\|_2^2 = \|D_{11}\|_{HS}^2 + \|[C_1 \ D_{12}] \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \bar{G}_{12}M\bar{Y}\bar{G}_{21} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \|_2^2 
= \|D_{11}\|_{HS}^2 + \|E_{12} \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \bar{G}_{12}M\bar{Y}\bar{G}_{21} \|_{E_{21}}^2. \tag{17}
$$

Here we used the fact that $\tilde{G}_{11}$ and $\bar{Y}$ are strictly proper. The Hilbert-Schmidt norm of $D_{11}$ follows from that of a general integral operator [18]:

$$
\|D_{11}\|_{HS}^2 = \text{trace} \int_a^T \int_0^t B_1^t e^{(T-t)A'}C_1'C_1e^{(T-t)A}B_1 \, dt. dt.
$$

In summary, the optimal cost can be found by

$$
\inf_Q \|T_{\Sigma^q}\|_2^2 = \|T_1\|_2^2 - \|\Pi_{\Omega_2} R_{11}\|_2^2 + \|R_{110} - T_{2o}(0)Q_0 T_{3c0}(0)\|_2^2,
$$

where $\|T_1\|_2$ is given in (17).

Next we look at how to find $Q_0$ solving (15). This is a least-square minimization problem and can be solved using matrix factorization theory.

For square and nonsingular matrices $T_{2o}(0)$ and $T_{3c0}(0)$, bring in factorizations

$$
T_{2o}(0) = U_2 R_2, \quad T_{3c0}(0) = R_3 U_3,
$$

where $U_2, R_2, U_3, R_3$ are all square, $U_2, U_3$ are orthogonal ($U_2'U_2 = I, U_3'U_3 = I$), and $R_2, R_3$ are lower-triangular. The existence and computation of such factorizations follow analogously from those of the well-known QR factorization. Recall that the 2-norm for matrices is induced by the inner product:

$$
(A, B) := \text{trace} (A'B).
$$

Thus the subspace $\Omega$ has its orthogonal complement $\Omega^\perp$ in the space of matrices of appropriate dimensions. Let $\Pi_\Omega$ and $\Pi_{\Omega_2}$ be the orthogonal projections to $\Omega$ and $\Omega^\perp$ respectively. It follows then that $\Pi_\Omega$ amounts to simply retaining the blocks corresponding to the unconstrained blocks in $\Omega$ and zeroing the other blocks.

**Lemma 3** *The optimal $Q(0)$ solving (15) is*

$$
Q_0 = R_2^{-1} \Pi_\Omega [U_2'R_{110}U_3'] R_3^{-1}. \tag{18}
$$

Note that the matrix $Q_0$ in (18) is indeed in $\Omega = \Omega(m, n)$: The matrices $R_2^{-1}$ and $R_3^{-1}$ are lower-triangular and so they belong to $\Omega(m, m)$ and $\Omega(n, n)$ respectively.
\textbf{Proof} Substituting in the factorizations for $T_{2o}(0)$ and $T_{3co}(0)$, we get that (15) is
\[
\min_{Q(0)\in\Omega} \|R_{110} - U_2 R_2 Q(0) R_3 U_3^*\|_2 \\
= \min_{Q(0)\in\Omega} \|U_2' R_{110} U_3' - R_2 Q(0) R_3^*\|_2 \\
\geq \|\Pi_{\Omega^+}[U_2' R_{110} U_3']\|_2
\]
The inequality follows from the fact that $R_2 Q(0) R_3 \in \Omega$ (since $R_2$ and $R_3$ are lower-triangular) and becomes equality if we select $Q(0)$ as in (18). \hfill \text{QED}

With Assumptions 1-4, let us recap and summarize briefly the steps in design:

\textbf{Step 1} Compute the constant matrices $E_{11}$, $E_{12}$, and $E_{21}$.

\textbf{Step 2} Compute a coprime factorization of $G_{22}$ and spectral and co-spectral factors $T_{2o}$ and $T_{3co}$.

\textbf{Step 3} Compute the (noncausal) transfer function $R_{11}$.

\textbf{Step 4} Solve the matrix optimization of (15) for $Q_0$ by Lemma 3.

\textbf{Step 5} The optimal $Q$ is given by (14) and the optimal $K_d$ by (10).

Finally, we conclude this section by showing how to do Step 1, namely, presenting the explicit formulas for $E_{11}$, $E_{12}$, and $E_{21}$. From their definitions, we have
\[
E_{11} = \begin{bmatrix} C_1^* D_{11} B_1^* & C_1^* D_{11} D_{21}^* \\ D_{12}^* D_{11} B_1^* & D_{12}^* D_{11} D_{21}^* \end{bmatrix}
\]
\[
E_{12}' E_{12} = \begin{bmatrix} C_1^* C_1 & C_1^* D_{12} \\ (C_1^* D_{12})' & D_{12}^* D_{12} \end{bmatrix}
\]
\[
E_{21} E_{21}' = \begin{bmatrix} B_1 B_1^* & B_1 D_{21}^* \\ (B_1 D_{21}^*)' & D_{21}^* D_{21} \end{bmatrix}
\]

With the functions $\Psi_j$ and $\Phi_i$ defined in (7) and (8), the individual blocks can be found to be
\[
C_1^* D_{11} B_1^* = \int_0^T \int_0^t e^{tA'} C_1^* C_1 e^{(t-\tau)A} B_1 B_1' e^{(T-\tau)A'} d\tau dt,
\]
\[
C_1^* D_{11} D_{21}^* = \int_0^T \int_0^t e^{tA'} C_1^* C_1 e^{(t-\tau)A} B_1 \begin{bmatrix} \Psi_0'(t) & \cdots & \Psi_{n-1}'(t) \end{bmatrix} d\tau dt,
\]
\[
D_{12}^* D_{11} B_1^* = \int_0^T \int_0^t \begin{bmatrix} \Phi_0'(t) \\ \vdots \\ \Phi_{m-1}'(t) \end{bmatrix} C_1 e^{(t-\tau)A} B_1 B_1' e^{(T-\tau)A'} d\tau dt,
\]

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\begin{align*}
D_{12}^* D_{11} D_{21}^* &= \int_0^T \int_0^t \left[ \begin{array}{c} \Phi_0(t) \\
\vdots \\
\Phi_{m-1}(t) \end{array} \right] C_1 e^{(t-\tau)A} B_1 \left[ \begin{array}{c} \Psi_0(t) \\
\cdots \\
\Psi_{n-1}(t) \end{array} \right] d\tau \, dt, \\
C_1^* C_1 &= \int_0^T e^{tA'} C_1' C_1 e^{tA} \, dt, \\
C_1^* D_{12} &= \int_0^T e^{tA'} C_1' \left[ \begin{array}{c} \Phi_0(t) \\
\cdots \\
\Phi_{m-1}(t) \end{array} \right] dt, \\
D_{12}^* D_{12} &= \int_0^T \left[ \begin{array}{c} \Phi_0(t) \\
\vdots \\
\Phi_{m-1}(t) \end{array} \right] \left[ \begin{array}{c} \Phi_0(t) \\
\cdots \\
\Phi_{m-1}(t) \end{array} \right] dt, \\
B_1 B_1^* &= \int_0^T e^{(T-\tau)A} B_1 B_1' e^{(T-\tau)A'} \, d\tau, \\
B_1 D_{21}^* &= \int_0^T e^{(T-\tau)A} B_1 \left[ \begin{array}{c} \Psi_0(\tau) \\
\cdots \\
\Psi_{n-1}(\tau) \end{array} \right] d\tau, \\
D_{21} D_{21}^* &= \int_0^T \left[ \begin{array}{c} \Psi_0(\tau) \\
\vdots \\
\Psi_{n-1}(\tau) \end{array} \right] \left[ \begin{array}{c} \Psi_0(\tau) \\
\cdots \\
\Psi_{n-1}(\tau) \end{array} \right] d\tau.
\end{align*}

With the two symmetric matrices $E_{12}' E_{12}$ and $E_{21}' E_{21}$ computed, there are many choices for $E_{12}$ and $E_{21}$; for example, we can take them as the square roots or Cholesky factors of the two symmetric matrices respectively.

5 \textbf{ $\mathcal{H}_\infty$-Optimal Control}

In this section we shall study the multirate $\mathcal{H}_\infty$ control problem: Design an admissible $K_d$ to provide internal stability and achieve a pre-specified level of $\mathcal{H}_\infty$ performance, i.e., the $L_2$-induced norm of $T_{zw}$, denoted $\|T_{zw}\|$, is less than $\gamma$, where $\gamma$ is positive. By proper scaling, we can always take $\gamma = 1$.

In principle, the multirate lifting procedure in Section 3 could be employed to reduce the problem to a discrete-time $\mathcal{H}_\infty$ problem with causality constraint. However, in this section we shall present a simpler reduction process which is based on recent single-rate results [4, 21] and the discrete lifting. Then the constrained discrete $\mathcal{H}_\infty$ problem is solved explicitly.

With the state model of $G$ in (4), Assumptions 1-3 made in Section 4 are in force in this section. Let $D_{11h} : L_2[0, h) \to L_2[0, h)$ be defined by

\[(D_{11h} w)(t) = C_1 \int_0^t e^{(t-\tau)A} B_1 w(\tau) \, d\tau.\]

An additional assumption is needed:

$4'$. $\|D_{11h}\| < 1$. 

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This is a necessary condition for $\|T_{zw}\| < 1$; its computation was studied in [4].

Corresponding to the two integers $m$ and $n$, introduce the discrete sampling operator $S_m : \ell \to \ell$ defined via

$$\psi = S_m \phi \iff \psi(k) = \phi(mk)$$

and the discrete hold operator $H_n : \ell \to \ell$ via

$$\psi = H_n \phi \iff \psi(kn + j) = \phi(k), \quad j = 0, 1, \ldots, n - 1.$$ 

Now we bring in a discrete LTI system

$$G_d := \begin{bmatrix} A_d & B_{1d} & B_{2d} \\ C_{1d} & D_{11d} & D_{12d} \\ C_{2d} & 0 & 0 \end{bmatrix}. \tag{19}$$

Here $G_d$ is an equivalent system for the single-rate $\mathcal{H}_\infty$ sampled-data problem with sampling period $\tau$; several sets of realization matrices were given in several recent papers, e.g., [4, 21]. Define the discrete system $T_{\zeta \omega} : \omega \leftrightarrow \zeta$ as in Figure 6, where $K_d$ is as in Section 2 and

![Diagram](image)

Figure 6: An LTI discrete system

$$G_d = \begin{bmatrix} I_{mn} & 0 \\ 0 & I_n S_m \end{bmatrix} G_d \begin{bmatrix} I_{m-1} & 0 \\ 0 & H_n I_{m-1} \end{bmatrix}.$$ 

It is not hard to check that $G_d$ is LTI, causal, and finite-dimensional. (In fact, based on (19) a state-space representation of $G_d$ is not difficult to derive.) Thus Figure 6 is LTI. The following result establishes the connection between the multirate $\mathcal{H}_\infty$ problem and a discrete $\mathcal{H}_\infty$ problem.

**Theorem 2** Under Assumptions 1-3 and 4', we have

(i) $K_d$ internally stabilizes $G$ iff $K_d$ internally stabilizes $G_d$;

(ii) $\|T_{zw}\| < 1$ iff $\|T_{\zeta \omega}\|_\infty < 1$. 

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Before we prove the result, it is beneficial to introduce a notation: Given an operator $K$ and an operator matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

the associated linear fractional transformation is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$  

Of course, the domains and co-domains of the operators must be compatible and the inverse must exist.

**Proof of Theorem 2** The first statement is relatively easy to establish (see Lemma 2); so we look at the second.

Note that $S_{mh}$ and $H_{nh}$ can be written

$$S_{mh} = S_mS_h, \quad H_{nh} = H_hH_n.$$  

So Figure 2 can be viewed as a single-rate system with time-varying control:

$$T_{zw} = \mathcal{F}(G, H_{nh}K_dS_{mh}) = \mathcal{F}(G, H_hK_{d1}S_h),$$

where $K_{d1} := H_hK_dS_m$ is time-varying. The single-rate result in [4] or [21] does not require the digital controller to be time-invariant and so can be applied now:

$$||\mathcal{F}(G, H_hK_{d1}S_h)|| < 1 \Leftrightarrow ||\mathcal{F}(G_d, K_{d1})|| < 1,$$

the latter norm being the $\ell_2$-induced one. From the definitions of $G_d$ and $K_d$ we get

$$T_{\zeta w} = \mathcal{F}(G_d, K_d) = L_{mn}\mathcal{F}(G_d, K_{d1})L_{mn}^{-1}.$$  

Since $L_{mn}$ is norm-preserving, we have

$$||T_{\zeta w}||_\infty = ||T(G_d, K_{d1})||.$$  

This completes the proof. \textbf{QED}

Theorem 2 provides a way to analyze a multirate sampled-data system: For a given admissible $K_d$, checking the $H_\infty$ performance condition $||T_{zw}|| < 1$ amounts to checking if the $H_\infty$ norm of the real-rational matrix $T_{\zeta w}(\lambda)$ is $< 1$.

Moreover, and more importantly, the theorem also implies that the multirate $H_\infty$ problem can be recast as a constrained $H_\infty$ model-matching problem. To see this, we note that the $(2, 2)$ block in $G_d$, $G_{22d}$, is in $\Omega_s(n, m)$. Parametrize all the stabilizing and admissible controllers $K_d$ for $G_{22d}$ as in Section 4 to get

$$T_{\zeta w} = T_1 - T_2QT_3,$$

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where $T_1, T_2, T_3$ are real-rational matrices in $\mathcal{H}_\infty$ and can be found from $G_d$. Then the multirate $\mathcal{H}_\infty$ problem is equivalent to the discrete $\mathcal{H}_\infty$ model-matching problem of finding a $Q \in \mathcal{RH}_\infty$ with the constraint $Q(0) \in \Omega(m,n)$ such that

$$||T_1 - T_2 QT_3||_\infty < 1.$$  

If such a $Q$ exists, we say the multirate $\mathcal{H}_\infty$ problem is *solveable*.

From now on we shall focus on this constrained $\mathcal{H}_\infty$ problem. For regularity, we need an assumption similar to Assumption 4 in Section 4:

5'. For every $\lambda$ on the unit circle, $T_2(\lambda)$ and $T_3^\sim(\lambda)$ are both injective.

Under this assumption, perform an inner-outer factorization $T_2 = T_2^\lambda T_2^o$ and an co-inner-outer factorization $T_3 = T_3^\lambda T_3^c$, where $T_2^o$ and $T_3^c$ are both invertible over $\mathcal{RH}_\infty$. Apply unitary transformations to $T_1 - T_2 QT_3$ and define

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} := \begin{bmatrix} T_2^\sim & T_3^\sim \\ I - T_2^\lambda T_3^\sim & I - T_3^\lambda T_3^\sim \end{bmatrix} T_1 \begin{bmatrix} T_3^\sim & I - T_3^\lambda T_3^\sim \end{bmatrix}. $$

The constrained model-matching problem is equivalent to the following four-block problem of finding a $Q \in \mathcal{RH}_\infty$ with $Q(0) \in \Omega(m,n)$ such that

$$|| \begin{bmatrix} R_{11} - T_2^o QT_3^c & R_{12} \\ R_{21} & R_{22} \end{bmatrix} ||_\infty < 1. $$ (20)

We shall consider the causality constraint at a later stage; let us now drop this constraint on $Q(0)$ and look at the unconstrained four-block problem. This allows us to use the powerful result in [17] to parametrize all $Q$ in $\mathcal{RH}_\infty$ achieving (20).

**Lemma 4** [17] There exists a $Q \in \mathcal{RH}_\infty$ such that (20) holds iff

$$|| \begin{bmatrix} \Pi_{\mathcal{H}_\infty^+} & 0 \\ 0 & I \end{bmatrix} R|_{\mathcal{H}_2 \oplus \mathcal{L}_2} ||_\infty < 1. $$ (21)

Moreover, if (21) is satisfied, then there exists an $\mathcal{RH}_\infty$ matrix

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

with $K_{12}^{-1}, K_{21}^{-1} \in \mathcal{RH}_\infty$ and $\|K_{22}\|_\infty < 1$ such that all $Q \in \mathcal{RH}_\infty$ satisfying (20) are characterized by

$$Q = F(K, Q_1), \quad Q_1 \in \mathcal{RH}_\infty, \quad \|Q_1\|_\infty < 1. $$ (22)
We refer to [17] for the details of checking inequality (21) and the expression of $K$. Hereafter, we shall assume that (21) is true. This is also necessary for the solvability of the multirate $\mathcal{H}_\infty$ problem.

Look at the constant term in (22):

$$Q(0) = K_{11}(0) + K_{12}(0)Q_1(0)[I - K_{22}(0)Q_1(0)]^{-1}K_{21}(0).$$

This is too complicated to study the causality constraint; but it would reduce to an affine mapping from $Q_1(0)$ to $Q(0)$ if we had $K_{22}(0) = 0$. Next we shall make this happen. Introduce another linear fractional transformation:

$$Q_1 = \mathcal{F}(V, Q_2).$$

Here $V$, partitioned as usual, is a constant unitary matrix. It follows that the mapping $Q_2 \mapsto Q_1$ is bijective from the open unit ball of $\mathcal{RH}_\infty$ onto itself [31]. Thus all $Q$ satisfying (20) can be re-parametrized by

$$Q = \mathcal{F}[K, \mathcal{F}(V, Q_2)]$$
$$= \mathcal{F}(L, Q_2), \quad Q_2 \in \mathcal{RH}_\infty, \quad \|Q_2\|_\infty < 1,$$

where $L$, partitioned as usual, can be written in terms of $K$ and $V$:

$$L = \begin{bmatrix}
K_{11} + K_{12}V_{11}(I - K_{22}V_{11})^{-1}K_{21} & K_{12}(I - V_{11}K_{22})^{-1}V_{12} \\
V_{21}(I - K_{22}V_{11})^{-1}K_{21} & V_{21}(I - K_{22}V_{11})^{-1}K_{22}V_{12} + V_{22}
\end{bmatrix}.$$

Setting

$$V = \begin{bmatrix}
K_{22}(0) \\
[I - K_{22}(0)K_{22}(0)]^{1/2} \\
-K_{22}(0)
\end{bmatrix}$$

achieves $L_{22}(0) = 0$; furthermore, $L_{12}(0)$ and $L_{21}(0)$ are still nonsingular. (Since $\|K_{22}(0)\| < 1$, this $V$ is well-defined and is unitary.)

To recap, the set of all $Q \in \mathcal{RH}_\infty$ achieving (20) is parametrized by

$$Q = \mathcal{F}(L, Q_2), \quad Q_2 \in \mathcal{RH}_\infty, \quad \|Q_2\|_\infty < 1.$$ 

Here $L$ has the additional properties that $L_{22}(0) = 0$, $L_{12}(0)$ and $L_{21}(0)$ are nonsingular. Thus

$$Q(0) = L_{11}(0) + L_{12}(0)Q_2(0)L_{21}(0). \quad (23)$$

Now we bring in the causality constraint on $Q(0)$. Our goal is to find the necessary and sufficient condition for the existence of a $Q_2 \in \mathcal{RH}_\infty$ with $\|Q_2\|_\infty < 1$ such that $Q(0)$ in (23) lies in $\Omega(m, n)$. Since $Q(0)$ depends only on $Q_2(0)$ and in general $\|Q_2\|_\infty \geq \|Q_2(0)\|$, the problem is the same as searching a constant matrix $Q_2(0)$ with $\|Q_2(0)\| < 1$ such that $Q(0) \in \Omega(m, n)$, the norm being the largest singular value of $Q_2(0)$.

As in Section 4, introduce matrix factorizations

$$L_{12}(0) = R_1U_1, \quad L_{21}(0) = -U_2R_2,$$
where $R_1, R_2, U_1, U_2$ are all square, $R_1, R_2$ are lower-triangular, and $U_1, U_2$ are orthogonal. Substitute the factorizations into (23) and pre- and post-multiply by $R_1^{-1}$ and $R_2^{-1}$ respectively to get

$$R_1^{-1}Q(0)R_2^{-1} = R_1^{-1}L_{11}(0)R_2^{-1} - U_1Q_2(0)U_2.$$  

Define

$$W := R_1^{-1}L_{11}(0)R_2^{-1}, \quad P := U_1Q_2(0)U_2.$$  

It follows that $||Q_2(0)|| < 1$ iff $||P|| < 1$ and $Q(0) \in \Omega(m, n)$ iff $R_1^{-1}Q(0)R_2^{-1} \in \Omega(m, n)$ (Section 4). Therefore, we arrive at the following equivalent matrix problem: Given $W$, find $P$ with $||P|| < 1$ such that $W - P \in \Omega(m, n)$.

Partition $W$ and $P$ as required in $\Omega(m, n)$. Apparently, $P$ must cancel the $\Omega^\perp$-part of $W$. So the blocks in $P$ corresponding to the zero blocks in $\Omega(m, n)$ are fixed and equal to those blocks in $W$; the other blocks are free for choice. The question is when it is possible and then how to choose the free blocks to make $||P|| < 1$.

Before we attack the general case, let us look at the example with $m = 3$ and $n = 2$ (Section 2). Then $W$ and $P$ are given by

$$W = \begin{bmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \\ W_{20} & W_{21} \end{bmatrix}, \quad P = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \\ X_{20} & X_{21} \end{bmatrix},$$

where $X_{ij}$ denote the free blocks in $P$. Obviously,

$$\min_{X_{ij}} ||P|| = || \begin{bmatrix} W_{01} \\ W_{11} \end{bmatrix} ||.$$  

So the matrix problem is solvable iff the right-hand side is $< 1$ and when this is so we choose all $X_{ij}$ to be zero to attain the minimum.

The general case is much more complicated. First, let us distinguish two cases: The fixed blocks in $P$, or the zero blocks in $\Omega(m, n)$, take the (block) row-echelon form if $m < n$ and the (block) column-echelon form if $n < m$. Next, we need to locate all the maximum fixed submatrices of $P$, namely, the submatrices which consist of only the fixed blocks and have maximum sizes. To do this, denote the integer part of a positive real number $x$ by $[x]$. If $m < n$, let $l = m$ and for $k = 0, 1, \ldots, l - 1$, define

$$M_k = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \begin{bmatrix} I \end{bmatrix} k + 1 \text{ blocks}$$

$$m \text{ blocks}$$

$$N_k = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \begin{bmatrix} I \end{bmatrix} N \text{ blocks}$$

$$n - 1 - \left[ \frac{kn}{m} \right] \text{ blocks}$$

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If $n < m$, define $l = n - 1$ and for $k = 0, 1, \ldots, l - 1$, define

$$M_k = \begin{bmatrix} I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ I & \cdots & I \end{bmatrix}^{(k+1)m} \text{ blocks}$$

$$N_k = \begin{bmatrix} 0 \\ I \\ \vdots \\ I \end{bmatrix}^{n \text{ blocks}}$$

Then it can be checked that $M_kW_Nk, k = 0, 1, \ldots, l - 1$, are exactly those maximum fixed submatrices of $P$. Define

$$\mu := \max\{\|M_kW_Nk\| : k = 0, 1, \ldots, l - 1\}.$$

**Theorem 3** Under Assumptions 1-3 and 4'-5', the multirate $\mathcal{H}_\infty$ problem is solvable, i.e., there exists a matrix $P$ with $\|P\| < 1$ such that $W - P \in \Omega(m, n)$, iff $\mu < 1$.

The proof requires a result on norm preserving dilations from operator theory, which is specialized to constant matrices below.

**Lemma 5** [29], [10] Assume that $A, B, C$ are fixed matrices of appropriate dimensions. Then

$$\inf_X \| \begin{bmatrix} C & A \\ X & B \end{bmatrix} \| = \max\{\| \begin{bmatrix} C & A \end{bmatrix} \|, \| \begin{bmatrix} A \\ B \end{bmatrix} \|\} =: \alpha.$$

Moreover, a minimizing $X$ is given by

$$X = -B(\alpha^2I - AA^*)^{-1}C.$$

**Proof of Theorem 3** The necessity follows easily since each $M_kW_Nk$ is a fixed submatrix of $P$. In what follows, we shall prove the sufficiency when $m = 3$ and $n = 5$; the general case is no harder conceptually and follows similarly.

When $m = 3$ and $n = 5$, the matrix $P$ is of the form

$$P = \begin{bmatrix} X_{00} & W_{01} & W_{02} & W_{03} & W_{04} \\ X_{10} & X_{11} & W_{12} & W_{13} & W_{14} \\ X_{20} & X_{21} & X_{22} & X_{23} & W_{24} \end{bmatrix}.$$

Thus the three fixed maximum matrices $M_kW_Nk, k = 0, 1, 2$, are

$$\begin{bmatrix} W_{01} & W_{02} & W_{03} & W_{04} \end{bmatrix}, \begin{bmatrix} W_{02} & W_{03} & W_{04} \\ W_{12} & W_{13} & W_{14} \end{bmatrix}, \begin{bmatrix} W_{04} \\ W_{14} \\ W_{24} \end{bmatrix}.$$
respectively. It suffices to show that we can choose $X_{ij}$ in $P$ to achieve $\|P\| = \mu$. First, choose $X_{11}$ via Lemma 5 so that $\|M_1 PN_0\|$ is minimized:

$$\|M_1 PN_0\| = \max\{\|M_0 WN_0\|, \|M_1 WN_1\|\}.$$ 

Fix this $X_{11}$. Next, choose $\begin{bmatrix} X_{21} & X_{22} & X_{23} \end{bmatrix}$ again via Lemma 5 so that $\|M_2 PN_0\|$ is minimized:

$$\|M_2 PN_0\| = \max\{\|M_1 PN_0\|, \|M_2 WN_2\|\}.$$ 

Finally, set $X_{00}, X_{10}, X_{20}$ to zero. These choices of $X_{ij}$ yield $\|P\| = \|M_2 PN_0\| = \mu$. \textbf{QED}

The proof provides a constructive procedure to determine the free blocks in $P$ to get $\|P\| = \mu$; this is done by sequentially applying Lemma 5 as was illustrated in the proof.

To summarize, let us list the solvability conditions for the multirate $\mathcal{H}_\infty$ control problem $\|T_{zw}\| < 1$:

(a) $\|D_{11}\| < 1$;

(b) $\| \begin{bmatrix} \mathcal{H}_2 & 0 \\ 0 & I \end{bmatrix} R|_{\mathcal{H}_2 \otimes \mathcal{L}_2} \| < 1$;

(c) $\mu < 1$.

Condition (a) was studied in detail in [4]. Condition (b) is the solvability condition for a standard four-block $\mathcal{H}_\infty$ problem, see, e.g., [17] for checking this condition. When conditions (a-b) hold, a necessary and sufficient test for condition (c) is given in Theorem 3; it ammounts to computing the norms (largest singular values) of several constant matrices.

Finally, by recapping the steps in the reduction process described in this section, one can obtain an implementable procedure for computing an admissible and stabilizing $K_d$ to achieve $\|T_{zw}\| < 1$, if the solvability conditions are met.

\section{Concluding Remarks}

In this paper we have addressed causality constraints in direct designs of multirate sampled-data control systems using $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures. Explicit solutions are given for the $\mathcal{H}_2$-optimal controller and the $\mathcal{H}_\infty$-suboptimal controllers which achieve the performance requirement $\|T_{zw}\| < 1$. $\mathcal{H}_\infty$ controllers which are arbitrarily close to optimality can be computed based on the solvability conditions (a-c) (with proper scaling) and a standard bisection search.

A good project for future research is to do some case studies and study the trade-offs between choices of sampling and hold rates and performance of the system. This is quite feasible using the results of this paper. Finally, we remark that although the setup in this paper has a uniform sampling rate and a uniform hold rate, extension to the more general setup in, e.g., [28, 30], is quite possible using the techniques developed in this paper.
References


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